The Showalter – Sidorov and the Cauchy problems for a linear Dzekzer equation with Wentzell and Roben boundary conditions in a bounded domain

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Keywords: Dzekzer equation, deterministic and stochastic Sobolev type equations, Nelson – Gliklikh derivative, Wentzell condition, Showalter – Sidorov condition, Cauchy condition.

MSC2010 codes: 93E10

Introduction. Let $\Omega \subset \mathbb{R}^n$, $n \in \mathbb{N} \setminus \{1\}$, be a bounded connected domain with the boundary $\partial\Omega$ of the class C^∞. In cylinder $Q_T = \Omega \times (0,T)$, $T \in \mathbb{R}_+$, let us consider the linear Dzekzer equation

$$(\lambda - \Delta) \overset{\circ}{\eta} (\omega, t) = \alpha_0 \Delta\eta(\omega,t) - \beta_0 \Delta^2 \eta(\omega,t) - \\ -\gamma\eta(\omega,t) + f, \ (\omega,t) \in Q_T. \tag{1}$$

which modeling the evolution of the free surface of the filtered liquid [1]. In particular, the authors are interested in the solutions, which must satisfy both to Wentzell boundary conditions

$$\Delta\eta(\omega,t) + \alpha_1 \frac{\partial\eta}{\partial\nu}(\omega,t) + \beta_1\eta(\omega,t) = 0, \ (\omega,t) \in \partial\Omega \times (0,T), \tag{2}$$

and to Roben boundary conditions

$$\alpha_2 \frac{\partial\eta}{\partial\nu}(\omega,t) + \beta_2\eta(\omega,t) = 0, \ (\omega,t) \in \partial\Omega \times (0,T), \tag{3}$$

as well as the initial Cauchy condition

$$\lim_{t\to 0+} (\eta(t) - \eta_0) = 0 \tag{4}$$

Here $\lambda \in \mathbb{R}$, α_k, β_k, $\gamma \in \mathbb{R}_+$, $k = 0, 1$ are real parameters, characterizing the medium; $\eta = \eta(t)$ is a stochastic process; $\overset{\circ}{\eta}$ is the NelsonвЂ“ Gliklikh derivative of the process $\eta(t)$; f is a «white noise», which we understand the Nelson – Gliklich derivative an one-dimensional Wiener process, $\nu = \nu(x)$ is a external unit normal to $\partial\Omega$.

In this report the deterministic and stochastic the Showalter – Sidorov and the Cauchy problems for the Dzekzer equation with Wentzell – Roben boundary conditions was considered by the authors. Note that for the filtration model under study, the Wentzell condition is considered, which is not a classical condition. In recent years, the boundary condition has been considered in the mathematical literature from two points of view (classical and neoclassical). Since Cauchy and Shoulter – Sidorov initial conditions have been studied earlier in various situations, in this work, in the particular case of classical Wentzell and Roben conditions, by methods of the theory of degenerate holomorphic semigroups, exact solutions have been constructed, which allow to determine quantitative predictions of changes in geochemical regime of groundwater under unpressurized filtration. Nelson – Glicklich derivative theory was used in stochastic case. In particular, the investigation of the set problems in the context of Wentzell boundary conditions allowed to determine the processes occurring at the boundary of two media (in the region and at its boundary). In particular, for suitable spaces the following theorem is proved in stochastic case.

Theorem 1. Let $\lambda \in \sigma(\Delta)$ and the coefficients $(\alpha_0, \gamma) \in \mathbb{R}^2$ and $\beta \in \mathbb{R}_+$ are such, that no eigenvalue $\lambda_k \in \sigma(\Delta)$ is the root of the equation $\beta_0\xi^2 - \alpha_0\xi + \gamma = 0$. Then for any $\eta_0 \in \mathbf{U_K L_2}$

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there exists a stochastic $\mathbf{K}$-process $\eta \in C^\infty(\mathbb{R}_+; \mathbf{U_K L}_2)$, each trajectory of which is the unique solution of the Showalter–Sidorov problems. Moreover, the stochastic $\mathbf{K}$-process. $\eta = \eta(t)$ has the following form

$$\eta(t) = -\sum_{\lambda=\nu_k} \lambda_k \overset{\circ}{\beta}_k (t) + U^t \xi_0 + \int_0^t U^{t-s} \overset{\circ}{W}_k (s) ds.$$

Otherwise, if $\lambda \notin \sigma(\Delta)$. *Then for any* $\eta_0 \in \mathbf{U_K L}_2$ *there exists a stochastic* $\mathbf{K}$*-process* $\eta \in C^\infty(\mathbb{R}_+; \mathbf{U_K L}_2)$ *whose every trajectory is a unique solution to the Cauchy and Showalter-Sidorov problem. Moreover, the stochastic* $\mathbf{K}$*-process.* $\eta = \eta(t)$ *has the following form*

$$\eta(t) = U^t \eta_0 + \int_0^t U^{t-s} \overset{\circ}{W}_k (s) ds.$$

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