Solution of a cauchy problem for a diffusion equation in a Hilbert space by a Feynman formula

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Solution of a Cauchy Problem for a Diffusion Equation in a Hilbert Space by a Feynman Formula

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Abstract. The Cauchy problem for a class of diffusion equations in a Hilbert space is studied. It is proved that the Cauchy problem in well posed in the class of uniform limits of infinitely smooth bounded cylindrical functions on the Hilbert space, and the solution is presented in the form of the so-called Feynman formula, i.e., a limit of multiple integrals against a gaussian measure as the multiplicity tends to infinity. It is also proved that the solution of the Cauchy problem depends continuously on the diffusion coefficient. A process reducing an approximate solution of an infinite-dimensional diffusion equation to finding a multiple integral of a real function of finitely many real variables is indicated.

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1. INTRODUCTION

The main result of the present paper claims that the Cauchy problem

$$\begin{cases} u_t'(t,x) = g(x)\,\mathrm{tr}\,(Au_{xx}''(t,x))\,; & t \geqslant 0, x \in H, \\ u(0,x) = u_0(x); & x \in H. \end{cases} \tag{1}$$

has a solution (which is unique in some class of function) given by

$$u(t,x) = \lim_{n\to\infty} \underbrace{\int_H \int_H \dots \int_H}_{n \text{ times}} u_0(y_1) \mu^{y_2}_{\frac{2t}{n}g(y_2)A}(dy_1) \mu^{y_3}_{\frac{2t}{n}g(y_3)A}(dy_2) \dots \mu^{y_n}_{\frac{2t}{n}g(y_n)A}(dy_{n-1}) \mu^{x}_{\frac{2t}{n}g(x)A}(dy_n), \tag{2}$$

and this solution continuously depends (in the uniform norm) on the functions g and u_0. Here H stands for a real separable Hilbert space, A is a linear positive trace-class selfadjoint operator on H, $u\colon [0,+\infty) \times H \to \mathbb{R}$ is the desired function, $u_0\colon H \to \mathbb{R}$ and $g\colon H \to \mathbb{R}$ are given functions (with $g(x) \geqslant g_0 \equiv \mathrm{const} > 0$ for every $x \in H$), $\mu^x_{(2t/n)g(x)A}$ is the Gaussian measure on H with the mean x and the correlation operator $(2t/n)g(x)A$. For details, see Theorem 4.

Relations of the form (2) in which some function is represented as a limit of a multiple integral as the multiplicity tends to infinity are referred to as Feynman formulas (see [21, 22]). The term "Feynman formula" was introduced in this context by Smolyanov [26]. For a survey of current results concerning Feynman formulas, see [27].

The paper is organized as follows. Section 2 contains the main notation, definitions, and auxiliary facts. In Sec. 3 we discuss properties of the operators used in the setting and the solution of problem (1), and the main theorem is proved in Sec. 4. In the final section we discuss the size of classes in which we seek solutions of the Cauchy problem.

2. NOTATION, DEFINITIONS, AND PRELIMINARIES

Let H denote a separable Hilbert space over $\mathbb{R}$ with the inner product $\langle\cdot,\cdot\rangle$. Denote by $C_b(H,\mathbb{R})$ the Banach space of all bounded continuous functions $H \to \mathbb{R}$ equipped with the uniform norm $\|f\| = \sup_{x\in H} |f(x)|$. Recall that a function $f\colon H \to \mathbb{R}$ is said to be *cylindrical* if there are vectors $e_1,\dots,e_n$ in H and a function $f^n\colon \mathbb{R}^n \to \mathbb{R}$ such that $f(x) = f^n(\langle x, e_1,\rangle, \dots, \langle x, e_n,\rangle)$ for every $x \in H$. Let $D = C^\infty_{b,c}(H,\mathbb{R})$ denote the space of all bounded cylindrical functions $H \to \mathbb{R}$ which are infinitely Fréchet differentiable at any point of H and whose Fréchet derivatives of any order are bounded (and continuous).

If $f\colon H \to \mathbb{R}$, then $f''(x)$ stands for the second-order Fréchet derivative of f evaluated at x. and $A\colon H \to H$ a linear trace-class[1]selfadjoint positive nondegenerate operator with the domain H. Introduce the differential operator Δ_A by the rule $(\Delta_A f)(x) := \operatorname{tr} Af''(x)$. Let $X = \overline{C^\infty_{b,c}(H,\mathbb{R})}$ be the closure of D in $C_b(H,\mathbb{R})$. For any function $g \in X$ bounded away from zero (i.e., $g(x) \geqslant g_0 \equiv \text{const} > 0$), introduce the operator $g\Delta_A$ by the rule $(g\Delta_A f)(x) := g(x)(\Delta_A f)(x) = g(x) \operatorname{tr} Af''(x)$.

For a function $u\colon [0,+\infty) \times H \to \mathbb{R}$ we pose the Cauchy problem for the diffusion equation (1). The abstract Cauchy problem for a function $U\colon [0,+\infty) \to X$ of the form

$$\begin{cases} (d/dt)U(t) = g\overline{\Delta_A}U(t); \quad t \geqslant 0, \\ U(0) = u_0, \end{cases} \tag{3}$$

where $g\overline{\Delta_A}$ stands for the closure of the unbounded differential operator $g\Delta_A$ and $(g\Delta_A f)(x) = g(x)\operatorname{tr}(Af''(x))$ is closely related to (1). Following [23], by a *strong* solution of problem (3) we mean a function $U\colon [0,+\infty) \to X$ such that

$$U \in C^1([0,+\infty),X), \quad U(t) \in D_1;\ t \geqslant 0, \quad (d/dt)U(t) = g\overline{\Delta_A}U(t);\ t \geqslant 0, \quad U(0) = u_0, \tag{4}$$

where $D_1 = \{f \in X : \exists (f_j) \subset D : \lim_{j\to\infty} f_j = f, \exists \lim_{j\to\infty} \operatorname{tr} A(f_j)''\}$ stands for the space of functions $f \in X$ uniformly approximable by cylindrical functions $f_j \in D$ for which the sequence $\operatorname{tr} A(f_j)''$ converges uniformly. By a *mild* solution of problem (3) we mean a function $U\colon [0,+\infty) \to X$ such that

$$U \in C([0,+\infty),X), \quad \int_0^t U(s)ds \in D_1;\ t \geqslant 0, \quad U(t) = g\overline{\Delta_A}\int_0^t U(s)ds + u_0;\ t \geqslant 0. \tag{5}$$

Problem (1) is reduced to (3) as follows. A function $u\colon (t,x) \mapsto u(t,x)$ of (t,x) is treated as a function $u\colon t \mapsto [x \mapsto u(t,x)]$ of t with the range in some function space of the argument x. Then $u(t,x) = (U(t))(x)$, $t \geqslant 0, x \in H$. The terms “strong solution” and “mild solution” of problem (1) are now transferred from solutions of problem (3) using this correspondence.

The following theorem is used below.

Theorem 1 (Chernoff product formula, see, e.g., [23, Th. III,5.2])**.** *Let X be a real Banach space and let $L_b(X,X)$ be the space of all bounded linear operators on X equipped with the operator norm. Let a function $S\colon [0,+\infty) \to L_b(X,X)$ be given such that $S_0 = I$, where I stands for the identity operator, let $\|(S_t)^m\|$ be bounded for some constant M for any $t \geqslant 0$ and $m \in N$, and let the strong limit $\lim_{t\downarrow 0} t^{-1}(S_t\varphi - \varphi) =: L\varphi$ exist for any $\varphi \in D \subset X$, where D and $(\lambda_0 I - L)(D)$ are dense subspaces of X for some $\lambda_0 > 0$. Then the closure $\overline{L}$ of L is an infinitesimal generator of some bounded strongly continuous operator semigroup $(T_t)_{t\geqslant 0}$ given by the formula $T_t\varphi = \lim_{n\to\infty}(S_{t/n})^n\varphi$, where the limit exists for any $\varphi \in X$ and is uniform with respect to $t \in [0,t_0]$ for any $t_0 > 0$.*

Let $x \in H$ and $B\colon H \to H$ a linear operator. By the symbol μ^x_B we denote the Gaussian probability measure on a Borel sigma algebra in H with the mean x and the correlation operator B. Let $(S_t\varphi)(x) := \int_H \varphi(x+y)\mu_{2tg(x)A}(dy)$ for $t > 0$ and $S_0\varphi := \varphi$ for $\varphi \in X$.

The objective of the paper is to prove, using Theorem 1, that the solution of the Cauchy problem (1) is given by the Feynman formula (2). We develop and strengthen results of [15] obtained there for the simplest diffusion equation with a variable coefficient at the higher derivative, namely, we consider here an infinite-dimensional case and prove the existence of a solving semigroup. We also discuss the reduction of an approximate evaluation of integrals over an infinite-dimensional space

[1]For the definition and properties of trace-class operators in a Hilbert space, see, e.g., G. J. Murphy, *C^*-Algebras and Operator Theory* (Academic Press, Inc., Boston, MA, 1990), § 2.4.

in the Feynman formula to evaluation of integrals over finite-dimensional spaces and the size of function classes in use.

Recall some notation, definitions, and facts needed below. Unless otherwise stated, the symbol X stands for an abstract real Banach space. We use the apparatus of strongly continuous one-parameter semigroups of bounded linear operators in Banach spaces and their infinitesimal generators; for the definitions and the main properties, we use the book [23]. For a closed linear operator $\overline{L}$, we refer to the problem

$$\begin{cases} (d/dt)F(t) = \overline{L}F(t); \quad t \geqslant 0, \\ F(0) = F_0, \end{cases} \tag{6}$$

for a function $F\colon [0,+\infty) \to X$ as the abstract Cauchy problem associated with the closed linear operator $\overline{L}\colon X \supset \mathrm{Dom}(\overline{L}) \to X$ and the vector $F_0 \in X$. A function $F\colon [0,+\infty) \to X$ is said to be a *classical solution* of the abstract Cauchy problem (6) if the function F has continuous derivative $F'\colon [0,+\infty) \to X$ in the strong operator topology for every $t \geqslant 0$, $F(t) \in \mathrm{Dom}(\overline{L})$ for every $t \geqslant 0$, and (6) holds; a strongly continuous function $F\colon [0,+\infty) \to X$ is referred to as a *mild solution* of (6) if $\int_0^t F(s)ds \in \mathrm{Dom}(\overline{L})$ and $F(t) = \overline{L}\int_0^t F(s)ds + F_0$ for every $t \geqslant 0$. Recall that, if an operator $(\overline{L}, \mathrm{Dom}(\overline{L}))$ is the infinitesimal generator of a strongly continuous semigroup $(T_s)_{s\geqslant 0}$, then, for every $F_0 \in \mathrm{Dom}(\overline{L})$, there is a unique classical solution of (6) given by the formula $F(t) = T(t)F_0$ and, for every $F_0 \in X$, there is a unique mild solution of (6) given by the formula $F(t) = T(t)F_0$ (see [23, II, Proposition 6.2]).

A strongly continuous operator semigroup $(T_t)_{t\geqslant 0}$ is said to be *contractive* (or a contraction semigroup) if $\|T_t\| \leqslant 1$ for every $t \geqslant 0$. A linear operator $L\colon X \supset \mathrm{Dom}(L) \to X$ on a Banach space X is said to be *dissipative* if $\|Lx - \lambda x\| \geqslant \lambda\|x\|$ for every $\lambda > 0$ and any $x \in \mathrm{Dom}(L)$. Recall [23, II, Proposition 3.14] that a linear dissipative operator $L\colon X \supset \mathrm{Dom}(L) \to X$ on a Banach space X whose domain $\mathrm{Dom}(L)$ is dense in X admits the closure, $\overline{L} : X \supset \mathrm{Dom}(\overline{L}) \to X$, which is also dissipative. By the Lumer–Phillips theorem (see, e.g., [23, Th. II.3.15]), the closure $\overline{L}$ of a dissipative operator $(L, \mathrm{Dom}(L))$ on a Banach space X is an infinitesimal generator of a contraction semigroup if and only if the image of the operator $\lambda I - L$ is dense in X for some (and hence for any) $\lambda > 0$.

Let $(e^{\overline{L_n}t})_{t\geqslant 0}$ be strongly continuous operator semigroups on a Banach space X with infinitesimal generators $(\overline{L_n}, \mathrm{Dom}(\overline{L_n}))$ satisfying the stability condition $\|e^{\overline{L_n}t}\| \leqslant Me^{wt}$ for constants $M \geqslant 1$, $w \in \mathbb{R}$, and all $t \geqslant 0$ and $n \in \mathbb{N}$. Let there be a densely defined operator $(L, \mathrm{Dom}(L))$ on X such that $\overline{L_n}x \to Lx$ for all x in a core D of L such that the image of the operator $(\lambda_0 I - L)$ is dense in X for some $\lambda_0 > 0$. Then the semigroups $(e^{\overline{L_n}t})_{t\geqslant 0}, n \in \mathbb{N}$ converge strongly (and uniformly with respect to $t \in [0, t_0]$ for any $t_0 > 0$) to a strongly continuous semigroup $(e^{t\overline{L}})_{t\geqslant 0}$ with the infinitesimal generator $\overline{L}$ [23, Th. III.4.9].

We also need some fundamentals concerning Fréchet differentiation in Banach and Hilbert spaces (for details, see [8]. In particular, the second Fréchet derivative of a function $f\colon H \to \mathbb{R}$ is regarded as a mapping $f''\colon H \to L_b(H,H)$. For $n+1$ times Fréchet differentiable function f, we have the Taylor expansion

$$f(x+h) = f(x) + (1/1!)f'(x)h + (1/2!)f''(x)(h,h) + \cdots + (1/n!)f^{(n)}(x)(h,\ldots,h) + R_n(x,h), \tag{7}$$

where $R_n(x,h) \in \overline{\mathrm{conv}\{(n+1)!^{-1}f^{(n+1)}(x+\theta h)(h,\ldots,h) : \theta \in (0,1)\}} \subset \mathbb{R}$, and therefore

$$|R_n(x,h)| \leqslant (\|h\|^{n+1}/(n+1)!) \sup_{z\in[x,x+h]} \|f^{(n+1)}(z)\|. \tag{8}$$

The following statement is used below.

Proposition 1. *Let f be a cylindrical real-valued function on H and let $(e_k)_{k\in\mathbb{N}}$ be an orthonormal basis in H such that f does not depend on the coefficients $\langle f, e_k\rangle$ with $k > n$ for some $n \in \mathbb{N}$. The the function f has a derivative in a direction h if and only if the function f^n has a derivative in the direction $(\langle h, e_1\rangle, \dots, \langle h, e_n\rangle) \in \mathbb{R}^n$, and $f'(x)h = \langle h, (\partial_1 f^n(\langle x, e_1\rangle, \dots, \langle x, e_n\rangle), \ \dots, \ \partial_n f^n(\langle x, e_1\rangle, \dots, \langle x, e_n\rangle), \ 0, \ 0, \ 0, \dots)\rangle$, where the symbol $\partial_j f^n$ stands for the partial derivative with respect to the jth argument of the function $f^n\colon \mathbb{R}^n \to \mathbb{R}$. If the function f has Fréchet derivative at x, then*

$$f'(x) = (\partial_1 f^n(\langle x, e_1\rangle, \dots, \langle x, e_n\rangle), \ \dots, \ \partial_n f^n(\langle x, e_1\rangle, \dots, \langle x, e_n\rangle), \ 0, \ 0, \ 0, \dots)^T, \tag{9}$$

and f has Fréchet derivative on H if and only if f^n has Fréchet derivative on $\mathbb{R}^n$. If $A\colon H \to H$ is a trace-class operator, then

$$\operatorname{tr} A f''(x) = \sum_{s=1}^{n}\sum_{k=1}^{n} \langle Ae_s, e_k\rangle \big(\partial_k \partial_s f^n(\langle x, e_1\rangle, \dots, \langle x, e_n\rangle)\big).$$

The proof is straightforward.

We also need some definitions and facts concerning Gaussian probability measures on a Hilbert space [SS]. Recall the definitions of measurable sets and of measures on H. A set $B \subset H$ is said to be cylindrical if there are an n-dimensional linear subspace $H_n \subset H$ and a Borel set $B_n \subset B$ such that $B = P^{-1}(B_n)$, where $P\colon H \to H_n$ stands for the orthogonal projection. For a fixed n-dimensional subspace $H_n \subset H$, the family $\{P^{-1}(B_n) : B_n \text{ is a Borel set in } H_n\}$ is a σ-algebra in H. If H is infinite-dimensional, then the family of cylindrical sets in H forms an algebra (rather than a σ-algebra, because, e.g., if (e_k) is a basis in H, then the countable intersection of the sets $\{x : 0 < \langle x, e_k\rangle < 1\}$ is not a cylindrical set). We refer to this set algebra as the *cylinder algebra* and to the smallest σ-algebra containing all cylinders as the *cylindrical σ-algebra*. Since H is separable, the cylindrical σ-algebra on H coincides with the Borel one.

By a cylindrical measure μ on H we mean a real finitely additive set function defined on the algebra of all cylindrical sets in H and such that, for every finite-dimensional linear subspace $H_n \subset H$ for the orthogonal projection $P\colon H \to H_n$, the restriction of the measure μ to the σ-algebra $\{P^{-1}(B_n) : \ B_n \text{ is a Borel subset of } H_n\}$ is a countably-additive measure.

For any finite-dimensional subspace $H_n \subset H$, the restriction of a cylindrical measure to the σ-algebra $\{P^{-1}(B_n) : B_n \text{ is a Borel subset of } H_n\}$ is a countably additive measure. Therefore, for the Borel cylindrical functions, the Lebesgue integral against a cylindrical measure has a definite meaning.

By the Fourier transform of a cylindrical measure μ on H we mean a function $\widetilde{\mu}\colon H \to \mathbb{C}$, $\widetilde{\mu}(z) := \int_H e^{i\langle z, y\rangle}\mu(dy)$. As is well known [9, p. 16], a cylindrical measure is defined by its Fourier transform uniquely. For a real separable Hilbert space H, a cylindrical measure μ_A^x defined on the cylinder algebra is referred to as a *nondegenerate Gaussian measure* if the Fourier transform of μ_A^x is of the form $\widetilde{\mu_A^x}(z) = \exp\big(i\langle z, x\rangle - \frac{1}{2}\langle Az, z\rangle\big)$, where $x \in H$ is the so-called expectation of μ_A^x and $A\colon H \to H$ is the so-called correlation operator of μ_A^x. If $x = 0$, then the measure is said to be centered, and we write $\mu_A^0 = \mu_A$.

The correlation operators of the Gaussian measures used below are selfadjoint positive trace-class operators. By [4, Ch. II, §2, 3°]), if a function $\varphi\colon H \to \mathbb{R}$ is cylindrical, continuous, and bounded, if $e_1, \dots, e_n$ is a full family of eigenvectors of an operator C, and $c_1, \dots, c_n$ are the corresponding eigenvalues, then

$$\int_H \varphi(y)\mu_A(dy) = (2\pi)^{-n/2}\Big(\prod_{i=1}^{n} c_i\Big)^{-1/2} \int_{\mathbb{R}^n} \varphi^n(z_1, \dots, z_n) \exp\Big(-\sum_{i=1}^{n} z_i^2/(2c_i)\Big) dz_1 \dots dz_n. \tag{10}$$

By [4, Ch. II, §2, 1°], if H is a real separable Hilbert space, $A\colon H \to H$ is a trace-class symmetric positive-definite linear operator, μ_A is a centered Gaussian measure on H with the correlation operator A, and $B\colon H\to H$ is a bounded linear operator, then

$$\int_H \langle By,y\rangle \mu_A(dy)=\operatorname{tr}(AB), \tag{11}$$

$$\int_H (\langle By,y\rangle)^2\mu_A(dy)=(\operatorname{tr}(AB))^2+2\operatorname{tr}(AB)^2. \tag{12}$$

Lemma 1. *Let H be a real separable Hilbert space. Let $A\colon H\to H$ be a positive, trace-class, self-adjoint linear operator. Let μ_A be a centered Gaussian measure on H with the correlation operator A. Let $f\colon H\to\mathbb{R}$ be a continuous bounded function. Then*

$$\int_H f(x)\mu_{tA}(dx)=\int_H f(\sqrt{t}x)\mu_A(dx). \tag{13}$$

The proof uses the uniqueness of a measure having a given Fourier transform and the standard theorem concerning the change of measure, in a Lebesgue integral, caused by a measurable mapping.

Finally, recall the following result (see [6, Th. 4.3.1 and 4.3.2, and Corollary 4.3.4].

Lemma 2. *Let $a^{ij}\colon\mathbb{R}^n\to\mathbb{R}$, $i,j=1,\dots,n$, be a function in $C_b^\infty(\mathbb{R}^n,\mathbb{R})$ (the space of bounded real function on $\mathbb{R}^n$ having the bounded derivatives of all orders). Let the ellipticity condition $\sum_{i,j=1}^n a^{ij}(x)\xi_i\xi_j\geqslant\varkappa\|\xi\|^2$ hold for some $\varkappa>0$ and every $\xi=(\xi_1,\dots,\xi_n)\in\mathbb{R}^n$ and $x\in\mathbb{R}^n$. Let a constant $\lambda>0$ and a function $f\in C_b^\infty(\mathbb{R}^n,\mathbb{R})$ be chosen. Then there is a unique function $u\in C_b^\infty(\mathbb{R}^n,\mathbb{R})$ which is a solution of the equation*

$$\sum_{i=1}^n\sum_{j=1}^n a^{ij}(x)\frac{\partial^2}{\partial x_i\partial x_j}u(x)-\lambda u(x)=f(x), \tag{14}$$

and every function $v\in C_b^\infty(\mathbb{R}^n,\mathbb{R})$ satisfies the bound

$$\sup_{x\in\mathbb{R}^n}\left|\sum_{i=1}^n\sum_{j=1}^n a^{ij}(x)\frac{\partial^2}{\partial x_i\partial x_j}v(x)-\lambda v(x)\right|\geqslant\lambda\sup_{x\in\mathbb{R}^n}|v(x)|. \tag{15}$$

3. FAMILIES OF OPERATORS AND THEIR PROPERTIES

Theorem 2. *Let $g\in X$ be such that $g(x)\geqslant g_o\equiv const>0$ for every $x\in H$, and let g be a uniform limit of smooth cylindrical functions in D. Let $t>0$ and $\mu_{2t\,g(x)A}$ a centered Gaussian measure on H with the correlation operator $2t\,g(x)A$. For $t\geqslant 0$ and $\varphi\in C_b(H,\mathbb{R})$, write*

$$(S_t\varphi)(x):=\int_H\varphi(x+y)\mu_{2tg(x)A}(dy)\quad\text{as}\quad t>0\quad\text{and}\quad S_0\varphi:=\varphi. \tag{16}$$

Then $S_t\varphi\in C_b(H,\mathbb{R})$ for $t\geqslant 0$ and $\varphi\in C_b(H,\mathbb{R})$ $S_t\colon C_b(H,\mathbb{R})\to C_b(H,\mathbb{R})$ is a bounded linear operator for any $t\geqslant 0$ with unit norm. If $g\in D$ ($g\in X$), then D (X, respectively) is S_t-invariant for any $t\geqslant 0$. The representation $\int_H\varphi(x+y)\mu_{2tg(x)A}(dy)=\varphi(x)+tg(x)\operatorname{tr}(A\varphi''(x))+t^2r(x,t)$, where $\sup_{x\in H}|r(x,t)|\leqslant\frac{1}{6}\sup_{z\in H}\|\varphi^{(4)}(z)\|(\sup_{z\in H}|g(z)|)^2((\operatorname{tr}A)^2+2\operatorname{tr}A^2)$, holds for any $\varphi\in D$ and every $t>0$, and, finally, $\lim_{t\to0}\sup_{x\in H}\left|t^{-1}(S_t\varphi)(x)-\varphi(x)-(g\Delta_A\varphi)(x)\right|=0$ for every function $\varphi\in D$.

Proof. The integral (16) exists in the Lebesgue sense because the function is bounded and continuous and the (probability) measure is bounded. Let a number $t>0$ and a function $\varphi\in$

$C_b(H,\mathbb{R})$ be chosen. By Lemma 1, $\int_H \varphi(x+y)\mu_{2tg(x)A}(dy) = \int_H \varphi(x+\sqrt{2tg(x)}y)\mu_A(dy)$. It follows from the bound

$$\|S_t\varphi\| = \sup_{x\in H}\left|\int_H \varphi(x+\sqrt{2tg(x)}y)\mu_A(dy)\right| \leqslant \sup_{x\in H}|\varphi(x)|\cdot\mu_A(H) = \|\varphi\|\cdot 1 \tag{17}$$

that the function $S_t\varphi$ is bounded. We claim that the function $S_t\varphi$ is continuous. Let $x_j \to x$. Then $\varphi(x_j+\sqrt{2tg(x_j)}y) \to \varphi(x+\sqrt{2tg(x)}y)$ for any $y \in H$. Moreover, $|\varphi(x_j+\sqrt{2tg(x_j)}y)| \leqslant \|\varphi\| \equiv \text{const}$ and $|\varphi(x+\sqrt{2tg(x)}y)| \leqslant \|\varphi\| \equiv \text{const}$. Therefore, by the dominated convergence theorem, $\lim_{j\to\infty}\int_H \varphi(x_j+\sqrt{2tg(x_j)}y)\mu_A = \int_H \varphi(x+\sqrt{2tg(x)}y)\mu_A$, i.e., $(S_t\varphi)(x_j) \to (S_t\varphi)(x)$. Thus, $S_t\varphi \in C_b(H,\mathbb{R})$. The operator S_t is obviously linear. If 1 is regarded as a constant function on H, then $S_t 1 = 1$, and hence $\|S_t\| \geqslant 1$. The bound (17) shows that $\|S_t\| \leqslant 1$.

To prove that $D\ (X)$ is S_t-invariant, choose a $t > 0$. If $g \in D$, then the operator S_t takes a cylindrical function to a cylindrical one. Indeed, if φ is cylindrical, then $\varphi(x) = \varphi^n(\langle x,e_1\rangle,\dots,\langle x,e_n\rangle)$ for every $x \in H$ and for some $n \in \mathbb{N}$, some function $\varphi^n\colon \mathbb{R}^n \to \mathbb{R}$, and some $e_1,\dots,e_n$ in H. Similarly, since the function g is cylindrical by assumption, we have $g(x) = g^m(\langle x,e_{n+1}\rangle,\dots,\langle x,e_{n+m}\rangle)$ for every $x \in H$ and for some $m \in \mathbb{N}$, some function $g^m : \mathbb{R}^m \to \mathbb{R}$, and some $e_{n+1},\dots,e_{n+m}$ in H. By Lemma 1,

$$\int_H \varphi(x+y)\mu_{2tg(x)A}(dy) = \int_H \varphi(x+\sqrt{2tg(x)}y)\mu_A(dy). \tag{18}$$

Write

$$\Phi(x_1,\dots,x_{n+m}) = \int_H \varphi^n\big(x_1+\sqrt{2tg^m(x_{n+1},\dots,x_{n+m})}\langle y,e_1\rangle, \\ \dots, x_n+\sqrt{2tg^m(x_{n+1},\dots,x_{n+m})}\langle y,e_n\rangle\big)\mu_A(dy).$$

then $(S_t\varphi)(x) = \Phi(\langle x,e_1\rangle,\dots,\langle x,e_{n+m}\rangle)$ for every $x \in H$, which means that $S_t\varphi$ is cylindrical.

We claim now that, if φ is cylindrical and has bounded Fréchet derivative of all orders, then so is $S_t\varphi$. Let us apply Proposition 1. We claim first that Φ has Fréchet derivatives of all orders. To this end, we pass in the expression for Φ from the integral over H to the integral over $\mathbb{R}^n$ (Since φ is cylindrical). Introduce the following notation: $\Psi_n\colon H \ni h \mapsto (\langle h,e_1\rangle,\dots,\langle h,e_n\rangle) \in \mathbb{R}^n$ is a projection, $H_n = \operatorname{span}(e_1,\dots,e_n)$ is the related subspace of H, $I_n\colon H_n \ni h \mapsto (\langle h,e_1\rangle,\dots,\langle h,e_n\rangle) \in \mathbb{R}^n$ is an isomorphism, $P_n\colon H \ni h \mapsto \langle h,e_1\rangle e_1+\dots+\langle h,e_n\rangle e_n \in H_n$ is the projection in H. Then $\Psi_n = I_nP_n$ and $\varphi(x) = \varphi^n(\Psi_n x)$. Moreover, write $\vec{x}^{\,n}_1 = (x_1,\dots,x_n) \in \mathbb{R}^n$ and $\vec{x}^{\,m}_{n+1} = (x_{n+1},\dots,x_m) \in \mathbb{R}^m$.

Since A is nondegenerate and symmetric on H, it follows that P_nA is nondegenerate and symmetric on H_n, and therefore it can be diagonalized in some basis. We may assume without loss of generality that $e_1,\dots,e_n$ is this basis (because the change of a basis in H_n modifies φ^n only). The matrix of the operator P_nA on H_n coincides with the matrix of $C_n = I_nP_nAI_n^{-1}$ on $\mathbb{R}^n$. Let $c_1,\dots,c_n$ -be the eigenvalues of C_n corresponding to the eigenvectors $\Psi_ne_1,\dots,\Psi_ne_n$. Note that $c_i > 0$ and $g^m(\vec{x}^{\,m}_{n+1}) \geqslant g_0 \equiv \text{const} > 0$ for every $\vec{x}^{\,m}_{n+1} \in \mathbb{R}^m$. By (10),

$$\begin{aligned}\Phi(\vec{x}^{\,n}_1,\vec{x}^{\,m}_{n+1}) &= \int_H \varphi^n\Big(\vec{x}^{\,n}_1+\sqrt{2tg^m\big(\vec{x}^{\,m}_{n+1}\big)}\Psi_n(y)\Big)\mu_A(dy)\\ &= (2\pi)^{-n/2}\Big(\prod_{i=1}^n c_i\Big)^{-1/2}\int_{\mathbb{R}^n}\varphi^n\Big(\vec{x}^{\,n}_1+\sqrt{2tg^m\big(\vec{x}^{\,m}_{n+1}\big)}\ z\Big)\exp\Big(-\sum_{i=1}^n z_i^2/(2c_i)\Big)dz.\end{aligned}$$

Introduce a measure ν on $\mathbb{R}^n$ given by a density with respect to the Lebesgue measure as follows: set

$$\nu(\mathcal{A}) := (2\pi)^{-n/2}\Big(\prod_{i=1}^n c_i\Big)^{-1/2}\int_{\mathcal{A}}\exp\Big(-\sum_{i=1}^n z_i^2/(2c_i)\Big)dz$$

for every measure set $\mathcal{A} \subset \mathbb{R}^n$. It follows from this definition that

$$\Phi(\overset{\rightarrow n}{x}_1, \overset{\rightarrow m}{x}_{n+1}) = \int_{\mathbb{R}^n} \varphi^n\big(\overset{\rightarrow n}{x}_1 + \sqrt{2tg^m(\overset{\rightarrow m}{x}_{n+1})}\ z\big)\nu(dz).$$

The integrand is a composition of mappings with bounded continuous Fréchet derivative, and therefore has the same property. The Fréchet derivative of the integrand is uniformly bounded, while $(\mathbb{R}^n, \nu)$ is a locally compact and countable at infinity normed linear space equipped with a nonnegative Radon measure, and therefore we can apply Theorem 115 in [13] on the Fréchet differentiation under the sign of a Lebesgue integral. Repeating this argument for every$k \in \mathbb{N}$, we conclude that, since the integrand has continuous bounded Fréchet derivatives of order k everywhere on $\mathbb{R}^{n+m}$, it follows that the function Φ has the same property on $\mathbb{R}^{n+m}$. By Proposition 1, the function $S_t\varphi$ has continuous bounded Fréchet derivatives of order k on H for every $k \in \mathbb{N}$, and therefore $S_t\varphi \in D$.

Suppose now that $\varphi \in X$, i.e, $\varphi \in C_b(H,\mathbb{R})$ and there is a sequence $(\varphi_j) \subset D$ such that $\varphi_j \to \varphi$ uniformly. Let $g \in X$, i.e., $g \in C_b(H,\mathbb{R})$ and there is a sequence $(g_j) \subset D$ such that $g_j \to g$ uniformly. Choose a $t > 0$. Denote the operator S_t constructed from g_j by the symbol $(S_j)_t$. As was proved above, $(S_j)_t\varphi_j \in D$ for every $j \in \mathbb{N}$. We claim that $((S_j)_t\varphi_j)(x) \to (S_t\varphi)(x)$ uniformly with respect to $x \in H$, and therefore $S_t\varphi \in X$.

Any function in D has bounded first derivative, and therefore is uniformly continuous by (7). Therefore, all functions in X are uniformly continuous, including φ. Since $a \mapsto \sqrt{a}$ is uniformly continuous, it follows that so is the function $z \mapsto \varphi(z + \sqrt{2tzy})$. Further, for any chosen y, we have the convergence $\varphi_j\big(x + \sqrt{2tg_j(x)}y\big) \to \varphi\big(x + \sqrt{2tg(x)}y\big)$, which is uniform with respect to $x \in H$. There is an index j_0 such that $\big|\varphi_j\big(x + \sqrt{2tg_j(x)}y\big) - \varphi\big(x + \sqrt{2tg(x)}y\big)\big| \leqslant \|\varphi_j\| + \|\varphi\| \leqslant 2\|\varphi\| + 1$ for every $x \in H, y \in H$ and $j \geqslant j_0$. Therefore, the following sequence of numerical functions is well defined, namely, $Y_j = \big[y \mapsto \sup_{x\in H}\big|\varphi_j\big(x + \sqrt{2tg_j(x)}y\big) - \varphi\big(x + \sqrt{2tg(x)}y\big)\big|\big]$. As was proved above, $Y_j(y)$ converges to 0 pointwise. The functions Y_j are jointly bounded, and therefore $\int_H Y_j(y)\mu_A(dy) \to 0$ by the dominated convergence theorem. To sum up,

$$\begin{aligned}\big\|(S_j)_t\varphi_j - S_t\varphi\big\| &= \sup_{x\in H}\Big|\int_H \varphi_j\big(x + \sqrt{2tg_j(x)}y\big) - \varphi\big(x + \sqrt{2tg(x)}y\big)\mu_A(dy)\Big| \\ &\leqslant \int_H \sup_{x\in H}\big|\varphi_j\big(x + \sqrt{2tg_j(x)}y\big) - \varphi\big(x + \sqrt{2tg(x)}y\big)\big|\mu_A(dy) = \int_H Y_j(y)\mu_A(dy \to 0.\end{aligned}$$

Choose some $t > 0$ and $x \in H$ and consider the integral $\int_H \varphi(x+y)\mu_{2tg(x)A}(dy)$.

We approximate the integrand by its Taylor polynomial of order three centered at x. Let us stipulate that the remainder term $R(x,y)$ is not small, since the vector y ranges the entire space H, and the vector x is chosen. However, since φ is four times continuously Fréchet differentiable on H, it follows that the function $R(x,y)$ such that

$$\begin{aligned}\int_H \varphi(x+y)\mu_{2tg(x)A}(dy) = \int_H \Big\{&\varphi(x) + \langle\varphi'(x), y\rangle + \frac{1}{2!}\langle\varphi''(x)y, y\rangle \\ &+ \frac{1}{3!}\varphi'''(x)(y,y,y) + R(x,y)\Big\}\mu_{2tg(x)A}(dy)\end{aligned}$$

is defined on $H \times H$ everywhere and, by (8), satisfies the bound

$$|R(x,y)| \leqslant \frac{\|y\|^4}{(4)!}\sup_{z\in[x,x-y]}\big\|\varphi^{(4)}(z)\big\| \leqslant \frac{1}{4!}a_\varphi\|y\|^4, \tag{19}$$

where we write $a_\varphi = \sup_{z\in H}\|\varphi^{(4)}(z)\|$ and $a_g = \sup_{z\in H}|g(z)|$.

The sum can be integrated termwise, because every summand is dominated by a polynomial with respect to the variable y, which the polynomials are integrable with respect to any Gaussian measure with trace-class correlation operator ([4, p. 68]).

The integrals against the symmetric measure $\mu_{2tg(x)A}$ of functions $\langle\varphi'(x), y\rangle$ and $(1/3!)\varphi'''(x)(y,y,y)$ that are odd with respect to y vanish. The number $\varphi(x)$ does not depend on y and $\mu_{2tg(x)A}$ is a probability measure, and therefore the integral of $\varphi(x)$ is equal to $\varphi(x)$. By (11), $\int_H (1/2!)\langle\varphi''(x)y, y\rangle\mu_{2tg(x)A}(dy) = (1/2)\operatorname{tr}(2tg(x)A\varphi''(x)) = t\,g(x)\operatorname{tr}(A\varphi''(x))$. Finally,

$$\left|\int_H R(x,y)\mu_{2tg(x)A}(dy)\right| \overset{(18)}{\leqslant} \int_H \left|R(x,\sqrt{2tg(x)}y)\right|\mu_A(dy) \overset{(12)}{\leqslant} \int_H \frac{1}{4!}a_\varphi\left(\sqrt{2tg(x)}\right)^4\|y\|^4\mu_A(dy)$$

$$\leqslant (1/6)t^2 a_\varphi a_g^2\int_H \|y\|^4\mu_A(dy) \overset{(19)}{=} \frac{1}{6}[a_\varphi a_g^2((\operatorname{tr} A)^2 + 2\operatorname{tr} A^2)]t^2.$$

This implies that $\lim_{t\to 0}\sup_{x\in H}\left|t^{-1}((S_t\varphi)(x)-\varphi(x))-(g\Delta_A\varphi)(x)\right| = \lim_{t\to 0} t\sup_{x\in H}|r(x,t)| = 0$.

Theorem 3. *Let $g \in X$, let g be bounded and bounded away from zero, and let g be the uniform limit of smooth cylindrical functions in D. Then $g\Delta_A\varphi \in D$ for $g \in D$ and $\varphi \in D$ and $g\Delta_A\varphi \in X$ for $g \in X$ and $\varphi \in D$. Moreover, the operator $\lambda I - g\Delta_A$, where I stands for the identity operator, is surjective on D for any $\lambda > 0$ if $g \in D$ (in particular, $(\lambda I - g\Delta_A)(D) = D$ is then dense in X), and, if $g \in X$, then the operator $(g\Delta_A, D)$ is dissipative and closable, and the closure of (Δ_A, D) is $(\overline{\Delta_A}, D_1)$, which is also dissipative.*

Proof. Let $\varphi \in D$, i.e., $\varphi(x) = \varphi^n(\langle x, e_1\rangle, \dots, \langle x, e_n\rangle)$ for some orthonormal family $e_1, \dots, e_n$ in H and for some function $\varphi^n\colon \mathbb{R}^n \to \mathbb{R}$ with bounded derivatives of all orders (see Proposition 1). This proposition implies the equation

$$(\Delta_A\varphi)(x) = \operatorname{tr} A\varphi''(x) = \sum_{s=1}^n\sum_{k=1}^n \langle Ae_s, e_k\rangle\Big(\partial_k\partial_s\varphi^n(\langle x, e_1\rangle, \dots, \langle x, e_n\rangle)\Big). \tag{20}$$

The function on the right-hand side of (20) is cylindrical, and the corresponding function on $\mathbb{R}^n$ is a finite sun of functions each of which has bounded derivatives of all orders (together with φ^n). By Proposition 1, $\Delta_A\varphi \in D$.

Since D is a function algebra, it contains products of its elements. If $g \in X$, there is a sequence $(g_j) \subset D$ such that $\|g - g_j\| \to 0$; we have $\|g\Delta_A\varphi - g_j\Delta_A\varphi\| \leqslant \|\Delta_A\varphi\| \cdot \|g - g_j\| \to 0$, and hence $g\Delta_A\varphi \in X$.

Let $g \in D$. Choose a $\lambda > 0$, consider an arbitrary function $\varphi \in D$, and show that there is a function $f \in D$ solving the equation

$$\lambda f(x) - g(x)\operatorname{tr} Af''(x) = \varphi(x). \tag{21}$$

Let vectors $e_1, \dots, e_n$ be such that

$$\varphi(x) = \varphi^n(\langle x, e_1\rangle, \dots, \langle x, e_n\rangle) \text{ and } g(x) = g^n(\langle x, e_1\rangle, \dots, \langle x, e_n\rangle) \tag{22}$$

for any $x \in H$, where $\varphi_n\colon \mathbb{R}^n \to \mathbb{R}$ and $g_n\colon \mathbb{R}^n \to \mathbb{R}$ are smooth functions bounded together with all their derivatives. We seek a solution of (21) in the form

$$f(x) = f^n(\langle x, e_1\rangle, \dots, \langle x, e_n\rangle), \tag{23}$$

where $f^n\colon \mathbb{R}^n \to \mathbb{R}$ is a smooth function bounded together with all its derivatives.

By (20), (22), and (23), equation (21) for the function f of the above form is equivalent to the following equation for the function f^n:

$$g^n(x_1, \dots, x_n)\sum_{s=1}^n\sum_{k=1}^n \langle Ae_s, e_k\rangle\partial_k\partial_s f^n(x_1, \dots, x_n) - \lambda f^n(x_1, \dots, x_n) = -\varphi^n(x_1, \dots, x_n), \tag{24}$$

where x_j stands for $\langle x, e_j\rangle$.

Equation (24) is a finite-dimensional partial differential equation. Note that $g_n(x_1,\dots,x_n) \geqslant g_0 \equiv \mathrm{const} > 0$. Moreover, the quadratic form given by the $n\times n$ matrix $(\langle Ae_s, e_k\rangle)$ is positive definite because A is positive. Therefore, the operator in (24) is elliptic, and equation (24) is of the form (14). The functions g^n and φ^n are bounded together with all their derivatives. Hence, Lemma 2 can be applied to equation (24), and therefore equation (24) has a continuous solution f^n which is bounded together with all its derivatives. By Proposition 1, the function f defined by (23) is in D.

Thus, for any $\lambda > 0$, the operator $\lambda I - g\Delta_A$ is surjective on D, because the preimage of a function $\varphi\in D$ contains at least the function $f(x) = f^n(\langle x, e_1\rangle,\dots,\langle x, e_n\rangle)$, where $f^n\colon \mathbb{R}^n\to\mathbb{R}$ is a solution of equation (24).

Suppose now that $g\in D$. We claim that $g\Delta_A$ is dissipative. Let $f\in D$ and $\lambda > 0$. As above, f and g are cylindrical functions if and only if $f(x) = f^n(\langle x, e_1\rangle,\dots,\langle x, e_n\rangle)$ and $g(x) = g^n(\langle x, e_1\rangle,\dots,\langle x, e_n\rangle)$, where $f^n\colon\mathbb{R}^n\to\mathbb{R}$ and $g^n\colon\mathbb{R}^n\to\mathbb{R}$ are smooth functions bounded together with all their derivatives.

As above, the value of a function obtained by applying $g\Delta_A$ to f at a point $x\in H$ is equal to the value of the function obtained by applying the above finite-dimensional second-order elliptic differential operator to f^n at the point $(\langle x, e_1\rangle,\dots,\langle x, e_n\rangle)\in R^n$. The operator A is positive definite, and the function g satisfies the inequality $g(x)\geqslant g_0\equiv \mathrm{const} > 0$, and therefore Lemma 2 can be applied to the finite-dimensional operator, which turns out to be dissipative, and thus the operator $g\Delta_A$ is dissipative as well. Formally,

$$\|g\Delta_A f - \lambda f\| = \sup_{x\in H}|g(x)\,\mathrm{tr}(Af''(x)) - \lambda f(x)|$$

$$\sup_{(x_1,\dots,x_n)\in\mathbb{R}^n}\left|g^n(x_1,\dots,x_n)\sum_{s=1}^n\sum_{k=1}^n\langle Ae_s,e_k\rangle\partial_k\partial_s f^n(x_1,\dots,x_n) - \lambda f_n(x_1,\dots,x_n)\right|$$

$$\overset{(15)}{\geqslant} \lambda \sup_{(x_1,\dots,x_n)\in\mathbb{R}^n}|f^n(x_1,\dots,x_n)| = \lambda\sup_{x\in H}|f(x)| = \lambda\|f\|,$$

where x_j stands for $\langle x, e_j\rangle$. The inequality

$$\|g\Delta_A f - \lambda f\| \geqslant \lambda\|f\| \tag{25}$$

thus obtained means that $g\Delta_A$ is dissipative.

Next, by [23, Proposition II.3.14], it follows from the density of the domain of $g\Delta_A$ and its dissipativity that $g\Delta_A$ is closable. By setting $g(x)\equiv 1$, we see that the operator (Δ_A, D) is also dissipative and closable. It follows from the definition of the space D_1 that the closure of (Δ_A, D) has the domain D_1.

Let $g\in X$, and let $(g_j)\subset D$ be a sequence such that $\|g - g_j\|\to 0$. The images of the operators $(\lambda I - g\Delta_A)$ and $(g\Delta_A - \lambda I)$ coincide. It suffices to prove that the image of $(g\Delta_A - \lambda I)$ is dense in D (then it is dense in X as well). Choose a $\lambda > 0$ and a function $\psi\in D$, which we intend to approximate by values of the operator $(g\Delta_A - \lambda I)$. Since the image of $(g_j\Delta_A - \lambda I)$ is D, for any $j\in\mathbb{N}$ there is a function $f_j\in D$ such that

$$g_j\Delta_A f_j - \lambda f_j = \psi. \tag{26}$$

We claim that, if $j\to\infty$, then $g\Delta_A f_j - \lambda f_j\to\psi$. To this end, we shall prove first that, for chosen λ, ψ, and g, the number set $\left(\sup_{x\in H}\left|\mathrm{tr}\,Af_j''(x)\right|\right)_{j\in\mathbb{N}}$ is bounded. Since $g(x)\geqslant g_0\equiv\mathrm{const} > 0$ and $g_j\to g$, there is an index j_0 such that

$$g_j(x)\geqslant g_0/2 \text{ for } j > j_0. \tag{27}$$

This gives the following bound which holds for $j > j_0$:

$$\sup_{x\in H}\left|\mathrm{tr}\,Af_j''(x)\right| \overset{(26)}{=} \sup_{x\in H}\frac{|\psi(x) + \lambda f_j(x)|}{g_j(x)} \overset{(27)}{\leqslant} \frac{2}{g_0}\left(\sup_{x\in H}|\psi(x)| + \lambda\sup_{x\in H}|f_j(x)|\right)$$

$$\overset{(25)}{\leqslant} \frac{2}{g_0}\left(\|\psi\| + \|g_j\Delta_A f_j - \lambda f_j\|\right) \overset{(26)}{=} \frac{4}{g_0}\|\psi\|.$$

Writing $C = \max\Big(\|\operatorname{tr} Af_1''\|, \dots, \|\operatorname{tr} Af_{j_0}''\|, \frac{4}{g_0}\|\psi\|\Big)$, we see that

$$\sup_{j\in\mathbb{N}} \sup_{x\in H} \left|\operatorname{tr} Af_j''(x)\right| \leqslant C \equiv \text{const.} \tag{28}$$

It remains to show that, if $j \to \infty$, then $g\Delta_A f_j - \lambda f_j \to \psi$. However, the following bound holds:

$$\begin{aligned}\|g\Delta_A f_j - \lambda f_j - \psi\| &= \|g\Delta_A f_j - g_j\Delta_A f_j + (g_j\Delta_A f_j - \lambda f_j - \psi)\|\\ &\overset{(26)}{=} \big\|(g-g_j)\operatorname{tr} Af_j'' + 0\big\| \leqslant \|g-g_j\|\cdot\|\operatorname{tr} Af_j''\| \overset{(28)}{\leqslant} \|g-g_j\|\cdot C,\end{aligned}$$

where $\|g-g_j\| \to 0$ as $j\to\infty$.

Let $g \in X$, and let $(g_j) \subset D$ be as above. Then, as was proved above, in accordance with (33), the inequality $\|g_j\Delta_A f - \lambda f\| \geqslant \lambda\|\varphi\|$ holds for any function $\varphi \in D$ and every $\lambda > 0$. Passing to the limit as $j\to\infty$, we obtain $\|g\Delta_A\varphi - \lambda\varphi\| \geqslant \lambda\|\varphi\|$, which means that $g\Delta_A$ is a dissipative operator. By [23, Proposition II.3.14], the densely defined (on D) dissipative operator $(g\Delta_A, D)$ in X admits the closure $(\overline{g\Delta_A}, \operatorname{Dom}(\overline{g\Delta_A}))$, which is also dissipative.

We claim that $\operatorname{Dom}(\overline{g\Delta_A}) = D_1$. Indeed, let $f \in \operatorname{Dom}(\overline{g\Delta_A})$. This means that there is a sequence $(f_j) \subset D$ such that $f_j \to f$ and the sequence $g\Delta_A f_j$ converges. However, since the bound $g_0 \leqslant g(x) \leqslant \|g\|$ holds for any $x \in H$, it follows that the sequence $g\Delta_A f_j$ converges if and only if the sequence $\Delta_A f_j$ converges, which holds if and only if $f \in \operatorname{Dom}(\overline{\Delta_A})$. It immediately follows from the definition of the space D_1 that $\operatorname{Dom}(\overline{\Delta_A}) = D_1$. It remains to note that $g\overline{\Delta_A} f = g\lim_{j\to\infty}\Delta_A f_j = \lim_{j\to\infty} g\Delta_A f_j = \overline{g\Delta_A} f$, and therefore $g\overline{\Delta_A} = \overline{g\Delta_A}$.

4. FEYNMAN FORMULA SOLVES THE CAUCHY PROBLEM

Theorem 4. *The Cauchy problem* (1) *has a unique strong solution for every* $u_0 \in D_1$*, the Cauchy problem* (1) *has a unique mild solution for every* $u_0 \in X$*, these solutions are norm continuous* (*for the norm in* X)*, uniformly continuous with respect to* $t \in [0, t_0]$ *for any* $t_0 > 0$*, continuously depend on* u_0 *and* g*, satisfy the bound* $\sup_{x\in H}|u(t,x)| \leqslant \sup_{x\in H}|u_0(x)|$ *for any* $t \geqslant 0$*, and are given by*

$$u(t,x) = \lim_{n\to\infty}\underbrace{\int_H\int_H\dots\int_H\int_H}_{n} u_0(y_1)\mu^{y_2}_{\frac{2t}{n}g(y_2)A}(dy_1)\mu^{y_3}_{\frac{2t}{n}g(y_3)A}(dy_2)\dots\mu^{y_n}_{\frac{2t}{n}g(y_n)A}(dy_{n-1})\mu^{x}_{\frac{2t}{n}g(x)A}(dy_n). \tag{29}$$

Proof. Let us verify the validity of the conditions of Theorem 1 for the operator family $(S_t)_{t\geqslant 0}$ and the operator $g\Delta_A$. First, $S_0 = I$ by the very definition of the family $(S_t)_{t\geqslant0}$. Second, by Theorem 2, we have $\|S_t\| = 1$ for any $t \geqslant 0$, and therefore $\|(S_t)^m\| \leqslant \|S_t\|\cdot\ldots\cdot\|S_t\| = 1\cdot\ldots\cdot 1 = 1$ for any $m\in\mathbb{N}$. Third, by the same theorem, the strong limit $\lim_{t\to0}\frac{S_t\varphi-\varphi}{t} = g\Delta_A\varphi$ exists in X for every function $\varphi \in D$. Fourth, the space D is automatically dense in its closure X. Moreover, by Theorem 3, for every $\lambda > 0$, the space $(\lambda I - g\Delta_A)(D)$ is dense in X. Therefore, by Theorem 1, the closure of $(g\Delta_A, D)$ (coinciding with the operator $(g\overline{\Delta_A}, D_1)$ by Theorem 3) generates a strongly continuous semigroup $e^{tg\overline{\Delta_A}}$ defined for every $\varphi \in X$ by the rule $e^{tg\overline{\Delta_A}}\varphi = \lim_{n\to\infty}\big(S_{t/n}\big)^n\varphi$, and the limit is uniform with respect to $t \in [0, t_0]$ for any chosen t_0. Note that, by Theorem 3, the Lumer–Phillips theorem [23, Th. II.3.15] can be applied to the semigroup $e^{tg\overline{\Delta_A}}$, and hence $e^{tg\overline{\Delta_A}}$ is a contraction semigroup.

Let us now introduce a function $U\colon [0,+\infty) \to X$ by the rule $(U(t))(x) = u(t,x)$. If we set $(U_0)(x) = u_0(x)$, then problem (1) with $u_0 \in D_1$ becomes equivalent to problem (3) with $u_0 \in D_1$ and problem (1) with $u_0 \in X$ to problem (3) with $u_0 \in X$. It follows from what was proved above, from definitions after formula (6), and from Proposition II.6.2 in [23] that, for every $U_0 \in D_1$, there is a unique solution of the Cauchy problem (3) given by the formula $U(t) = e^{tg\overline{\Delta_A}}u_0$, and, for every

$U_0 \in X$, there is a unique solution of the Cauchy problem (5) given by the formula $U(t) = e^{tg\overline{\Delta_A}}u_0$. Since the operator semigroup $e^{tg\overline{\Delta_A}}$ is strongly continuous, it follows that, by Proposition I.1.2 in [23], the function $t \mapsto e^{tg\overline{\Delta_A}}\varphi(x)$ is differentiable with respect to t at every point $t \geqslant 0$ uniformly with respect to $x \in H$ for every function $\varphi \in D_1$.

Thus, the Cauchy problem (6) has a unique solution for every $u_0 \in D_1$, which is given by the formula $u(x,t) = (e^{tg\overline{\Delta_A}}u_0)(x) = \lim_{n\to\infty}\left(\left(S_{t/n}\right)^n u_0\right)(x)$, and the Cauchy problem (7) has a unique solution for every $u_0 \in X$, which is given by the formula $u(x,t) = (e^{tg\overline{\Delta_A}}u_0)(x) = \lim_{n\to\infty}\left(\left(S_{t/n}\right)^n u_0\right)(x)$.

Let us explain how the formula $u(x,t) = \lim_{n\to\infty}\left(\left(S_{t/n}\right)^n u_0\right)(x)$ leads to formula (37). The change of variable $\int_H \psi(y)\mu_A(dy) = \int_H \psi(y-x)\mu_A^x(dy)$ holds for every continuous bounded function $\psi\colon H \to \mathbb{R}$ and every point $x \in H$. Applying this rule and replacing A by $2t\,g(x)A$, we obtain

$$(S_t\varphi)(x) = \int_H \varphi(x+y)\mu_{2tg(x)A}(dy) = \int_H \varphi(x+(y-x))\mu^x_{2tg(x)A}(dy) = \int_H \varphi(y)\mu^x_{2tg(x)A}(dy).$$

Therefore, the expression whose limit is taken in (37) for $n = 2$ is

$$\left(\left(S_{t/2}\right)^2\varphi\right)(x) = \left(S_{t/2}\left(S_{t/2}\varphi\right)\right)(x) = \int_H\left(\int_H \varphi(y_1)\mu^{y_2}_{\frac{2t}{2}g(y_2)A}(dy_1)\right)\mu^x_{\frac{2t}{2}g(x)A}(dy_2).$$

The expressions for $n > 2$ can be obtained in a similar way. This proves formula (37).

Since the semigroup operators in the semigroup $e^{tg\overline{\Delta_A}}$ are continuous, it follows that the solution u strongly continuously (in X) depends on u_0. The bound $\sup_{x\in H}|u(t,x)| \leqslant \sup_{x\in H}|u_0(x)|$ holds because $e^{tg\overline{\Delta_A}}$ is a contraction semigroup.

It remains to show that the solution continuously depends on g. Let $(g_j) \subset X$ and $g_j \to g$. Then we automatically have $g_j(x) \geqslant \text{const} > 0$ beginning with some index j_0. As was proved above, the operators $g_j\overline{\Delta_A}$ are infinitesimal generators of the semigroups $e^{tg_j\overline{\Delta_A}}$. Since $g_j\Delta_A\varphi \to g\Delta_A\varphi$ for any $\varphi \in D$, we can apply Theorem III.4.9 in [23] and conclude that the semigroups $e^{tg_j\overline{\Delta_A}}$ converge strongly (and uniformly with respect to $t \in [0,t_0]$ for any chosen $t_0 > 0$) to a strongly continuous semigroup $e^{tg\overline{\Delta_A}}$ with the infinitesimal generator $g\overline{\Delta_A}$, i.e., for every function $\varphi \in X$ we have $\lim_{j\to\infty} e^{tg_j\overline{\Delta_A}}\varphi = e^{tg\overline{\Delta_A}}\varphi$ uniformly with respect to $t \in [0,t_0]$ for any chosen $t_0 > 0$. Therefore, the solution of the Cauchy problem (6) corresponding to g_j converges as $j \to \infty$ to the solution of the Cauchy problem (6) corresponding to g uniformly with respect to $x \in H$ and uniformly with respect to $t \in [0,t_0]$ for any chosen $t_0 > 0$.

Remark 1. The functions u_0 and g belong to the space X, and therefore can be approximated uniformly by cylindrical functions. Substituting cylindrical approximations $(u_0)_j$ and g_j into the Feynman formula (29) (instead of the functions u_0 and g themselves), then the integrands become cylindrical functions, and therefore the integral over the infinite-dimensional space H can then be replaced by the integral over a finite-dimensional subspace of H by (17). The passage to the limit as $j \to +\infty$ and $n \to +\infty$ gives the function $u(t,x)$ again. This means that we obtain a sequence of cylindrical functions uniformly approximating the function u, and the functions in the sequence can be obtained as integrals of finite multiplicity over finite-dimensional spaces.

5. HOW LARGE ARE FUNCTION CLASSES D_1 AND X?

Proposition 2. *Let $\alpha_k\colon \mathbb{R} \to \mathbb{R}$ be a family of infinitely smooth functions that are uniformly bounded together with their first and second derivatives,* $\sup_{p\in\{0,1,2\}}\sup_{k\in\mathbb{N}}\sup_{t\in\mathbb{R}}\left|d^p\alpha_k(t)/dt^p\right| \leqslant M \equiv \text{const}$. *For example, the functions $\alpha_k(t) = \sin(d_k(t-t_k))$ and $\alpha_k(t) = \exp\left(-d_k(t-t_k)^2\right)$, where d_k and t_k are constants, $0 < d_k \leqslant 1$, satisfy these conditions. Let a number series $\sum_{k=1}^{\infty} b_k$ converge* *absolutely. Let* $(e_k)_{k=1}^{\infty}$ *be an orthonormal basis in* H*. Then the function* $f(x)=\sum_{k=1}^{\infty} b_k\alpha_k(\langle x,e_k\rangle)$ *belongs to the class* D_1*.*

Proof. The sequence of cylindrical functions $f_j(x)=\sum_{k=1}^{j} b_k\alpha_k(\langle x,e_k\rangle)$ converges uniformly to f. By Proposition 1, the sequence $\operatorname{tr} Af_j''(x)$ is of the form

$$\operatorname{tr} Af_j''(x)=\sum_{k=1}^{j}\langle Ae_k,e_k\rangle b_k\alpha''(\langle x,e_k\rangle),$$

and it converges as $j\to\infty$ to the function $\sum_{k=1}^{\infty}\langle Ae_k,e_k\rangle b_k\alpha''(\langle x,e_k\rangle)$, because

$$\Big|\sum_{k=n_1}^{n_2}\langle Ae_k,e_k\rangle b_k\alpha''(\langle x,e_k\rangle)\Big|\leqslant\|A\|M\sum_{k=n_1}^{n_2}|b_k|,$$

and the series $\sum_{k=1}^{\infty} b_k$ converges absolutely.

Proposition 3. *A nonconstant function in* X *cannot have any limit at infinity. In particular, the function* $x\mapsto\exp(-\|x\|^2)$ *belongs to* $C_b(H,\mathbb{R})$ *but not to* X*.*

Proof. Let $f\in D$. Then there is an n-dimensional subspace $H_n\subset H$ such that $f(x)=f(Px)$ for every $x\in H$, where $P\colon H\to H_n$ is the orthogonal projection. Since $f\neq\text{const}$, there is a number $\varepsilon_0>0$ and points $x_1,x_2\in H_n$ such that $|f(x_1)-f(x_2)|>\varepsilon_0$. Then $|f(x_1+y)-f(x_2+y)|>\varepsilon_0$ for every $y\in\ker P$, and, in particular, for $\|x_1+y\|\geqslant R$ and $\|x_2+y\|\geqslant R$, which contradicts the existence of a limit of f at infinity.

Suppose now that $f\in X$. Then there is a sequence of functions $(f_j)\subset D$ which converges uniformly to f. There is an index j such that $\|f-f_j\|<\frac{\varepsilon_0}{8}$. Therefore, $|f_j(x_1)-f_j(x_2)|>\frac{3\varepsilon_0}{4}$ and $|f_j(x_1+y)-f_j(x_2+y)|>\frac{3\varepsilon_0}{4}$ for every $y\in\ker P$ and for two points x_1,x_2 of the space H_n constructed from the function f_j. Since $\|f-f_j\|<\frac{\varepsilon_0}{8}$, it follows that $|f(x_1+y)-f(x_2+y)|>\varepsilon_0/2$, which contradicts the existence of a limit at infinity.

Remark 2. Recall that, by the proof of Theorem 2, any function $f\in X$ is uniformly continuous.

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