Limit theorems for compositions of random operator valued process

V. Zh. Sakbaev[1], E. V. Shmidt[2]

Keywords: random linear operators; random operator valued process; law of large numbers; generalized convergence in distribution; Chernoff theorem

MSC2010 codes: 60B12, 60B20, 47B80, 47H40

Introduction. Limit properties of a distribution of the sum of random variables with values in topological vector spaces can be described by limit theorems [1]. In particular, the law of large numbers describes the convergence in probability of the sequence of averaged sum of independent identically distributed (iid) random vectors to the limit of the mean value of the sum. The central limit theorem gives conditions of the convergence in distribution for the sequence of averaged sum of iid random vector valued variables to the Gaussian random vector.

We study the sequence of compositions of iid random variables with values in the Banach algebra of bounded linear operators $B(H)$ acting in the separable Hilbert space H. In the commutative case of operators of an argument shift on a random vector the limit distribution of averaged composition can be described by limit theorems for the sum of vector valued variables. CLT and LLN for the products of independent random matrices or linear operators in finite-dimensional Euclidean space are obtained in [2]. Some results on the LLN and CLT for the averaged composition of independent random linear operators in infinite-dimensional Hilbert space was obtained in [3, 4]. We obtain the analogs of LLN and CLT for the sequence of compositions of iid random semigroups for $B(H)$-valued random processes with non-commutative values.

The relation between the operator-valued function in the space $H = L_2(E)$ and the random processes with values in the space E or in the algebra $B(E)$ is introduced and studied. This relation has the important role in the proof of CLT for the composition of random linear operators in algebra $B(H)$

Law of Large numbers for compositions of random semigroups. Let $\mathbf{A}_j$, $j \in \mathbb{N}$, be the sequence of independent identically distributed random variables with the values in Banach space of bounded linear operators $B(H)$ in some Hilbert space H. We are studying the asymptotic behavior of the probability distribution of the averaged random variables

$$\bar{\mathbf{A}}_n = (\mathbf{A}_n)^{\frac{1}{n}} \circ \ldots \circ (\mathbf{A}_1)^{\frac{1}{n}},$$

when $n \to \infty$. The fractional power of the operator is defined by means of spectral decomposition for self-adjoin operator. The fractional power for the operator $\mathbf{U}(t)$ belonging to the set of semigroup values $\{\mathbf{U}(t),\ t \geq 0\}$ is defined according to its dynamical sense: $(\mathbf{U}(t))^{\frac{1}{n}} = \mathbf{U}(\frac{t}{n})$.

Let $Y_s(H)$ be the topological vector space of the maps $[0, +\infty) \to B(H)$ which is continuous in the strong operator topology. The topology τ_s of the space $Y_s(H)$ is generated by the family of seminorms $\Phi_{T,v}$, $T \geq 0$, $v \in H:\ \Phi_{T,v}(\mathbf{U}) = \sup\limits_{t\in[0,T]} \|\mathbf{U}(t)v\|_H$, $\mathbf{U} \in Y_s$.

The random semigroup is defined as the measurable mapping $\mathbf{U}:\ \Omega \to Y_s(H)$ of the probability space $(\Omega, \mathcal{A}, \mu)$ into the measurable space $(Y_s, \mathcal{B}_s)$ such that the values of this map are C_0-semigroups. Here $\mathcal{B}_s$ is the Borel σ-algebra of subsets of the topological space $(Y_s(H), \tau_s)$.

Theorem 1. Let $\mathbf{A}$ be a random variable with values in the set of self-adjoint operators in the space H. Let $\mathcal{D} \subset H$ be the dense linear manifold such that $\int\limits_\Omega \|\mathbf{A}(\omega)u\|_H d\mu(\omega) < \infty\ \forall\ u \in \mathcal{D}$.

[1]Moscow Institute of Physics and Technology, Department of General Mathematics, Russia, Dolgoprudny. Email: sakbaev.vzh@phystech.edu

[2]Moscow Institute of Physics and Technology, Department of General Mathematics, Russia, Dolgoprudny. Email: eugenelighting@yandex.ru

Let operator $\bar{\mathbf{A}}u = \int\limits_{\Omega} \mathbf{A}(\omega)ud\mu(\omega),\ u \in \mathcal{D}$ *be essentially self-adjoint. Let* $\{\mathbf{U}_n\}$ *be the sequence of independent identically distributed random semigroups such that distribution of each of them coincides with the distribution of random semigroup* $\mathbf{U}(t) = \exp(i\mathbf{A}t),\ t \geq 0$*. Then the sequence* $\{\mathbf{U}_n \circ ... \circ \mathbf{U}_1\}$ *of its compositions satisfies the LLN in the strong operator topology:*

$$\lim_{n\to\infty} [\sup_{t\in[0,T]} P(\|(\mathbf{U}_n^{\frac{1}{n}} \circ ... \circ \mathbf{U}_1^{\frac{1}{n}} - M[\mathbf{U}_n^{\frac{1}{n}} \circ ... \circ \mathbf{U}_1^{\frac{1}{n}}])x\|_H > \epsilon)] = 0 \quad \forall x \in H,\ \forall\ \epsilon > 0.$$

Generalized weak convergence and convergence in distribution for compositions of operator valued random processes. Let E be a Hilbert space. Let $B(E)$ be a Banach space of bounded linear operators in the space E endowed with some operator topology. Let $ca(B(E), \mathcal{B}(B(E)))$ be a Banach space of Borel measure with bounded variation on the measurable space $(B(E), \mathcal{B}(B(E)))$. Let X be a locally convex space of complex valued functions on the space E. Let $\mathcal{L}(X)$ be a locally convex space of linear operators acting in the space X.

Definition. A sequence of measures $\{\mu_n\} : \mathbb{N} \to ca(B(E), \mathcal{B}(B(E)))$ converges $\mathcal{L}(X)$-weakly to the measure $\mu \in ca(B(E), \mathcal{B}(B(E)))$ if the sequence of operators $\{\Psi_{\mu_n}\}$ converges in the space $\mathcal{L}(X)$ to the operator Ψ_μ: $\Psi_\mu u(x) = \int\limits_{B(E)} u(\mathbf{A}x)d\mu(\mathbf{A}),\ u \in X,\ x \in E$. Here $\Psi_{\mu_n} u(x) = \int\limits_{B(E)} u(\mathbf{A}x)d\mu_n(\mathbf{A}),\ u \in X,\ x \in E$.

Definition. The sequence $\{\xi_n\}$ of random variables with values in the space $B(E)$ converges in the distribution $\mathcal{L}(X)$-weakly to the random variable ξ if the sequence of Borel measures $\{\mu_n\}$: $\mu_n(A) = P(\xi_n^{-1}(A)),\ A \in \mathcal{B}(B(E)),\ n \in \mathbb{N}$, converges $\mathcal{L}(X)$-weakly to the measure μ: $\mu(A) = P(\xi^{-1}(A)),\ A \in \mathcal{B}(B(E))$.

Let $\mathcal{A}_c$ be an algebra generated by the cylindrical sets of the space $B(E)^{\mathbb{R}_+}$. Let $a(B(E)^{\mathbb{R}_+}, \mathcal{A}_c)$ be a linear space ob complex valued finitely-additive measures on the measurable space $(B(E)^{\mathbb{R}_+}, \mathcal{A}_c)$.

Theorem 2. There is the injective mapping $\mathbf{\Lambda} :\ B(H)^{\mathbb{R}_+} \to a(B(E)^{\mathbb{R}_+}, \mathcal{A}_c)$.

We obtain the explicit expression for the mappings $\mathbf{\Lambda}$ and $\mathbf{\Lambda}^{-1}$.

Theorem 3. Let $\xi(t),\ t \geq 0$, *be a random process with values in the space* $B(E)$*. Let* $\{\xi_n\}$ *be a sequence of iid random processes such that any of them has the same distribution. Let* X *be a Banach space of functions* $u :\ E \to \mathbb{C}$ *such that for any* $t \geq 0$ *the linear operator* $u \to \mathbf{F}(t)u = \mathrm{M}(u(\xi(t)\cdot))$ *is satisfied on the space* X*. If the function* $\mathbf{F}(t),\ t \geq 0$, *satisfies the conditions of Chernoff theorem then the sequence of random processes* $\{\eta_n(t),\ t \geq 0\}$, *where* $\eta_n(t) = \xi_n(\frac{t}{n}) \circ ... \circ \xi_1(\frac{t}{n})$, *converges in distribution with respect to the space* $(B(X), \tau_{sot})$ *to the Markov random processes corresponding with the distribution* $\mathbf{\Lambda}^{-1}(\exp(\mathbf{F}'(0)\cdot))$.

References:

[1] W. Feller: An Introduction to Probability Theory and its Applications. Volume 2. — J. Wiley, New York, 1971.

[2] M. Berger. Central Limit Theorem for Products of Random Matrices// Trans. of AMS. 1984. Vol. 285. No 2. P. 777-803.

[3] Yu.N. Orlov, V.Zh. Sakbaev, E.V. Shmidt. Operator approach to weak convergence of measures and limit theorems// Lobachevskii J. of Math. 2021. Vol. 42 (10). P. 2413-2426.

[4] Yu.N. Orlov, V.Zh. Sakbaev, O.G. Smolyanov. Feynman Formulas and the Law of Large Numbers for Random One-Parameter Semigroups// Proc. Steklov Inst. Math. 2019. Vol. 306. P. 196-211.