One Sobolev type equation in Hilbert spaces of differential forms with stochastic coefficients

D. E. Shafranov[1], O. G. Kitaeva[2], G. A. Sviridyuk[3]

Keywords: Sobolev type equations; differential forms; stochastic equations; Nelson – Gliklikh derivative.

MSC2010 codes: 35R60

Our spaces. Let $\mathcal{M}$ – be a smooth compact oriented Riemannian manifold without boundary with local coordinates $x_1, x_2, ..., x_n$. By $H_k = H_k(\mathcal{M}, \Omega)$ we denote the space of smooth differential k-forms $k = 0, 1, 2, ..., n$. In our consideration, the coefficients of the k-forms are allowed to contain time $t \in [0, +\infty)$, but differentials of time are impossible in the collection of k-forms; i.e., the differential forms have the form with stochastic coefficients

$$\chi_{i_1,i_2,...,i_k}(t, x_1, x_2, ..., x_n, \omega) = \sum_{|i_1,i_2,...,i_k|=k} a_{i_1,i_2,...,i_k}(t, x_{i_1}, x_{i_2}, ..., x_{i_k}, \omega) dx_{i_1} \wedge dx_{i_2} \wedge ... \wedge dx_{i_k},$$

where the $a_{i_1,i_2,...,i_k}(t, x_{i_1}, x_{i_2}, ..., x_{i_k}, \omega)$ – are coefficients depending, among other variables, on time, and, $|i_1, i_2, ..., i_k|$ is a multi-index.

One has the standard inner product

$$(\xi, \varepsilon)_0 = \int_{\mathcal{M}} \xi \wedge *\varepsilon, \quad \xi, \varepsilon \in H_k. \tag{1}$$

on the spaces H_k. Here $*$ is the Hodge operator and $\wedge$ is the operator of exterior multiplication of k-forms.

Completing the space H_k by continuity in the norm $\|\cdot|_0$ corresponding to the inner product (1), we obtain the space $\mathfrak{H}_k^0$. Introducing inner product in the spaces of differentiable or twice differentiable (in the NelsonB–Gliklikh sense) k-forms and completing the space in the norms corresponding to these inner products, we construct the spaces $\mathfrak{H}_k^1$ and $\mathfrak{H}_k^2$, respectively.

For these Hilbert spaces, one has continuous embedding other like $\mathfrak{H}_k^2 \subseteq \mathfrak{H}_k^1 \subseteq \mathfrak{H}_k^0$.

In the spaces constructed, we can use a generalization of the Laplace operator -B— the LaplaceB–Beltrami operator$\Delta u = d\delta + \delta du$, where d – is the operator of exterior differentiation of differential forms and the operator $\delta = *d*$ – is the adjoint of the operator d.

Theorem 1. [2] (Hodge – Kodaira) For the space $\mathfrak{H}_k^l, l = 0, 1, 2$, one has the following decomposition into the direct sum of subspaces:

$$\mathfrak{H}_k^l = \mathfrak{H}_{kd}^l \oplus \mathfrak{H}_{k\delta}^l \oplus \mathfrak{H}_{k\Delta}^l, l = 0, 1, 2,$$

where $\mathfrak{H}_{kd}$ is the space of potential forms, $\mathfrak{H}_{k\delta}$ is the space of solenoidal forms, and, $\mathfrak{H}_{kd}$ is the space of harmonic forms.

Corollary 1. Under the assumptions of the theorem, one has the decomposition

$$\mathfrak{H}_k^l = (\mathfrak{H}_{k\Delta}^l)^\perp \oplus \mathfrak{H}_{k\Delta}^l, l = 0, 1, 2.$$

[1]South Ural State University(NRI), Department of Mathematical Physics Equations, Russia, Chelyabisk City. Email: shafranovde@susu.ru

[2]South Ural State University(NRI), Department of Mathematical Physics Equations, Russia, Chelyabisk City. Email: kitaevaog@susu.ru

[3]South Ural State University(NRI), Department of Mathematical Physics Equations, Russia, Chelyabisk City. Email: sviridiukga@susu.ru

We can construct the *spaces of random* **K**-*variables* and the *spaces of* **K**-*"noises"* , defined on the manifold $\mathcal{M}$. Let $\mathbf{K} = \{\lambda_k\}$ be a sequence such that $\sum\limits_{k=1}^{\infty} \lambda_k^2 < +\infty$. By $\{\varphi_k\}$ and $\{\psi_k\}$ we denote the systems of eigenvectors of the LaplaceB–Beltrami operator orthonormal with respect to the inner products $< \cdot, \cdot >_0$ and $< \cdot, \cdot >_2$ in these spaces, These systems form bases in the spaces $\mathfrak{H}_k^0$ and $\mathfrak{H}_k^2$. The elements of the spaces $\mathbf{H}^0_{k\mathbf{K}}\mathbf{L}_2$ and $\mathbf{H}^2_{k\mathbf{K}}\mathbf{L}_2$ are vectors $\chi = \sum\limits_{k=1}^{\infty} \lambda_k \xi_k \varphi_k$ and $\kappa = \sum\limits_{k=1}^{\infty} \lambda_k \zeta_k \psi_k$, in which the sequences of random variables $\{\xi_k\} \subset \mathbf{L}_2$ and $\{\zeta_k\} \subset \mathbf{L}_2$ are such that the variances satisfy the inequalities $\mathbf{D}\xi_k \le const$ and $\mathbf{D}\zeta_k \le const$. we construct the set of continuous processes $\mathbf{C}(\mathfrak{I}; \mathbf{H}^0_{k\mathbf{K}}\mathbf{L}_2)$ and the set of continuously NelsonB–Gliklikh differentiable processes $\mathbf{C}^1(\mathfrak{I}; \mathbf{H}^0_{k\mathbf{K}}\mathbf{L}_2)$.

Equation with relatively operator. Further, let us proceed to the existence and stability of solutions of Ginzburg – Landau equation in the spaces $\mathbf{H}^0_{k\mathbf{K}}\mathbf{L}_2$. To this end, we define operators $L,\ M : \mathbf{H}^0_{k\mathbf{K}}\mathbf{L}_2 \to \mathbf{H}^2_{k\mathbf{K}}\mathbf{L}_2$ by the formulas

$$L = \lambda + \Delta,\ M = \nu\Delta - id\Delta^2 \tag{2}$$

Ginzburg – Landau equation are reduce to the equation

$$L \overset{\circ}{\chi} = M\chi. \tag{3}$$

Lemma 1. For any $\nu,\ \lambda,\ d \in \mathbb{R}$ the operator M is strongly $(L, 0)$-radial.

Theorem 2. (i) If $\lambda \notin \{\sigma_k\}$, then the phase space of equation (3) coincides with the space $\mathbf{H}^0_{k\mathbf{K}}\mathbf{L}_2$.

(ii) If $\lambda \in \{\sigma_k\}$, then the phase space of equation (3) is the space $\mathcal{P} = \{\varepsilon \in \mathbf{H}^0_{k\mathbf{K}}\mathbf{L}_2 :< \varepsilon, \varphi_l >= 0, \sigma_l = \lambda\}$.

The relative spectrum of the operator M is representable in the form of two disjoint components $\sigma^L(M) = \sigma^L_+(M) \bigcup \sigma^L_-(M)$, where

$$\sigma^L_+(M) = \left\{\mu_k = \frac{\nu\sigma_k - id\sigma_k^2}{\lambda + \sigma_k},\ \sigma_k < -\lambda\right\}, \sigma^L_-(M) = \left\{\mu_k = \frac{\nu\sigma_k - id\sigma_k^2}{\lambda + \sigma_k},\ \sigma_k > -\lambda\right\}. \tag{4}$$

Theorem 3. (i) For any $\nu,\ \lambda \in \mathbb{R}_-$ and $d \in \mathbb{R}$ there exists a finite-dimensional unstable and an infinite-dimensional stable invariant space of equation (3) and the solutions of equations (3) have an exponential dichotomy.

(ii) For any $\nu \in \mathbb{R}_-$, $\lambda \in \mathbb{R}_+$ and $d \in \mathbb{R}$ the phase space of equation (3) coincides with the stable invariant space.

References

[1] O.G. Kitaeva, D.E. Shafranov, G.A. Sviridyuk. *Exponential dichotomies in Barenblatt–Zheltov–Kochina model in spaces of differential forms with "noise"*. // Bulletin of SUSU. Ser. MathMod,ProgCompSoftw. 2019. Vol. 12. No. 2. P. 47-B–57.

[2] O.G. Kitaeva, D.E. Shafranov, G.A. Sviridyuk. *Degenerate holomorphic semigroups of operators in spaces of k-"noises" on Riemannian manifolds.* // Semigroups of Operators – II Theory and Applications SOTA 2018. Springer Proc. Math. Stat., Cham: Springer. 2020. Vol. 325. P. 279-B–292.

[3] D.E. Shafranov, O.G. Kitaeva, G.A. Sviridyuk. *Stochastic equations of Sobolev Type with relatively p-radial operators in spaces of differential forms.* // Differential equations. 2021. Vol. 57. No. 4. P. 526-B–535.