Tomography on locally compact groups

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Keywords: locally compact groups, projective unitary representations, quantum tomography

MSC2010 codes: 81P18, 20C25

Let $G \ni g \to U_g$ be a unitary representation of a locally compact Abelian group G with the Haar measure μ in a Hilbert space $\mathcal{H}$. Given a unit vector $f \in \mathcal{H}$

$$F(g) = \langle f, U_g f \rangle, \ g \in G,$$

is a characteristic function of some probability distribution $\{\pi_\chi, \ \chi \in \hat{G}\}$ which can be determined by the formula

$$\pi_\chi = \int_G \overline{\chi(g)} F(g) d\mu(g), \tag{1}$$

where μ is the Haar measure on G and $\hat{G}$ is the dual group consisting of characters χ on G. The knowledge of the probability distribution (1) only is not sufficient for reconstructing f. We define a one-parameter set of unitary representations of groups being sections of the group $\hat{G} \times G$. The corresponding set of probability distributions allowes to restore f. The entire set of unitary representations forms a projective unitary representation of $\hat{G} \times G$. Thus we introduce the tomography of a state f on G [1]. Three examples in which $G = \mathbb{R}$, $\mathbb{Z}_n$ and $\mathbb{T}$ (the circle group) are considered. As an application we study tomography of output states of quantum Weyl channels [2-3].

The main idea of tomography can be explained as follows. Let us define a projective unitary representation π of the group $\hat{G} \times G$ in the Hilbert space $\mathcal{H} = L^2(G)$ by the formula

$$(\pi(\chi, g) f)(a) = \chi(a) f(a + g), \ \chi \in \hat{G}, \ g \in G, \ f \in \mathcal{H}.$$

Then, the following statements holds true

Proposition 1. Given fixed $\chi \in \hat{G}, \ g \in G$ the set $G_{\chi,g} = \{(\chi', g') : \ \chi'(g) = \chi(g')\}$ is a subgroup of $\hat{G} \times G$.

Proposition 2. The map $G_{\chi,g} \ni (\chi', g') \to [\chi'(g')]^{1/2} \pi(\chi', g')$ is a unitary representation of $G_{\chi,g}$ in $\mathcal{H}$.

The unitary representation of $G_{\chi,g}$ determined by Proposition 2 results in the probability distribution $\pi^{(\chi,g)}$. The set of all distributions $(\pi^{(\chi,g)}, \ \chi \in \hat{G}, \ g \in G)$ is said to be a quantum tomogram. In the partial case $G = \hat{G} = R$

$$G_{x,y} = \{x', y' : \ x' = t\cos\varphi, \ y' = t\sin\varphi, \ \tan\varphi = \frac{y}{x}, \ t \in \mathbb{R}\} \equiv G_\varphi$$

appears to be a one-parameter group indexed by $\varphi \in [0, 2\pi]$.

Acknowledgments. This work is supported by the Russian Science Foundation under grant 19-11-00086.

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