List of open problems in one-parameter operator (semi)group theory (collected by participants of the OPSO 2021 online conference)

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February 18, 2022

DRAFT VERSION OF THE TEXT

Problem 1 (Wolfgang Arendt)**.** Let 0≤S(t)≤T(t) on a Banach lattice X. If $(T(t))_{t\geq 0}$ is holomorphic, does it follow that $(S(t))_{t\geq 0}$ is holomorphic?

Comments. For $X = L^p(\mathbb{R}^n)$, $1 \leq p < \infty$, $T(t) = e^{t\Delta}$, $S(t) = e^{t\Delta - V}$, $0 \leq V : \mathbb{R}^n \to \mathbb{R}$ measurable, the answer is yes.

References:

• T. Kato: L^p-theory of Schrödinger operators, in aspect of positivity in functional analysis. R. Nagel, U. Shlotterbeck, M. Wolff (eds.) pp. 63-78. North-Holland: Amsterdam 1986.

• W. Arendt, C. J. K. Batty: Absorption C_0-semigroups and Dirichlet boundary conditions, Math. Ann. 295 (1993), 427-448.

Source: R. Nagel's list of problems collected in 2003 at the workshop in Bari.

Problem 2 (Charles J. K. Batty)**.** Prove or disprove that non-analytic growth bound $\zeta(T)$ of a C_0-semigroup T coincides with critical growth bound $\omega_{crit}(T)$

Comments. One has

$$\zeta(T) \leq \omega_{ess}(T) \quad \text{and} \quad \zeta(T) = \omega_{crit}(T)$$

in each of the following cases:

- T is a C_0-semigroup on Hilbert space,
- T is a C_0-semigroup on a Banach space, T has L^p-resolvent, $p \in (1, \infty)$,
- T is eventually differentiable.

Comments. (Charles Batty, April 2021)

This problem has not been solved so far. Note that there is some vague similarity to Open Problem 4.

Relevance. This problem is interesting because $\zeta(T)$ and $\omega_{crit}(T)$ are both candidates to be variants of the exponential growth bound $\omega_0(T)$, modulo analytic functions and spectral bounds of the generator modulo horizontal strips. Some variants of standard results have been obtained using $\zeta(T)$ instead of $\omega_(T)$ have been established. See, for example, Section 5 of my article in

Perspectives in operator theory, 39вЂ“53, Banach Center Publ., 75, Polish Acad. Sci. Inst. Math., Warsaw, 2007.

References: C. J. K. Batty, M. D. Blake, S. Srivastava: A non-analytic growth bound of Laplace transforms and semigroup of operators. Int. Eq. Op. Th. 45 (2003), no.2, 125-154.

Source: R. Nagel's list of problems collected in 2003 in the workshop in Bari.

Problem 3 (Charles J. K. Batty and Klaus-Jochen Engel)**.** Prove or disprove that semigroup is immediatly norm continuous if and only if
$\|R(is, A)\| \to 0$ as $|s| \to \infty$.

Comment (Charles Batty). Open Problem 3 was answered negatively, by Ralph Chill and Yuri Tomilov (J. Funct. Anal. 256 (2009), no. 2, 352-384), and independently by Tamás Mátrai (Israel J. Math. 168 (2008), 1-28).
References:

- O. El-Mennaoui, K.J. Engel: Towards a characterization of eventually norm continuous semigroup on Banach spaces, Quaest. Math. 19 (1996), 183-190.
- O. Blasco, J. Martines: Norm continuity and related notions for semigroup on Banach spaces, Archiv Math. 66 (1996), 470-478.
- V. Goerrsmeyer, L. Weis: Norm continuity of C_O-semigroups, Studia Math. 134 (1999), 169-178.

Source: R. Nagel's list of problems collected in 2003 in the workshop in Bari.

Problem 4 (Jerome A. Goldstein)**.** Which C_0-semigrops are "asymptotically analytic"?

Comment. First, we explain the notion of "asyptotically analytic semigroups". Let B be a positive selfadjoint operator in Hilbert space X, and let $a > 0$. Let u be a solution of the graph equation

$$u'' + 2au' + Bu = 0.$$

This problem is governed by a C_0-semigroup on energy space besed on X. Eckstein-Goldstein-Leggas [EJDE, Proc, Conf. 03, 1999] proved that

$$u(t) = v(t) + w(t),$$

where u satisfies the heat equation

$$2av' + Bv = 0$$

and $\|w(t)\| = o(\|v(t)\|)$ as t tends to infinity. This leads to the notion of "asymptotically analytic".

Let $(T(t))_{t\geq 0}$ be a C_0-semigroup on a Banach space X, let $S := (s(t))_{t\geq 0}$ be an analytic C_0-semigroup on a Banach space Y and let be P a (somehow natural) bounded linear operator from X to Y. We call S "asymptotically analytic" if there exist S, P as above such that from all $f \in X$ there is $g \in Y$ so that

$$T(t)f = S(t)g + w(t), \quad t \geq 0,$$

where $\|w(t)\| = o(\|S(t)g\|)$ as t tends to infinity.

Thus asymptotically analytic semigroups have the asymptotics of analytic semigroups, except for errors that are relatively small asymptotically. As a first step it would be of interest to have some results in the case of $Y = X$ and $P = I$. More generally, which C_0-groups (such as those governing second order equations) and which nonanalytic semigroups governing FDE's are asyptotically anatilic?

(Charles Batty, April 2021) Note that Open Problem 2 and Open Problem 4 have some vague similarity.
Source: R. Nagel's list of problems collected in 2003 at the workshop in Bari.

Problem 5 (Birgit Jacob, Hans Zwart)**.** Let $(T(t))_{t\geq 0}$ be a contraction semigroup with generation A on a Hilbert space H. Consider the following properties.

(i) There exist $m \geq 0$ such that

$$\|(\lambda - A)x\| \geq m|Re\lambda|\|x\| \quad \text{for all } Re\lambda < 0 \text{ and all } x \in H.$$

(ii) There exist $m_1 > 0$ such that

$$\|T(t)x\| \geq m_1\|x\| \quad \text{for all } t \geq 0, x \in H$$

Does (i) imply (ii)?

Comment. Condition (ii) always implies (i). This is not true if $(T(t))_{t\geq 0}$ is only bounded, but it is true if $\lambda \in \rho(A)$ with $Re\lambda < 0$.

Comment (Charles Batty, April 2021) I do not know of an answer to this problem. In a paper by Xu and Shang (Systems Control Lett. 58 (2009), no. 8, 561-566, a related result on Banach spaces is stated in Theorem 2.4. Their proof is seriously flawed, but a proof is given in a paper by Geyer and myself (J. Operator Theory 78 (2017), no. 2, 473-500); see Theorem 5.4 and Example 5.6.
Source: R. Nagel's list of problems collected in 2003 at the workshop in Bari.

Problem 6 (Yuri Latushkin)**.** Consider the following properties of the generation A of a C_0-semigroups $(T(t))_{t\geq 0}$.

1. rg A closed.
2. $\text{rg}(1 - T(t))$ closed for one $t > 0$.

Does (ii) imply (i)?

Comment. This is a version of a "special inclusion theorem". The converse implication does not hold.
Source: R. Nagel's list of problems collected in 2003 at the workshop in Bari.

Problem 7 (Alessandra Lunardi)**.** Let A and B be generators of C_0 semigroups. Under which condition does

$$C := \overline{A^2 + B^2}$$

generate an analytic semigroup?

Source: R. Nagel's list of problems collected in 2003 at the workshop in Bari.

Problem 8 (Alessandra Lunardi)**.** Let $(T(t))_{t\geq 0}$ be a not necessary strongly continious semigroup on a Banach space X and consider the following condition.

(i) $(0, \infty) \ni t \mapsto T(t) \in L(X)$ is analytic.

(ii) $t \mapsto T(t)$ is analytic on a sector containing $\mathbb{R}_+$

(iii) $\|T(t)\| \leq Me^{t\omega}$, $\frac{d}{dt}T(t) \in L(X)$ and $\|\frac{d}{dt}T(t)\| \leq \frac{M}{t}e^{t\omega}$ for some constants ω, M.

Under which assumption added to (i), (ii) or (iii) does there exist a sectorial operator A generating $(T(t))_{t\geq 0}$?

Source: R. Nagel's list of problems collected in 2003 at the workshop in Bari.

Problem 9 (Alessandra Lunardi)**.** Study the "backward uniqueness property", i.e., characterize injective C_0-semigroups. Apply the result to the backward uniqueness property for non-autonomous Cauchy problems $u'(t) = A(t)u(t), A(t)$ sectorial, by looking at the corresponding evolution semigroup.

Comments. Why this problem is important

Source: R. Nagel's list of problems collected in 2003 at the workshop in Bari.

Problem 10 (Alessandra Lunardi)**.** Consider "non-C_0-semigroups", e.g., bi-continious semigroup and describe appropriate regularization properties.

Comments. Compare the Ornstein-Uhlebeck semigroup in $C_b(\mathbb{R}^n)$.

Source: R. Nagel's list of problems collected in 2003 at the workshop in Bari.

Problem 11 (R. Nagel)**.** Let A and B the generators of two communiting C_0-semigroups on a Banach space and let G be the generator of corresponding product semigroup. Find (the most general) conditions implying

$$D(G) = D(A) \cap D(B).$$

Comments. This yields abstract "maximal regularity" results.

Source: R. Nagel's list of problems collected in 2003 at the workshop in Bari.

Problem 12 (R. Nagel)**.** Let $(T(t))_{t\geq 0}$ be a C_0-semigroup whith growth bound

$$\omega_0 := \inf\{\omega \in \mathbb{R} : \|T(t)\| \leq M^\omega \cdot e^{t\omega} \text{ for } t \geq 0\}$$

Find condition such that ω_0 is minimum, i.e.,

$$\|T(t)\| \leq M_0 \cdot e^{t\omega_0} \text{ for } t \geq 0$$

Comments. This corresponds to a characterization of boundedness for semigroups.

Source: R. Nagel's list of problems collected in 2003 at the workshop in Bari.

Problem 13 (H. Zwart)**.** (i) Does every bounded C_0-semigroup on a Hilbert space have a bounded rational calculus?

(ii) When is a C_0-semigroups on a Hilbert space similar to a contraction semigroup?

Comments. The answer to the first question is no. A bounded semigroup on a Hilbert space has a bounded rational calculus if and only if it has a bounded H_∞-calculus, see thesis of Markus Haase.

Comments.(Charles Batty)

In Problem (i) and the Comment above, the "boundedness" of the rational calculus is being interpreted as meaning boundedness with respect to the H^∞-norm. There are one or two issues as to what is the domain of those functions and whether or not the generator is injective, but basically the answer is correct. An alternative to Haase's thesis is his functional calculus book, Sections 5.3.4 and 5.3.5.

Instead of considering the H^∞-norm, one may consider Banach algebras in different norms that are embedded in $H^\infty(\mathbb{C}_+)$, where $\mathbb{C}_+$ is the open right half-plane. One example is the (Hille)-Phillips calculus, where the norm comes from measures on $[0,\infty)$. Alexander Gomilko, Yuri Tomilov and I have recently shown that if $-A$ is the generator of a bounded C_0-semigroup on a Hilbert space, then there is a bounded $\mathcal{B}$-calculus for A. Here $\mathcal{B}$ is a Banach algebra of "analytic Besov" functions on the right half-plane. This algebra is considerably bigger than the Phillips algebra, and the $\mathcal{B}$-norm is considerably smaller than the Phillips norm but it is bigger than the H^∞-norm.

On Banach spaces, the $\mathcal{B}$-calculus exists if and only if A satisfies the condition introduced by Gomilko, and independently by Shi and Feng, in 1999 and 2000. In particular, it exists if A is sectorial of angle less than $\pi/2$, so $-A$ generates a bounded holomorphic C_0-semigroup. For those operators, there are two further calculi, $\mathcal{D}$-calculus and $\mathcal{H}$-calculus, which extend the calculus to larger classes of functions than $\mathcal{B}$, with smaller norms.

Relevance. This problem was/is related to the inverse generator problem and questions concerning the powers of the co-generator of bounded semigroups. For both problems, the answers have been negative in general, and some positive partial answers have been obtained, but the answer for bounded semigroups on Hilbert spaces is unknown. The extended calculi provide systematic ways to approach such problems, instead of using ad hoc methods each time. For the functions $((z-1)(z+1)^{-1})^n$ the Hille-Phillips norm grows like $n^{1/2}$, the $\mathcal{B}$-norm grows like $\log n$, and the $\mathcal{D}$-norms are uniformly bounded.

Source: R. Nagel's list of problems collected in 2003 at the workshop in Bari.

Problem 14 (R. Nagel)**.** $A+B$ problem. Let A and B be generators of C_0-semigroups on a Banach space X.

(i) Define the sum $C = A + B$ such that C becomes a (maximal) closed operator on X and $Cx = \tilde{A}x + \tilde{B}x$ for all $x \in X$ and some extrapolated operators A and B.

(ii) Find assumptions on A and B such that C remains a generator on X, thereby unifying known perturbation results.

(iii) A test case is the following. Let $\mathcal{X} = C_0(\mathbb{R}, X)$ or $\mathcal{X} = L^p(\mathbb{R}, X)$, and take $Af = f'$ and $Bf(s) = C(s)f(s)$ for appropriate $f \in \mathcal{X}$ and closed operator $C(s)$ on X.

Source: R. Nagel's list of problems collected in 2003 at the workshop in Bari.

Problem 15 (R. Nagel)**. Non-autonomous abstract Cauchy problems.** For unbounded linear operators $A(t)$ on a Banach space X and for a starting time t_0, characterize the well-posedness of the non-autonomous Cauchy problem

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) \quad \text{for } t \geq t_0 \\ x(t_0) &= x_0 \end{aligned}$$

by a Hille-Yosida type condition for an operator G generating an evolution semigroup on $\mathcal{X} = C_0(\mathbb{R}, X)$ or $\mathcal{X} = L^p(\mathbb{R}, X)$.

Source: R. Nagel's list of problems collected in 2003 at the workshop in Bari.

Problem 16 (Communicated by Rainer Nagel)**.** Characterizing Koopman groups on Hilbert spaces:

Use the Perron Frobenius spectral theory of positive C_0-groups to characterize unitary C_0-groups on a Hilbert space H that are unitarily isomorphic to a Koopman group on an L^2-space.

References:

• Alexandre I. Danilenko, and Mariusz Lemańczyk: Spectral multiplicities for ergodic flows. Discrete Contin. Dyn. Syst. 33, No. 9, 4271-4289 (2013).

• D. V. Anosov: pectral multiplicity in ergodic theory, Proc. Steklov Inst. Math. 290, Suppl. 1, S1-S44 (2015); translation from Sovrem. Probl. Mat. 2003, No. 3, 3вЂ“84 (2003).

Source: Manuscript of R. Nagel provided during OPSO 2021 conference.

Problem 17 (D. Seifert)**.** Let $(T(t))_{t\geq 0}$ be a bounded C_0-semigroup on a (complex) Banach space X, and let A denote its generator. Suppose that $\sigma(A) \cap i\mathbb{R} = \emptyset$ and that there exists $\alpha > 0$ such that $\|(is - A)^{-1}\| = O(|s|^\alpha)$ as $|s| \to \infty$. Find $\beta \in [0, 1]$ depending on the geometric properties of the space X (for instance its Fourier type, its type or its cotype) such that

$$\|T(t)A^{-1}\| = O\left(\frac{\log(t)^{\beta/\alpha}}{t^{1/\alpha}}\right), \qquad t \to \infty.$$

Comments. It was shown by Batty and Duyckaerts (2008) that one may always take $\beta = 1$; see also Chill and Seifert (2016). Batty and Duyckaerts moreover showed that negative values of β are in general not permissible. It is reasonable, therefore, to restrict attention to values of β lying in $[0, 1]$. Borichev and Tomilov (2010) showed that one may take $\beta = 0$, yielding the best possible upper bound, if X is a Hilbert space. This result may be viewed as a special case of more recent results appearing in papers by Batty, Chill and Tomilov (2016) and Rozendaal, Seifert and Stahn (2019). Borichev and Tomilov also showed, by considering the left-shift semigroup on a certain subspace of $\mathrm{BUC}(\mathbb{R}_+)$ with an appropriate norm, that one may have

$$\limsup_{t\to\infty} \frac{t^{1/\alpha}}{\log(t)^{1/\alpha}} \|T(t)A^{-1}\| > 0.$$

Hence one cannot in general hope to do better than $\beta = 1$ unless one imposes additional assumptions on X; see also Debruyne and Seifert (2019).

Relevance. This problem is important from a theoretical point of view, as its solution would elegantly complement our current understanding of polynomial stability of C_0-semigroups. Furthermore, there are likely to be interesting applications to concrete evolution equations on L^p-spaces and other (non-Hilbertian) Banach spaces with non-trivial geometric properties.

References:

- C.J.K. Batty, R. Chill, and Y. Tomilov. Fine scales of decay of operator semigroups. *J. Eur. Math. Soc. (JEMS)*, 18(4):853–929, 2016.
- C.J.K. Batty and T. Duyckaerts. Non-uniform stability for bounded semi-groups on Banach spaces. *J. Evol. Equ.*, 8(4):765–780, 2008.
- A. Borichev and Y. Tomilov. Optimal polynomial decay of functions and operator semigroups. *Math. Ann.*, 347(2):455–478, 2010.

- R. Chill and D. Seifert. Quantified versions of Ingham's theorem. *Bull. Lond. Math. Soc.*, 48(3):519–532, 2016.
- G. Debruyne and D. Seifert. An abstract approach to optimal decay of functions and operator semigroups. *Israel J. Math.*, 233(1):439–451, 2019.
- G. Debruyne and D. Seifert. Optimality of the quantified Ingham-Karamata theorem for operator semigroups with general resolvent growth. *Arch. Math. (Basel)*, 113(6):617–627, 2019.
- J. Rozendaal, D. Seifert, and R. Stahn. Optimal rates of decay for operator semigroups on Hilbert spaces. *Adv. Math.*, 346:359–388, 2019.

Problem 18 (A.E. Teretenkov)**.** Let $\mathcal{B}$ be a Banach space. Let $\mathcal{P}$ be a projection on the *finite-dimensional* Banach subspace of $\mathcal{B}$. Let $\mathcal{L}^0 + \lambda\mathcal{L}$ be a generator of C_0-semigroup $\mathcal{U}_t^\lambda$ on $\mathcal{B}$ for all $\lambda \in [0, \lambda_{\mathrm{sup}})$. Let $\mathcal{U}_t^0$ leave both $\mathcal{PB}$ and its complement $(I - \mathcal{P})\mathcal{B}$ invariant, let $\mathcal{PLP} = 0$. Denote $\mathcal{L}_t \equiv (\mathcal{U}_t^0)^{-1}\mathcal{L}\mathcal{U}_t^0$. Let the integrals

$$\int_{-\infty}^{t} dt_1 \ldots \int_{-\infty}^{t_{k-1}} dt_k \mathcal{P}\mathcal{L}_{t_1} \ldots \mathcal{L}_{t_k}\mathcal{P}, \qquad k = 1, \ldots, n+1$$

finite for all $t \geqslant 0$. If is it possible to find such a λ-dependent operator $r^{n,\lambda}$ on $\mathcal{PB}$, which is polynomial in λ, and λ-dependent semigroup $u_t^{n,\lambda}$ on $\mathcal{PB}$, whose generator is polynomial in λ, such that

$$\mathcal{P}(\mathcal{U}^0_{\frac{t}{\lambda^2}})^{-1}\mathcal{U}^\lambda_{\frac{t}{\lambda^2}}\mathcal{P} = u_t^{n,\lambda} r^{n,\lambda} + O(\lambda^{2n+2}), \qquad \lambda \to +0$$

for all $t > 0$? Let us emphasize that we do not assume here asymptotic behavior to be uniform in t, it is just assumed for each fixed $t > 0$. If there are counterexamples, what further restrictions should be assumed to obtain such asymptotic estimate?

Relevance. This problem is important for derivation of perturbative corrections to Markovian quantum master equations. It seems to be a possible direction of generalization of the classical results by E.B. Davies to higher orders of perturbation theory in λ and seems to be hold in a simple example discussed in arXiv:2008.02820. It also seems to be necessary for strict perturbative derivation of master equations recently obtained by A.S. Trushechkin.

References:

- E.B. Davies. *Commun. Math. Phys.* 39, no. 2 (1974): 91–110.
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