Inverse problem for incomplete Sobolev type equation of higher order and application

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Introduction. Let $\mathcal{U}, \mathcal{F}, \mathcal{Y}$ be Banach spaces, operators $L,\ M \in \mathcal{L}(\mathcal{U};\mathcal{F})$, $C \in \mathcal{L}(\mathcal{U};\mathcal{Y})$, $\ker L \neq \{0\}$, given functions $\chi : [0,T] \to \mathcal{L}(\mathcal{Y};\mathcal{F})$, $f : [0,T] \to \mathcal{F}$, $\Psi : [0,T] \to \mathcal{Y}$. Consider the following problem for $t \in [0,T]$

$$Lv^{(n)}(t) = Mv(t) + \chi(t)q(t) + f(t), \tag{1}$$

$$v(0) = v_0,\ \ldots,\ v^{(n-1)}(0) = v_{n-1}, \tag{2}$$

$$Cv(t) = \Psi(t). \tag{3}$$

The problem of finding a pair of functions $v \in C^n([0,T];\mathcal{U})$ and $q \in C^1([0,T];\mathcal{Y})$ from relations (1) – (3) is called the inverse problem.

Definition 1. The operator M is called (L,σ)-bounded if

$$\exists a \in \mathbb{R}_+ \ \ \forall \mu \in \mathbb{C} \ \ (|\mu| > a) \Rightarrow \left((\mu L - M)^{-1} \in \mathcal{L}(\mathcal{F};\mathcal{U})\right).$$

Definition 2. If the operator M is (L,σ)-bounded, and ∞ is a pole of order $p \in \{0\} \cup \mathbb{N}$ of the L-resolvent of the operator M, then the operator M is called (L,p)-bounded.

Existence of solutions. Let the operator M be (L,p)-bounded, then $v(t)$ can be represented as $v(t) = Pv(t) + (\mathbb{I} - P)v(t)$. Denote $Pv(t) = u(t), (\mathbb{I} - P)v(t) = \omega(t)$. Suppose that $\mathcal{U}^0 \subset \ker C$. Then, by virtue of [2,4], problem (1) – (3) is equivalent to the problem of finding the functions $u \in C^n([0,T];\mathcal{U}^1)$, $\omega \in C^n([0,T];\mathcal{U}^0)$, $q \in C^1([0,T];\mathcal{Y})$ from the relations $t \in [0,T]$

$$u^{(n)}(t) = Su(t) + (L^1)^{-1}Q\chi(t)q(t) + (L^1)^{-1}Qf(t), \tag{4}$$

$$u(0) = u_0,\ \ldots,\ u^{(n-1)}(0) = u_{n-1}, \tag{5}$$

$$Cu(t) = \Psi(t) \equiv Cv(t), \tag{6}$$

$$H\omega^{(n)}(t) = \omega(t) + (M^0)^{-1}(\mathbb{I} - Q)\chi(t)q(t) + (M^0)^{-1}(\mathbb{I} - Q)f(t), \tag{7}$$

$$\omega(0) = \omega_0,\ \ldots,\ \omega^{(n-1)}(0) = \omega_{n-1}, \tag{8}$$

where $H = (M^0)^{-1}L^0$, $u_0 = Pv_0,\ \ldots,\ u_{n-1} = Pv_{n-1}$, $\omega_0 = (\mathbb{I} - P)v_0,\ \ldots, \omega_{n-1} = (\mathbb{I} - P)v_{n-1}$.

Theorem 1. [2] Let the operator M be (L,p)-bounded, $p \in \mathbb{N}_0$, operator $C \in \mathcal{L}(\mathcal{U};\mathcal{Y})$, $\mathcal{U}^0 \subset \ker C$, $\chi \in C^{n(p+1)}([0,T];\mathcal{L}(\mathcal{Y};\mathcal{F}))$, $f \in C^{n(p+1)}([0,T];\mathcal{F})$, $\Psi \in C^{n(p+2)}([0,T];\mathcal{Y})$, for any $t \in [0,T]$ operator $C(L^1)^{-1}Q\chi$ be invertible, with $\left(C(L^1)^{-1}Q\chi\right)^{-1} \in C^{n(p+1)}([0,T];\mathcal{L}(\mathcal{Y}))$, the condition $Cu_{n-1} = \Psi^{(n-1)}(0)$ be satisfied at some initial value $u_{n-1} \in \mathcal{U}^1$, and the initial values $w_k = (\mathbb{I} - P)v_k \in \mathcal{U}^0$ satisfy

$$w_k = -\sum_{j=0}^{p} H^j (M^0)^{-1} \frac{d^{nj+k}}{dt^{nj+k}} \Big[(\mathbb{I} - Q)(\chi(t)q(t) + f(t))\Big]\Big|_{t=0}, \quad k = 0, 1, \ldots, n-1.$$

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Then there exists a unique solution (v, q) of inverse problem (1) – (3), where $q \in C^{n(p+1)}([0,T];\mathcal{Y})$, $v = u + w$, whence $u \in C^n([0,T];\mathcal{U}^1)$ is the solution of (4) – (6) and the function $w \in C^n([0,T];\mathcal{U}^0)$ is a solution of (7) – (8) given by

$$w(t) = -\sum_{j=0}^{p} H^j (M^0)^{-1} \frac{d^{nj}}{dt^{nj}} \Big[(\mathbb{I} - Q)(\chi(t)q(t) + f(t))\Big].$$

Applications. Let $G = G(\mathfrak{D}, \mathfrak{E})$ be a finite connected oriented graph, where $\mathfrak{D} = \{V_i\}$ is the set of vertices, and $\mathfrak{E} = \{E_j\}$ is the set of edges. Each edge is characterized by two numbers $l_j, d_j \in \mathbb{R}_+$, denoting the length and cross-sectional area of the edge E_j respectively. On a graph G consider the Boussinesq – Love equations [1]

$$(\alpha - \Delta)v_{tt} = \beta(\Delta - \gamma)v + qf, \quad v = (v_1, v_2, ..., v_j, ...),\ f = (f_1, f_2, ..., f_j, ...), \tag{9}$$

with the conditions at each vertex V_i of the graph

$$\sum_{E_j \in E^\alpha(V_i)} d_j v_{jx}(0,t) - \sum_{E_m \in E^\omega(V_i)} d_m v_{mx}(l_m, t) = 0, \tag{10}$$

$$v_j(0,t) = v_k(0,t) = v_m(l_m,t) = v_n(l_n,t), \tag{11}$$

initial conditions

$$v(x,0) = \varphi(x), \quad \varphi = (\varphi_1, \varphi_2, ..., \varphi_j, ...), \tag{12}$$

$$v_t(x,0) = \psi(x), \quad \psi = (\psi_1, \psi_2, ..., \psi_j, ...) \tag{13}$$

and overdetermination condition

$$< v(x,t), K(x) >= \Phi(t), \quad K = (K_1, K_2, ..., K_j, ...), \tag{14}$$

where $f(x,t),\ \varphi(x),\ \psi(x),\ K(x)$ are given vector-functions, $\Phi(t)$ is given function and

$$< v(x,t), K(x) >= \sum_{E_j} \int_0^{l_j} v_j(x,t) K_j(x) dx$$

is the inner product in space $L^2(G)$. Function $v_j(x,t)$ defines a longitudinal displacement at point x at moment t for the j-th element of construction. The coefficients α, β andγ characterize the properties of the rods material construction. Function $f(x,t)$ sets the known external load and $q(t)$ is its coefficient. Usually, (10) is the «flow balance» condition, and condition (11) means «continuity» of the solution $v(x,t)$. Condition (12) specifies the initial position, the condition (13) specifies the initial velocity. Condition (14) is necessary to restore the coefficient $q(t)$ in equation (9).

Theorem 2. [3] Let $K, u_0, u_1 \in \mathcal{U}^1, f \in C^2([0,T];\mathcal{L}(\mathcal{Y};\mathcal{F})), \Phi \in C^4([0,T];\mathcal{Y})$, one of the conditions $\alpha \notin \sigma(\Delta)$ or $(\alpha \in \sigma(\Delta)) \wedge (\alpha \neq \gamma)$ be fulfilled, the conditions:

$$\sum_{\lambda_k \neq \alpha} \frac{< f(\cdot,t), K(\cdot) >}{\alpha - \lambda_k} \neq 0, \quad \sum_{\lambda_k \neq \alpha} < u_1, K(\cdot) >= \Phi'(0)$$

be satisfied for initial value $u_1 \in \mathcal{U}^1$, and the initial values $w_k = (\mathbb{I} - P)v_k \in \mathcal{U}^0,\ k = 0,1$ satisfy

$$< w_0 + \frac{q(0)f(\cdot,0)}{\beta(\lambda_k - \gamma)}, \mathbb{X}_k >= 0 \text{ and } < w_1 + \frac{q(0)f_t(\cdot,0) + q'(0)f(\cdot,0)}{\beta(\lambda_k - \gamma)}, \mathbb{X}_k >= 0 \text{ for } k : \lambda_k = \alpha.$$

Then there exists a unique solution (v, q) of the inverse problem (9) – (14), where $q \in C^2([0,T];\mathcal{Y})$, $v = u+w$, whence $u \in C^2([0,T];\mathcal{U}^1)$ is a solution of (4) – (6) and the function $w \in C^2([0,T];\mathcal{U}^0)$ is a solution of (7) – (8) given by

$$w(t) = -\sum_{\lambda_k=\alpha} < \frac{q(t)f(\cdot,t)}{\beta(\lambda_k-\gamma)}, \mathbb{X}_k > \mathbb{X}_k.$$

Computational experiment. Let the graph G (Figure 1) consist of two edges with lengths $l_1 = l_2 = \pi$, connecting three vertices.

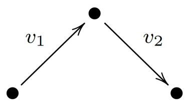

Fig. 1. Graph G

Set the parameters and functions $\alpha = 4$, $\beta = 1$, $\gamma = 1$, $\varepsilon = 0.8$, $n = 3$, $T = 10$, $l = \Big(\pi, \pi\Big)$,

$$\varphi(x) = \Big(\cos(x), \cos(x+\pi)\Big), \quad \psi(x) = \Big(\cos(5x), \cos\Big(5(x+\pi)\,\Big),$$

$$f(x) = \Big(\cos(x), \cos(x)\Big), \quad K(x) = \Big(\cos(x), \cos(x)\Big), \quad F(t) = \frac{4\cos(t)}{3}.$$

Therefore, all conditions of Theorem 2 are satisfied. The function q was obtained by the method of successive approximations.

$$q(t) = \frac{4\cos(t)(1762152484\cos^2(t) - 8037989418\cos(t) + 17837462559)}{20647703175}.$$

It is an approximate solution of the problem posed, reaching admissible error $0.6551933817 < \varepsilon$ at the 3-rd step of approximation.

Further, the required vector-function $v(x,t)$ was found using the algorithms developed for the direct problem [3]. Figure 2 show the graphs of the function $v(x,t)$ at different time t.

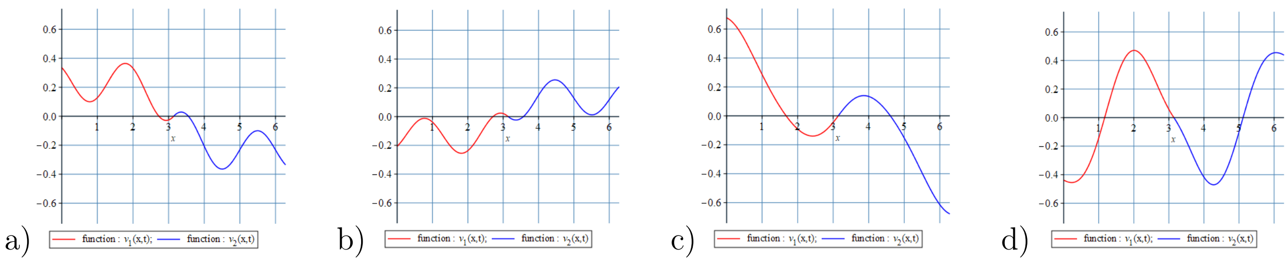

Fig. 2. Function $v(x,t)$ graph at: a) $t = 0$; b) $t = 3.33$ c) $t = 6.66$; d) $t = 10$

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