A Lumer–Phillips type generation theorem for bi-continuous semigroups

Christian Budde [1]

The famous 1960s Lumer-Phillips Theorem states that a closed and densely defined operator $A\colon D(A)\subseteq X\to X$ on a Banach space X generates a strongly continuous contraction semigroup if and only if $(A,D(A))$ is dissipative and the range of $\lambda - A$ is surjective in X for some $\lambda > 0$.

Despite its great success, the theory of C_0-semigroups of course does not apply to any evolution equation on any space. A well-known issue is that for some operators $(A,D(A))$ there is indeed a semigroup $T\colon [0,\infty)\to L(X)$ but its orbit maps fail to be continuous. A prototype example for this effect is the transport equation, i.e., $A=\frac{\mathrm{d}}{\mathrm{d}\xi}$, on the space $(\mathrm{C_b}(\mathbb{R}),\|\cdot\|_\infty)$ of bounded continuous functions on the real line. Whereas the classic theory works perfectly on the subspace of uniformly continuous functions, a quite natural expansion of the space destroys the strong continuity of the shift semigroup which gives the solutions to the corresponding Cauchy problem. A natural fix for this lack of continuity is to endow the space with a coarser topology, here for example the topology of pointwise convergence or compact convergence. The latter leads to the study of strongly continuous semigroups on locally convex topological vector spaces.

Since its outset in 2001, the theory of bi-continuous semigroups has grown and by now includes a Hille-Yosida type generation theorem, an approximation theory as well as a perturbation theory including inter- and extrapolation spaces [4, 2]. We add another piece to the puzzle by establishing a Lumer-Phillips type generation theorem for contraction semigroups. This is in particular desirable since in the C_0-setting the Lumer-Phillips theorem, see [3, Chapter II, Thm. 3.15], is one of the most important tools to prove that a given operator is a generator. Moreover, a Lumer-Phillips theorem for (equicontinuous) semigroups on general locally convex spaces has been established recently by Albanese, Jornet [1]. To demonstrate the applicability of our theorem we treat the prototype example from above as well as the heat equation, where we now are able to prove the generation property without using a priori knowledge on the semigroup. The latter was not possible with the previously existing generation theorems, see [5, Prop. 3.11 & 3.12] and [4, Thm. 2.4.2 & 2.4.4]. Our whole last chapter is then devoted to applications treating flows on infinite networks.

This is joint work with S.-A. Wegner.

References

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[1]University of the Free State

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