# Comments on the Chernoff estimate

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**Keywords:** Chernoff lemma; Semigroup theory; Product formula; Convergence rate.

**MSC2010 codes:** 34G10, 47B10, 47D06

**Introduction.** The Chernoff $\sqrt{n}$-Lemma is revised. This concerns two aspects: a re-examination of the Chernoff estimate in the strong operator topology and the operator-norm estimate for quasi-sectorial contractions. Applications to the Lie-Trotter product formula approximation $C_0$-semigroups are also discussed.

**Chernoff $\sqrt{n}$-Lemma**

*Lemma 1.* Let bounded operator $C$ on a Banach space $\mathfrak{X}$ be a contraction, i.e., $\|C\| \leq 1$. Then one has the estimate

$$\|(C^n - e^{n(C-1)})x\| \leq \sqrt{n}\, \|(C-1)x\| \;,\; x \in \mathfrak{X},\; n \in \mathbb{N}. \tag{1}$$

**Revised $\sqrt{n}$-Lemma**

*Proposition 1.* Let $C$ be contraction on a Banach space $\mathfrak{X}$. Then $\{e^{t(C-1}\}_{t\geq 0}$ is a norm-continuous contraction semigroup on $\mathfrak{X}$ and one has the estimate

$$\|(C^n - e^{n\,(C-1)})\,x\| \leq \frac{n}{\epsilon_n^2}\, 2\,\|x\| + \epsilon_n\, \|(1-C)\,x\,\|\,, \qquad n \in \mathbb{N}\setminus\{0\}\,, \tag{2}$$

for all $x \in \mathfrak{X}$ and $\epsilon_n > 0$. For optimal value of the *splitting* parameter $\epsilon_n$ :

$$\|(C^n - e^{n\,(C-1)})\,\| \leq \frac{3}{2}\,\sqrt[3]{n}\, \|2\,(1-C)\,\|^{2/3}\,, \tag{3}$$

which is the $\sqrt[3]{n}$ -Lemma.

*Proposition 2.* Let $C \in \mathcal{L}(\mathfrak{X})$ be contraction on a Banach space $\mathfrak{X}$. Then following estimate

$$\|(C^n - e^{n(C-1)})\,x\| \leq \frac{n}{2}\,(\|(C-1)^2\,x\| + \frac{e^2}{3}\,\|(C-1)^3\,x\|)\;, \tag{4}$$

holds for all $n \in \mathbb{N}$ and $x \in \mathfrak{X}$.

**Quasi-sectorial contractions**

*Definition 1.* A contraction $C$ on the Hilbert space $\mathfrak{H}$ is called *quasi-sectorial* with semi-angle $\alpha \in [0, \pi/2)$ with respect to the vertex at $z = 1$, if its numerical range $W(C) \subseteq D_\alpha$, where $D_\alpha := \{z \in \mathbb{C} : |z| \leq \sin\alpha\} \cup \{z \in \mathbb{C} : |\arg(1-z)| \leq \alpha \text{ and } |z-1| \leq \cos\alpha\}$ .

*Proposition 3.* Let $C$ be a quasi-sectorial contraction on $\mathfrak{H}$ with numerical range $W(C) \subseteq D_\alpha$, $0 \leq \alpha < \pi/2$. Then

$$\left\|C^n - e^{n(C-1)}\right\| \leq \frac{M_\alpha}{n^{1/3}}, \quad n \in \mathbb{N}, \quad M_\alpha > 0\,. \tag{5}$$

**References**

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