One-parameter subgroups of the isometry group of the space $\mathbb{H}^2_{\mathbb{C}a}$

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Keywords: octave algebra; Cayley hyperbolic plane; infinitesimal operators.

MSC2010 codes: 17A35, 53C27, 53C35

We consider the realization of the Cayley hyperbolic plane $F_4^*/\mathrm{Spin}(9)$ on the unit ball in $\mathbb{R}^{16}$ and write out an explicit form of the isometries of this space. The results presented are used to solve various problems in the theory of convolution equations on $F_4^*/\mathrm{Spin}(9)$ (see [1]-[3]).

Let $\mathcal{A}$ be an arbitrary finite-dimensional algebra over $\mathbb{R}$ in which a conjugation, i.e. some involutory antiautomorphism $a \to \overline{a}$ is given. Consider a vector space $\mathcal{A}^2$ which is a direct sum of two copies of a vector space $\mathcal{A}$, i.e. which consists of pairs of the form (α, β), where $\alpha, \beta \in \mathcal{A}$. We introduce into $\mathcal{A}^2$ multiplication as follows: $(\alpha, \beta)(\gamma, \delta) = (\alpha\gamma - \overline{\delta}\beta, \beta\overline{\gamma} + \delta\alpha)$. A simple check shows that relative to that multiplication the vector space $\mathcal{A}^2$ is an algebra. The algebra $\mathcal{A}^2$ is called a doubling of $\mathcal{A}$. We shall identify elements α and $(\alpha, 0)$ and thus assume the algebra $\mathcal{A}$ to be a subalgebra of $\mathcal{A}^2$. If $\mathcal{A}$ is a unit algebra, then the element $1 = (1, 0)$ will obviously be an identity element in $\mathcal{A}^2$ too. In addition, every element $(\alpha, \beta) \in \mathcal{A}^2$ is uniquely written as $\alpha + \beta e$, where $e = (0, 1)$.

For the procedure of a doubling to be iterated it is necessary to define a conjugation in $\mathcal{A}^2$. We shall do this by the formula $\overline{(\alpha, \beta)} = (\overline{\alpha}, -\beta)$. Then $\mathbb{C} := \mathbb{R}^2$, $\mathbb{Q} := \mathbb{C}^2$, $\mathbb{C}a := \mathbb{Q}^2$.

The basis of $\mathbb{C}$ consists of $\mathbf{i}_0 = 1$ and the imaginary unit $\mathbf{i}_1 = (0, 1)$. The basis of $\mathbb{Q}$ consists of $\mathbf{i}_0 = 1$ and three elements $\mathbf{i}_1 = (\mathbf{i}_1, 0)$, $\mathbf{i}_2 = (0, 1)$, $\mathbf{i}_3 = \mathbf{i}_1\mathbf{i}_2$. Analogously, the basis of $\mathbb{C}a$ consists of $\mathbf{i}_0 = 1$ and seven elements

$$\mathbf{i}_1 = (\mathbf{i}_1, 0),\ \mathbf{i}_2 = (\mathbf{i}_2, 0),\ \mathbf{i}_3 = (\mathbf{i}_3, 0),\ \mathbf{i}_4 = (0, 1),\ \mathbf{i}_5 = \mathbf{i}_1\mathbf{i}_4,\ \mathbf{i}_6 = \mathbf{i}_2\mathbf{i}_4,\ \mathbf{i}_7 = \mathbf{i}_3\mathbf{i}_4.$$

As the construction of a doubling is iterated the algebraic properties of the multiplication gradually deteriorate. In particular, the octave algebra $\mathbb{C}a$ is noncommutative and nonassociative.

For the Cayley algebra, we consider the vector space

$$\mathbb{C}a^2 = \{a = (a_1, a_2)\colon a_k \in \mathbb{C}a,\ k = 1, 2\}.$$

If $b = (b_1, b_2) \in \mathbb{C}a^2$, put

$$\Phi_{\mathbb{C}a}(a, b) = |a_1|^2|b_1|^2 + |a_2|^2|b_2|^2 + 2\mathrm{Re}\big((a_1a_2)(\overline{b_1b_2})\big).$$

We identify $\mathbb{C}a^2$ with $\mathbb{R}^{16}$ by the map $a = (a_1, a_2) \to x = (x_1, \ldots, x_{16})$, where

$$a_1 = x_1 + x_9\mathbf{i}_1 + x_5\mathbf{i}_2 + x_{13}\mathbf{i}_3 + x_3\mathbf{i}_4 + x_{11}\mathbf{i}_5 + x_7\mathbf{i}_6 + x_{15}\mathbf{i}_7,$$

$$a_2 = x_2 + x_{10}\mathbf{i}_1 + x_6\mathbf{i}_2 + x_{14}\mathbf{i}_3 + x_4\mathbf{i}_4 + x_{12}\mathbf{i}_5 + x_8\mathbf{i}_6 + x_{16}\mathbf{i}_7.$$

Setting $y = (y_1, \ldots, y_{16})$, $y_k \in \mathbb{R}$,

$$b_1 = y_1 + y_9\mathbf{i}_1 + y_5\mathbf{i}_2 + y_{13}\mathbf{i}_3 + y_3\mathbf{i}_4 + y_{11}\mathbf{i}_5 + y_7\mathbf{i}_6 + y_{15}\mathbf{i}_7,$$

$$b_2 = y_2 + y_{10}\mathbf{i}_1 + y_6\mathbf{i}_2 + y_{14}\mathbf{i}_3 + y_4\mathbf{i}_4 + y_{12}\mathbf{i}_5 + y_8\mathbf{i}_6 + y_{16}\mathbf{i}_7,$$

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we have

$$\Phi_{\mathbb{C}a}(a,b)=\Phi_{\mathbb{C}a}(x,y)=2\sum_{k=1}^{8}p_k(x)p_k(y)+p_9(x)p_9(y)+p_{10}(x)p_{10}(y)$$

with

$$p_1(x)=x_1x_2-x_3x_4-x_5x_6-x_7x_8-x_9x_{10}-x_{11}x_{12}-x_{13}x_{14}-x_{15}x_{16},$$
$$p_2(x)=x_1x_4-x_9x_{12}-x_5x_8-x_{13}x_{16}+x_3x_2+x_{11}x_{10}+x_7x_6+x_{15}x_{14},$$
$$p_3(x)=x_1x_6-x_9x_{14}+x_5x_2+x_{13}x_{10}+x_3x_8+x_{11}x_{16}-x_7x_4-x_{15}x_{12},$$
$$p_4(x)=x_1x_8+x_9x_{16}+x_5x_4-x_{13}x_{12}-x_3x_6+x_{11}x_{14}+x_7x_2-x_{15}x_{10},$$
$$p_5(x)=x_1x_{10}+x_9x_2+x_5x_{14}-x_{13}x_6+x_3x_{12}-x_{11}x_4-x_7x_{16}+x_{15}x_8,$$
$$p_6(x)=x_1x_{12}+x_9x_4-x_5x_{16}+x_{13}x_8-x_3x_{10}+x_{11}x_2-x_7x_{14}+x_{15}x_6,$$
$$p_7(x)=x_1x_{14}+x_9x_6-x_5x_{10}+x_{13}x_2+x_3x_{16}-x_{11}x_8+x_7x_{12}-x_{15}x_4,$$
$$p_8(x)=x_1x_{16}-x_9x_8+x_5x_{12}+x_{13}x_4-x_3x_{14}-x_{11}x_6+x_7x_{10}+x_{15}x_2,$$
$$p_9(x)=\sum_{k=1}^{8}x_{2k-1}^2,\quad p_{10}(x)=\sum_{k=1}^{8}x_{2k}^2.$$

For $x=(x_1,\dots,x_{16})\in\mathbb{R}^{16}$, $y=(y_1,\dots,y_{16})\in\mathbb{R}^{16}$, $i,j\in\{1,\dots,16\}$, we set

$$a_{ij}(x)=\delta_{i,j}(1-|x|^2)+\frac{1}{2}\frac{\partial^2}{\partial y_i\partial y_j}\big(\Phi_{\mathbb{C}a}(x,y)\big),\quad g_{ij}(x)=\frac{a_{ij}(x)}{(1-|x|^2)^2}.$$

The matrix $\|g_{ij}\|_{i,j=1}^{16}$ induces the structure of a Riemannian manifold on the unit ball $B^{16}=\{x\in\mathbb{R}^{16}\colon\ |x|<1\}$. Denote this manifold by $\mathbb{H}^2_{\mathbb{C}a}$.

Theorem 1. The Cayley hyperbolic plane $F_4^*/\mathrm{Spin}(9)$ of maximal sectional curvature -1 is isometric to the space $\mathbb{H}^2_{\mathbb{C}a}$.

Let $u=(t,\alpha)$, where $t\in\mathbb{R}$, $\alpha\in\mathbb{C}a$, and $t^2+|\alpha|^2=1$. Define the mapping

$$R_u(a)=\big(-ta_1+\overline{\alpha}\,\overline{a}_2,ta_2+\overline{a}_1\overline{\alpha}\big),\quad a=(a_1,a_2)\in\mathbb{C}a^2.$$

Take $x\in\mathbb{S}^{15}=\{x\in\mathbb{R}^{16}\colon|x|=1\}$ arbitrarily. We write x in the form $x=(\alpha,\beta)$, where $\alpha,\beta\in\mathbb{C}a$. Put

$$\tau_x=R_{u_x}\circ R_{v_x},$$

where $u_x=\big(0,-|\beta|\beta^{-1}\big)$, $v_x=\big(|\beta|,-|\beta|\beta^{-1}\alpha\big)$ if $\beta\neq0$, and $u_x=(0,\overline{\alpha})$, $v_x=(0,1)$ if $\beta=0$.

Next, let $a\in\mathbb{C}a^2$, $|a|<1$. Define

$$\sigma_a=\begin{cases}\tau_{a/|a|}\circ\varkappa_a\circ\tau^{-1}_{a/|a|} & \text{if}\quad a\neq0\\ \varkappa_a & \text{if}\quad a=0,\end{cases}$$

where $\varkappa_a$ is the mapping acting by the formula

$$\varkappa_a(z_1,z_2)=\big((z_1-|a|)(|a|z_1-1)^{-1},\sqrt{1-|a|^2}(|a|\overline{z}_1-1)^{-1}z_2\big),\ (z_1,z_2)\in\mathbb{C}a^2.$$

We also put

$$\Psi_{\mathbb{C}a}(z,w)=\Phi_{\mathbb{C}a}(z,w)-2\langle z,w\rangle_{\mathbb{R}}+1,\quad z,w\in\mathbb{C}a^2,$$

where $\langle z,w\rangle_{\mathbb{R}}$ is the Euclidean inner product of the vectors $z,w\in\mathbb{R}^{16}$.

Theorem 2. (i) $\sigma_a(0)=a$ and $\sigma_a(a)=0$.

(ii) The identity

$$(1-|a|^2)^2\Psi_{\mathbb{C}a}(z,w)=\Psi_{\mathbb{C}a}(z,a)\Psi_{\mathbb{C}a}(w,a)\Psi_{\mathbb{C}a}(\sigma_a(z),\sigma_a(w))$$

holds for all $z, w \in B^{16}$. In particular, $1 - |\sigma_a(z)|^2 = \dfrac{(1-|a|^2)(1-|z|^2)}{\Psi_{\mathbb{C}a}(z,a)}$, $z \in B^{16}$.

(iii) σ_a is an involution.

(iv) σ_a is an isometry of the space $\mathbb{H}^2_{\mathbb{C}a}$.

(v) The relation $\sigma_a\big((1+\sqrt{1-|a|^2})^{-1}a\big) = (1+\sqrt{1-|a|^2})^{-1}a$ holds. Moreover, σ_a fixes exactly one point of B^{16}, and no point of $\mathbb{S}^{15}$.

Corollary 1. Let

$$g_t(z_1, z_2) = \big((z_1 - \operatorname{th} t)(1 - (\operatorname{th} t)z_1)^{-1}, (\operatorname{ch} t - (\operatorname{sh} t)\overline{z}_1)^{-1} z_2\big),\ (z_1, z_2) \in \mathbb{C}a^2.$$

Then $\{g_t\}_{t\in\mathbb{R}}$ is a one-parameter subgroup of the isometry group of the space $\mathbb{H}^2_{\mathbb{C}a}$.

In conclusion, we note that Theorem 2 also allows us to write down an explicit form of the infinitesimal operators corresponding to involutive isometries of the space $\mathbb{H}^2_{\mathbb{C}a}$.

References:

[1] V.V. Volchkov: Integral Geometry and Convolution Equations. — Kluwer Academic Publishers, 2003.

[2] V.V. Volchkov, Vit.V. Volchkov: Harmonic Analysis of Mean Periodic Functions on Symmetric Spaces and the Heisenberg Group. — Springer, 2009.

[3] V.V. Volchkov, Vit.V. Volchkov: Offbeat Integral Geometry on Symmetric Spaces. — Birkhäuser, 2013.