Convergence the approximates solutions to initial-boundary value problem for IMBq equation in the *-weak sense

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Introduction. Let $\Omega \subset \mathbb{R}^n$ be a domain with the boundary $\partial\Omega$ of class C^∞ and $T \in \mathbb{R}_+$. In the cylinder $C = \Omega \times (0, T)$, consider the modified Boussinesq equation

$$(\lambda - \Delta)u_{tt} - \alpha^2\Delta u - \Delta(u^3) = y, \quad (x,t) \in \Omega \times (0,T) \tag{1}$$

with Dirichlet boundary condition

$$u(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,T) \tag{2}$$

and Cauchy conditions

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \Omega, \tag{3}$$

where $\lambda, \alpha \in \mathbb{R}$. The equation has many applications in various fields of natural science. For example, it simulates wave propagation in shallow water, taking into account capillary effects. In this case, the function $u = u(x,t)$ determines the wave height. In monograph [1] a linear mathematical model of wave propagation in shallow water is constructed. A (modified) mathematical model of wave propagation in shallow water in a one-dimensional region was investigated in [2] and a soliton solution of equation (1) was obtained. The existence of a unique global solution to the Cauchy problem for equation (1) was proved [3] for $\lambda = 1, \alpha = 1$. In [4], a similar solution was obtained for describing the interaction of shock waves. In all the works listed above, an essential condition is the continuous invertibility of the operator at the highest derivative with respect to t. However, the operator $\lambda - \Delta$ can be degenerate. Equations that are not solvable with respect to the highest time derivative, according to [5] are called Sobolev type equations.

Using the theory of p-bounded operators developed by G.A. Sviridyuk and his disciples [6, 7], it was shown in [8] that in appropriately chosen spaces the problem (1)–(3) can be reduced to the initial value problem

$$u(0) = u_0, \quad \dot{u}(0) = u_1 \tag{4}$$

for an abstract semilinear Sobolev type equation of second-order

$$L\ddot{u} - Mu + N(u) = y, \tag{5}$$

Then, using the phase space method, a theorem on the existence of a unique local solution was proved.

Definition 1. The set $\mathfrak{P}$ is called the phase space of equation (5) if
1) for any $(u_0, u_1) \in T\mathfrak{P}$ ($T\mathfrak{P}$ is the tangent bundle of $\mathfrak{P}$) there is a unique solution to problem (4), (5);
2) any solution $u = u(t)$ of equation (5) lies in $\mathfrak{P}$ as a trajectory.
Moreover, the notation $(u_0, u_1) \in T_{u_0}\mathfrak{P}$ should be understood as $u_0 \in \mathfrak{P}$ and $u_1 \in T_{u_0}\mathfrak{P}$.

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Let $\ker L \neq \{0\}$ and the operator M be $(L, 0)$ bounded, then, by the splitting theorem [7], equation (7) can be reduced to an equivalent system of equations

$$\begin{cases} (\mathbb{I} - Q)(M + N)(u) = (\mathbb{I} - Q)y, \\ \ddot{u}^1 = L_1^{-1}Q(M + N)(u), \end{cases}$$

where $u^1 = Pu$, P is some projector along $\ker L$. Then the phase space $\mathfrak{P}$ of equation (5) is the set [8]

$$\mathfrak{P} = \{u \in \mathfrak{U} : (\mathbb{I} - Q)(M + N)(u) = (\mathbb{I} - Q)y\}.$$

It was also noted that in the case of monotonicity of the operator N, the phase space would be a simple manifold.

Main result. First, let's solve problem (4), (5). To do this, we construct several function spaces. Let $H = (H, \langle \cdot, \cdot \rangle)$ be a real, separable, Hilbert space. We define dual pairs of reflexive Banach spaces $(\mathfrak{A}, \mathfrak{A}^*)$ and (L^p, L^q) with respect to duality $\langle \cdot, \cdot \rangle$ such that there is a chain of dense and continuous embeddings

$$\mathfrak{A} \hookrightarrow L^p \hookrightarrow H \hookrightarrow L^q \hookrightarrow \mathfrak{A}^*.$$

In the given spaces, we define operators L, M, N satisfying the conditions:
(C1) $L \in \mathcal{L}(\mathfrak{A}, \mathfrak{A}^*)$ is self-adjoint, non-negative definite, Fredholm and $L : \mathrm{coim}L \to \mathrm{im}L$ is compact;
(C2) $M \in \mathcal{L}(\mathfrak{A}, \mathfrak{A}^*)$ is self-adjoint, non-negative definite;
(C3) $N \in C^r(L^p, L^q), r \leq 1$ is s–monotone, p–coercive and homogeneous operator of order p, with symmetric Frechet derivative.
Due to the condition **(C3)**, the operator N satisfies the equality

$$\frac{d}{dt}\langle N(u), u \rangle = p\langle N(u), \dot{u} \rangle.$$

In addition, we define spaces of distributions $L^\infty(0, T; \mathfrak{A})$ and $L^\infty(0, T; \mathrm{coim}L)$, $\mathfrak{A} = \ker L \oplus \mathrm{coim}L$. The conjugate spaces are constructed according to the Dunford–Pettis theorem $L^\infty(0, T; A)^* \simeq L^1(0, T; A^*)$ and $(L^\infty(0, T; \mathrm{coim}L))^* \simeq L^1(0, T; A^*)$.

Theorem 1. Let conditions **(C1), (C2), (C3)** be fulfilled and $y \in L^q(0, T; L^q)$. Then for all $(u_0, u_1) \in T\mathfrak{P}$ where $u_0 \in \mathfrak{A}$, $u_1 \in \mathrm{coim}L$ there exists a solution to (4), (5) $u = u(x, t)$ such that $u \in L^\infty(0, T; \mathfrak{A})$ and $\dot{u} \in L^\infty(0, T; \mathrm{coim}L)$.

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