Global Existence of Non-cutoff Boltzmann Equation for Dilute Gas

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Introduction. This talk focus on the existence of global-in-time unique solution to the non-cutoff Boltzmann equation for hard potential on the whole space. We present a new approach of semigroup analysis and pseudo-differential calculus for deriving the regularizing estimate on non-cutoff linearized Boltzmann equation. We are able to obtain regularizing estimate of semigroup e^{tB} that is continuous from weighted Sobolev space $H(a^{-1/2})H^m_x$ to $H(a^{1/2})H^m_x$ with a sharp large time decay. This work develops the application of pseudo-differential calculus, spectrum analysis and semigroup theory to non-cutoff Boltzmann equation.

In the present paper, we consider the Boltzmann equation in d-dimension:

$$F_t + v\cdot\nabla_x F = Q(F,F), \tag{1}$$

where the unknown $F(x,v,t)$ represents the density of particles in phase space, with spatial coordinate $x\in\mathbb{R}^d$ and velocities $v\in\mathbb{R}^d$ with $d\geq 2$. The Boltzmann collision operator $Q(F,G)$ is a bilinear operator defined for sufficiently smooth functions F,G by

$$Q(F,G)(v) := \int_{\mathbb{R}^d}\int_{\mathbb{S}^{d-1}} B(v-v_*,\sigma)(F'_*G' - F_*G)\,d\sigma dv_*$$

where $F'_* = F(x,v'_*,t)$, $G' = G(x,v',t)$, $F_* = F(x,v_*,t)$, $G = G(x,v,t)$. (v,v_*) are the velocities of two gas particles before collision while (v',v'_*) are the velocities after collision satisfying the following conservation laws of momentum and energy,

$$v+v_* = v'+v'_*,\quad |v|^2+|v_*|^2 = |v'|^2+|v'_*|^2.$$

As a consequence, for $\sigma\in\mathbb{S}^{d-1}$, the unit sphere in $\mathbb{R}^d$, we have the σ-representation:

$$v' = \frac{v+v_*}{2} + \frac{|v-v_*|}{2}\sigma,\quad v'_* = \frac{v+v_*}{2} - \frac{|v-v_*|}{2}\sigma.$$

Also, we define the angle θ between $v-v_*$ and σ by

$$\cos\theta = \frac{v-v_*}{|v-v_*|}\cdot\sigma,$$

where $\cdot$ denotes the usual inner product in $\mathbb{R}^d$. The collision kernel B satisfies

$$B(v-v_*,\sigma) = |v-v_*|^\gamma b(\cos\theta),$$

for some $\gamma\in\mathbb{R}$ and function b. Without loss of generality, we can assume that $B(v-v_*,\sigma)$ is supported on $(v-v_*)\cdot\sigma\geq 0$ which corresponds to $\theta\in[0,\pi/2]$, since B can be replaced by its symmetrized form $\overline{B}(v-v_*,\sigma) = B(v-v_*,\sigma)+B(v-v_*,-\sigma)$. Moreover, we are going to work on the collision kernel without angular cut-off, which corresponds to the case of inverse power interaction laws between particles. That is,

$$b(\cos\theta)\approx\theta^{-d+1-2s}\quad\text{on }\theta\in(0,\pi/2), \tag{2}$$

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where we assume

$$s \in (0,1), \quad \gamma \in (-d, \infty). \tag{3}$$

For Boltzmann equation without angular cut-off, it's customary to use term soft potential when $\gamma + 2s < 0$, Maxwell molecules when $\gamma + 2s = 0$, and hard potential when $\gamma + 2s \geq 0$. For mathematical theory of Boltzmann equation, one may refer to [1, 2, 6, 16] for more introduction.

We will study the Boltzmann equation (1) near the global Maxwellian equilibrium

$$\mu(v) = (2\pi)^{-d/2} e^{-|v|^2/2}.$$

Set $F = \mu + \mu^{\frac{1}{2}} f$ and then the Boltzmann equation (1) becomes

$$f_t + v \cdot \nabla_x f = Lf + \mu^{-1/2} Q(\mu^{1/2} f, \mu^{1/2} f),$$

where L is called the linearized Boltzmann operator given by

$$Lf = \mu^{-1/2} Q(\mu, \mu^{1/2} f) + \mu^{-1/2} Q(\mu^{1/2} f, \mu).$$

The kernel of L is $\text{Span}\{\psi_i\}_{i=0}^{d+1}$ defined in

$$\psi_0 = \mu^{1/2},\ \psi_i = v_i \mu^{1/2}\ (i = 1, \cdots, d),\ \psi_{d+1} = \frac{1}{\sqrt{2d}}(|v|^2 - d)\mu^{1/2}. \tag{4}$$

and we denote the projection from L^2 onto $\text{Ker}L$ by

$$Pf := \sum_{i=0}^{d+1} (f, \psi_i)_{L^2} \psi_i.$$

Definition 1. We define a Hilbert space $H(M, \Gamma) := \{u \in \mathscr{S}' : \|u\|_{H(M,\Gamma)} < \infty\}$, where

$$\|u\|_{H(M,\Gamma)} := \int M(Y)^2 \|\varphi_Y^w u\|_{L^2}^2 |g_Y|^{1/2}\, dY < \infty, \tag{5}$$

and $(\varphi_Y)_{Y \in \mathbb{R}^{2d}}$ is any uniformly confined family of symbols which is a partition of unity.

We would like to apply the symbolic calculus in [3] for our study as the following. One may refer to the appendix as well as [11] for more information about pseudo-differential calculus. Let $\Gamma = |dv|^2 + |d\eta|^2$ be an admissible metric. Define

$$a(v, \eta) := \langle v \rangle^\gamma (1 + |\eta|^2 + |\eta \wedge v|^2 + |v|^2)^s + K_0 \langle v \rangle^{\gamma + 2s} \tag{6}$$

be a Γ-admissible weight, which is proved in [3], where $K_0 > 0$ is chosen as the following and $|\eta \wedge v| = |\eta||v| \sin \theta_0$ with θ_0 being the angle between η and v. By the invertibility of $(a^{1/2})^w$, we have equivalence

$$\|(a^{1/2})^w(\cdot)\|_{L^2} \approx \|\cdot\|_{H(a^{1/2})},$$

and hence we will equip $H(a^{1/2})$ with norm $\|(a^{1/2})^w(\cdot)\|_{L^2}$. Also, we will denote the weighted Sobolev norms that, for $l, n \in \mathbb{R}$,

$$\|f\|_{L^2_v H^m_x} := \|\langle D_x \rangle^m f\|_{L^2_{x,v}},$$
$$\|f\|_{H(a^{1/2}) H^m_x} := \|\langle D_x \rangle^m (a^{1/2})^w f\|_{L^2_{x,v}},$$

where $\langle D_x \rangle^m f = \mathscr{F}_x^{-1} \langle y \rangle^m \mathscr{F}_x f$, $L^2_{x,v} = L^2(\mathbb{R}^{2d}_{x,v})$.

Define an operator closure

$$B = \overline{-v \cdot \nabla_x + L}.$$

Then B generates a strongly continuous semigroup e^{tB} on L^2. The closure is necessary for generating the semigroup and hence it's necessary throughout our arguments. Then we can discuss the solution $e^{tB} f_0$ to the linearized Boltzmann equation:

$$\begin{cases} f_t = Bf, \\ f|_{t=0} = f_0. \end{cases}$$

This semigroup e^{tB} generated by the linearized Boltzmann operator plays an important role in the perturbation theory of Boltzmann equation and kinetic equation, since the solution to Boltzmann equation can be written into a perturbation form of the solution to its linearized equation, for instance [10, 14]. Our first main result gives an optimal regularizing effect on the linearized Boltzmann equation for hard potential and a large time decay estimate. With this regularizing estimate established, we can apply the energy estimate to obtain a global solution. Our method is building on the whole space $\mathbb{R}^d_x$, which is different from torus $\mathbb{T}^d_x$. Thus, the analysis on the spectrum structure on linearized Boltzmann operator is required and provides a new approach in the existence theory of Boltzmann equation without angular cutoff.

Theorem 1. Assume $\gamma + 2s \geq 0$. Fix $f \in \mathscr{S}(\mathbb{R}^{2d}_{x,v})$. Then for $k \geq 2$, $m \in \mathbb{R}$ $p \in [1, 2]$, we have

$$\|e^{tB} f\|^2_{H(a^{1/2})H^m_x} \lesssim \frac{e^{-2\sigma_y t}}{t^{2k}} \|f\|^2_{H(a^{-1/2})H^m_x} + \frac{1}{(1+t)^{d/2(2/p-1)}} \|a^{-1/2}(v, D_v) f\|^2_{L^2_v(L^p_x)},$$

where $\sigma_y > 0$ is some constant is independent of f and p.

Once the large time behavior of linearized Boltzmann equation is established, we can find out the existence of global-in-time unique solution to Boltzmann equation.

Theorem 2. Suppose $d \geq 3$, $m > \frac{d}{2}$, $\gamma + 2s \geq 0$. There exists $\varepsilon_0 > 0$ so small that if

$$\|f_0\|_X \leq \varepsilon_0,$$

where X is defined as

$$\|f\|^2_X = \delta \|f\|^2_{L^2_v H^m_x} + \int_0^\infty \|e^{\tau B} f\|^2_{L^2_v H^m_x} \, d\tau, \tag{7}$$

then there exists an unique global weak solution f to Boltzmann equation

$$f_t = Bf + \Gamma(f, f), \quad f|_{t=0} = f_0, \tag{8}$$

satisfying

$$\|f\|_{L^\infty([0,\infty); L^2_v H^m_x)} + \|f\|_{L^2([0,\infty); H(a^{1/2}) H^m_x)} \leq C\varepsilon_0,$$

with some constant $C > 0$.

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