Inertial manifolds and hidden oscillations in delay equations

M. M. Anikushin[1], A. O. Romanov[2]

Keywords: inertial manifolds; hidden oscillations; multistability; quadratic functionals; frequency theorem; delay equations; neutral equations; ENSO; lossless transmission lines

MSC2010 codes: 35B42, 37L25, 34K19, 34K13, 34K60, 34K11

In our work [3], we develop a geometric theory of inertial manifolds (these are, roughly speaking, normally hyperbolic and globally attracting invariant manifolds, which provide a decomposition of the system into fast and slow modes) based on the use of indefinite quadratic Lyapunov-like functionals. In applications, various versions of the Frequency Theorem provide a variational approach, which is concerned with the study of infinite-horizon quadratic optimization problems, for the construction of such functionals [4,5].

We plan to discuss nuances concerned with abstract applications of the theory to delay equations. These include the well-posedness of such equations in a proper Hilbert space setting (see [1]; J.A. Burns, T. L. Herdman and H.W. Stech [8]) and mainly concerned with some specificity of the quadratic regulator problem [4]. This allows to provide a unification of some results due to R.A. Smith on the Poincaré-Bendixson theory [13]; Yu.A. Ryabov, R.D. Driver and C. Chicone [10] for equations with small delays; S. Chen and J. Shen [9] for neutral equations with small delays and relate these results to classical results for semilinear parabolic equations [5] in the context of [3]. We refer to our works [1,3,6,7] for various discussions on the topic.

As for concrete applications to study applied models, we will show how the theory can be applied to provide analytical-numerical methods for studying complex systems, where multistability and presence of hidden oscillations may occur. We refer to the surveys of D. Dudkowski et al. [11] and G.A. Leonov and N.V. Kuznetsov [12] for various discussions on the theory of hidden oscillations in the context of finite-dimensional systems.

In this direction we apply the theory to study the delayed oscillator, which was proposed by M.J. Suarez and P.S. Schopf in [14] as a simple model for ENSO, given by the scalar delay equation

$$\dot{x}(t) = x(t) - \alpha x(t-\tau) - x^3(t), \tag{1}$$

where $\alpha \in (0,1)$ and $\tau > 0$ are parameters. We provide analytical-numerical evidence for the existence of two-dimensional inertial manifolds in the model for parameters from the region of linear stability, where the symmetric equilibria $\pm\sqrt{1-\alpha}$ are linearly stable. This allows to propose a qualitative description of dynamics in the region, where the presence of multistability with hidden and self-excited periodic orbits is possible. We also discuss how the influence of a small periodic forcing can cause quasi-periodic, almost automorphic (strange non-chaotic) [6] and chaotic behavior in the model. This relates the model to well-known oscillators on the plane. These results are described in our paper [2].

We also use the intuition obtained in the study of the Suarez-Schopf model to numerically detect hidden and self-excited asynchronous oscillations in the model describing a ring array of coupled lossless transmission lines, which was proposed by J. Wu and H. Xia in [15] and is described as the coupled system of neutral delay equations:

$$\begin{aligned} \frac{d}{dt}\left[D(q)x^k_t\right] = -ax^k(t) - bqx^k(t-\tau) - g(x^k(t)) + qg(x^k(t-\tau)) + \\ + dD(q)\left[x^{k+1}_t - 2x^k_t + x^{k-1}_t\right],\ k = 1,\dots N \pmod N \end{aligned} \tag{2}$$

[1]Saint Petersburg State University, Department of Applied Cybernetics, Russia, St. Petersburg. Email: demolishka@gmail.com

[2]Saint Petersburg State University, Department of Applied Cybernetics, Russia, St. Petersburg. Email: romanov.andrey.twai@gmail.com

where $\tau, b, q, d > 0$ are positive parameters, a is not sign definite and $D(q)\phi = \phi(0) - q\phi(-\tau)$. Putting $g(x) = x^3$, for certain values of parameters with $a < 0$ we show that hidden or self-excited asynchronous periodic regimes may arise.

References:

[1] M.M. Anikushin. Nonlinear semigroups for delay equations in Hilbert spaces, inertial manifolds and dimension estimates // J. Differ. Equations. (2022) (revision submitted)

[2] M.M Anikushin, A.O. Romanov. Hidden and unstable periodic orbits as a result of homoclinic bifurcations in the Suarez-Schopf delayed oscillator and the irregularity of ENSO // arXiv:2102.11711v3 [math.DS] (2022) (to appear)

[3] M.M. Anikushin. Invariant manifolds and foliations for asymptotically compact cocycles in Banach spaces // arXiv:2012.03821v3 [math.DS] (2022) (to appear)

[4] M.M. Anikushin. Frequency Theorem for the Regulator Problem with Unbounded Cost Functional and Its Applications for Constructing Invariant Manifolds of Nonlinear Delay Equations // arXiv:2003.12499v5 [math.OC] (2022) (to appear)

[5] M.M. Anikushin. Frequency theorem for parabolic equations and its relation to inertial manifolds theory // J. Math. Anal. Appl. 505:1 (2022) 125454.

[6] M.M. Anikushin. Almost automorphic dynamics in almost periodic cocycles with one-dimensional inertial manifold // Differ. Uravn. Protsessy Upr. 2 (2021) 13–48 [in Russian]

[7] M.M. Anikushin. A non-local reduction principle for cocycles in Hilbert spaces // J. Differ. Equations. 269:9 (2020) 6699–6731.

[8] J.A. Burns, T.L. Herdman, H.W. Stech. Linear functional differential equations as semigroups on product spaces // SIAM J. Math. Anal. 14:1 (1983) 98–116.

[9] S. Chen, J. Shen. Smooth Inertial Manifolds for Neutral Differential Equations with Small Delays // J. Dyn. Differ. Equ. (2021)

[10] C. Chicone. Inertial and slow manifolds for delay equations with small delays // J. Differ. Equations. 190:2 (2003) 364–406.

[11] D. Dudkowski, S. Jafari, T. Kapitaniak, N.V. Kuznetsov, G.A. Leonov, and A. Prasad. Hidden attractors in dynamical systems // Phys. Rep. 637 (2016) 1–50.

[12] G.A. Leonov and N.V. Kuznetsov. Hidden attractors in dynamical systems. From hidden oscillations in Hilbert-Kolmogorov, Aizerman, and Kalman problems to hidden chaotic attractor in Chua circuits // Int. J. Bifurcat. Chaos. 23:01 (2013) 1330002.

[13] R.A. Smith. Poincaré-Bendixson theory for certain retarded functional-differential equations // Differ. Integral Equ. 5:1 (1992) 213–240.

[14] M.J. Suarez, P.S. Schopf. A delayed action oscillator for ENSO // J. Atmos. Sci. 45:21 (1988) 3283–3287.

[15] J. Wu, H. Xia. Self-Sustained Oscillations in a Ring Array of Coupled Lossless Transmission Lines // J. Differ. Equations. 1 (1996) 247–278.