A new probabilistic formula for the gradient of solutions of some Hypoelliptic Dirichlet problems

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We are here concerned with the following Cauchy–Dirichlet problem,

$$\begin{cases} D_t u(t,x) = \frac{1}{2}\operatorname{Tr}\left[C D_x^2 u(t,x)\right] + \langle Ax, D_x u(t,x)\rangle, & (t,x)\in\mathbb{R}^+\times\overline{\mathcal{O}}, \\ u(t,x)=0, \quad t\ge 0,\ x\in\partial\mathcal{O}, \\ u(0,x)=\varphi(x), \quad x\in\overline{\mathcal{O}}. \end{cases} \tag{1}$$

A and C are $d\times d$ matrices with C semi definite positive and *singular* and $\mathcal{O}$ is a bounded convex open subset of $\mathbb{R}^d$. Our basic assumption is the following

Hypothesis [Hypoellipticity]
The matrix $Q_t := \int_0^t e^{sA} C\, e^{sA^*} ds$ is non singular for all $t>0$.

The formal solution $u(t,x)$ of (1) is given by $R_T^{\mathcal{O}}\varphi(x)$ where $R_T^{\mathcal{O}}\varphi$ is the stopped semigroup

$$R_T^{\mathcal{O}}\varphi(x) = \mathbb{E}\left[\varphi(X(T,x))\mathbb{1}_{T\le\tau_x}\right],\ T>0, \quad x\in\overline{\mathcal{O}}$$

where

$$X(T,x) = e^{tA}x + \int_0^T e^{(t-s)A}dW(s)$$

and $W(t),\ t\ge 0$, is an $\mathbb{R}^d$–valued standard Wiener process defined on a probability space $(\Omega,\mathcal{F},\mathbb{P})$. Moreover, τ_x is the exit time from $\overline{\mathcal{O}}$,

$$\tau_x = \inf\{t\ge 0:\ X(t,x)\in \overline{\mathcal{O}}^c\}.$$

We prove a new representation formula of the gradient of $R_T^{\mathcal{O}}\varphi(x)$, for all $T>0$, $x\in\overline{\mathcal{O}}$ and all φ bounded and Borel on $\overline{\mathcal{O}}$.

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