Open problems in operator semigroup theory (OPSO 2021 and 2022)

Editors: Jochen Glück, Rainer Nagel, Ivan Remizov

February 14, 2022

Please submit your open problem(s) to Jochen Glück:
jochen.glueck@alumni.uni-ulm.de

DRAFT

Generation problems

Problem 1 (Inverse generator problem)**.** Let H be a Hilbert space over $\mathbb{C}$ and let $A : D(A) \subseteq H \to H$ be the infinitisimal generator of a bounded semigroup on H. Assume further that A^{-1} exists as a densely defined, closed operator.

Is A^{-1} the infinitesimal generator of a bounded semigroup?

Comments. This problem was originally posed (for Banach spaces) by R. de Laubenfels in [1]. However, it is not hard to show that for general Banach spaces the answer to the problem is negative, see e.g. [4]. The counter example can be chosen such that the strongly continuous semigroup generated by A is a contraction semigroup, whereas in Hilbert spaces the answer to the problem is positive for a generator of a contraction semigroup (almost trivial to show).

There is a strong relation between the inverse generator problem and the question whether the Cayley transform of A is power bounded, i.e. if $\sup_n \|A_d^n\| < \infty$, where $A_d = (I + A)(I - A)^{-1}$, see [3]. The latter question is related to numerical analysis, since this Cayley transform pops up when applying the Crank-Nicolson scheme to the differential equation $\dot{x}(t) = Ax(t)$.

When the answer to the inverse generator problem is positive, then the (strong) stability of the semigroup generated by A is equivalent to the (strong) stability of the semigroup generated by A^{-1}. Furthermore, it is equivalent to the strong stability of A_d, i.e., $\lim_{n\to\infty} A_d^n x = 0$ for all $x \in H$, [3].

For finite-dimensional Hilbert spaces H, it is clear that the problem has a positive answer. For these spaces the question is; if there exists a constant c independent of the dimension of H such that $\sup_{t\geq 0} \|e^{A^{-1}t}\| \leq c \sup_{t\geq 0} \|e^{At}\|$.

In 2017, a nice survey on the problem appeared [2]. In that paper, the interested reader can find more results and references on the inverse generator problem.

Communicated by Hans Zwart.

References:

[1] R. deLaubenfels. Inverses of generators. *Proceedings of the American Mathematical Society*, Vol. 104, 2, pp. 443–448, 1988.

[2] A. Gomilko. Inverses of semigroup generators: a survey and remarks. *Banach Center Publications*, Vol. 102, pp. 107–142, 2017.

[3] B.-Z. Guo and H. Zwart. On the relation between stability of continuous- and discrete-time evolution equations via the Cayley transform. *Integral Equations and Operator Theory*, Vol. 54, 3, pp. 349 – 383, 2006.

[4] H. Zwart Is A^{-1} an infinitesimal generator? *Perspectives in Operator Theory. Banach Center Publications*, Vol. 75, pp. 303–313, 2007.

Problem 2 (An analytic semigroup generation problem)**.** This problem, arising in the theory of Gaussian open quantum systems, consists in proving that a dissipative operator G which is quadratic in creation and annihilation operators (or, after a unitary transformation, a differential operator quadratic in partial derivatives ∂_j and multiplication by coordinates x_k) generates an analytic semigroup. The solution has applications in the proof of strong positivity, irreducibility and regularity properties of Gaussian open quantum systems.

Let a_j, a_k^* be the annihilation and creation operators on $\ell^2(\mathbb{N}^d;\mathbb{C})$ defined by closure from their action on the canonical orthonormal basis $(e(n_1,\ldots,n_d))_{n\in\mathbb{N}^d}$

$$\begin{aligned} a_j\, e(n_1,\ldots,n_d) &= \sqrt{n_j}\; e(n_1,\ldots,n_{j-1},n_j-1,\ldots,n_d), \\ a_k^*\, e(n_1,\ldots,n_d) &= \sqrt{n_k+1}\; e(n_1,\ldots,n_{k-1},n_k+1,\ldots,n_d). \end{aligned}$$

Let H, L_ℓ be the closures of operators defined on the canonical basis by

$$\begin{aligned} H &= \sum_{j,k=1}^{d} \left(\Omega_{jk} a_j^* a_k + \frac{\kappa_{jk}}{2} a_j^* a_k^* + \frac{\overline{\kappa}_{jk}}{2} a_j a_k \right) + \sum_{j=1}^{d} \left(\frac{\zeta_j}{2} a_j^* + \frac{\bar{\zeta}_j}{2} a_j \right), \\ L_\ell &= \sum_{k=1}^{d} \left(\overline{v}_{\ell k} a_k + u_{\ell k} a_k^* \right) \qquad \ell = 1, \dots, 2d \end{aligned}$$

where $\Omega := (\Omega_{jk})_{1 \le j,k \le d} = \Omega^*$ and $\kappa := (\kappa_{jk})_{1 \le j,k \le d} = \kappa^{\mathrm{T}} \in M_d(\mathbb{C})$, are $d \times d$ complex matrices with Ω Hermitian and κ symmetric, $V = (v_{\ell k})_{1 \le \ell \le 2d, 1 \le k \le d}$, $U = (u_{\ell k})_{1 \le \ell \le 2d, 1 \le k \le d}$ are $2d \times d$ complex matrices $\zeta = (\zeta_j)_{1 \le j \le d} \in \mathbb{C}^d$.

It is not difficult to show (see e.g. [1] Proposition 4.9) that the closure of the operator G defined on the canonical basis by

$$G = -\mathrm{i}H - \frac{1}{2} \sum_{\ell=1}^{2d} L_\ell^* L_\ell$$

generates a C_0 contraction semigroup $P = (P_t)_{t \ge 0}$ on $\ell^2(\mathbb{N}^d; \mathbb{C})$.

Suppose that the non-degeneracy condition (block-matrix form)

$$\mathbb{K} = \begin{bmatrix} V^{\mathrm{T}} \\ U^* \end{bmatrix} \begin{bmatrix} \overline{V} & U \end{bmatrix} = \begin{bmatrix} V^{\mathrm{T}} \overline{V} & V^{\mathrm{T}} U \\ U^* \overline{V} & U^* U \end{bmatrix} > 0$$

holds, then sufficient conditions for P to be an analytic semigroup are also available.

The problem it to find a *classification* of the set of parameters $\Omega, \kappa, U, V, \zeta$ for which P is analytic.

Alternatively, by the unitary correspondence of the above basis with multidimensional Hermite polynomials (multiplied by $\exp(-|x|^2/2)$ normalized), one can formulate the problem with differential operators

$$a_j = \frac{1}{\sqrt{2}} \left(x_j + \frac{\partial}{\partial x_j} \right), \qquad a_k^* = \frac{1}{\sqrt{2}} \left(x_k - \frac{\partial}{\partial x_k} \right)$$

In this case strict positivity of $\mathbb{K}$ implies

$$\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \mathbb{K} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} > 0$$

from which one finds the strong ellipticity condition for the self-adjoint part $G_0 = -(1/2) \sum_{\ell=1}^{2d} L_\ell^* L_\ell$ of G

$$\mathrm{Re} \sum_{j,k=1}^{d} \left(U^* U + V^{\mathrm{T}} \overline{V} - V^{\mathrm{T}} U - U^* \overline{V} \right)_{jk} \overline{z}_j z_k > \epsilon \|z\|^2$$

for $z = (z_j)_{1\leq j\leq d} \in \mathbb{C}^d$.

Thinking of spectra of G_0 and H one wonders if the semigroup P generated by P is analytic when $\mathbb{K} > 0$ and H is bounded from below or from above.

Communicated by Franco Fagnola.

References:

[1] F. Fagnola, Quantum Markov Semigroups and Quantum Markov Flows. *Proyecciones* **18** n.3 1–144 (1999).

Problem 3 (Lumer–Phillips for transition semigroups)**.** Let Ω be a Polish space. We call a semigroup $T = (T(t))_{t\geq 0}$ of contractions on $C_b(\Omega)$ a *transition semigroup* if

(i) for every $f \in C_b(E)$ we have $T(t)f \to T(s)f$ locally uniformly whenever $t \to s$ and

(ii) for every $t \geq 0$ and every uniformly bounded sequence $(f_n)_{n\in\mathbb{N}} \subset C_b(\Omega)$ that converges locally uniformly to f, we have that $T(t)f_n$ converges locally uniformly to f.

Is there a chracterization of the generators of such a semigroup akin to the Lumer–Phillips theorem?

Comments. The name 'transition semigroup' is inspired by applications in probability theory, where semigroups with the above properties frequently appear as transition semigroups of Markov processes (typically, these semigroups are additionally positive). We should point out that semigroups of this kind (at least at first glance) do not fit into the theory of semigroups on locally convex spaces (see, e.g. [5]) as this theory requires equicontinuity of the operators involved. But even the simplest example of the heat semigroup on $C_b(\mathbb{R}^d)$ shows that one cannot expect equicontinuity with respect to the topology τ_{co} of uniform convergence on compact subsets of $\mathbb{R}^d$.

Consequently, in the literature several approaches were developed where this equicontinuity condition was weakened such as the 'theory of of weakly continuous semigroups' by Cerrai [1] (where instead of $C_b(\Omega)$ one works on the space $BUC(\Omega)$) or the 'theory of bi-continuous semigroups' by Kühnemund [3]; both approaches allow for a Hille–Yosida type generation result. On the other hand, [4, Theorem 4.4] shows that conditions (i) and (ii) the above definition already entail equicontinuity: Not with respect to τ_{co} but with respect to the so-called *strict topology* β_0 (which agrees with τ_{co} on $\|\cdot\|_\infty$-bounded subsets of $C_b(\Omega)$). This allows us to use the results from [5] after all to characterize the generators of transition semigroups.

Thus, characterizations of generators of transition semigroups are available in the literature. However, to the best of my knowledge, none of these Hille–Yosida type theorems was ever used to establish that a certain operator generates a transition semigroup (even though many examples of such semigroups and also their generators are known). This is not as surprising as it might seem, for even in the setting of strongly continuous semigroups the Hille–Yosida theorem is difficult to apply. This is due to the fact that this result requires us (in the case of bounded semigroups) to prove uniform boundedness of the family $\{\lambda^n R(\lambda, A)^n : \lambda > 0, n \in \mathbb{N}\}$, which is difficult in concrete examples. In the case of non-strongly continuous semigroups one would have to establish equicontinuity of this family of operators – an even harder task.

The 'weapon of choice' to prove that a given operator generates a strongly continuous semigroup is rather the Lumer–Phillips theorem (see [2, Theorem II.3.15]) which, however, only characterizes the generators of *contraction* semigroups. The main advantage of the Lumer–Phillips theorem is that one does not have to consider powers of the resolvent. Indeed, given dissipativity, we only need to check the so-called *range condition*, i.e. we need to prove that $\lambda - A$ has dense range for some $\lambda > 0$.

It would be very interesting to have a Lumer–Phillips type result for transition semigroups. Part of the problem is to find out how such a result should look like. Here, usability is (in my opinion) more important than generality. If we can obtain a sufficient condition for generation (which might make use of additional properties that one has at hand in many possible applications such as positivity of the resolvent or the strong Feller property for the resolvent or ...) this would be already be very nice.

Communicated by Markus Kunze.

References:

[1] S. Cerrai, *A Hille-Yosida theorem for weakly continuous semigroups*, Semigroup Forum, 49 (1994), pp. 349 – 367.

[2] K. Engel and R. Nagel, *One-parameter semigroups for linear evolution equations*, vol. 194 of Graduate Texts in Mathematics, Springer-Verlag, New York, 2000.

[3] F. Kühnemund, *A Hille-Yosida theorem for bi-continuous semigroups*, Semigroups Forum, 67 (2003), pp. 205 – 225.

[4] M. Kunze, *Continuity and equicontinuity of semigroups on norming dualpairs*, Semigroup Forum, 79 (2009), pp. 540 – 560.

[5] K. Yosida, *Functional Analysis*, vol. 123 of Grundlehren der mathematischen Wissenschaften, Springer, 1980.

Long-term behaviour of semigroups

The answer to this problem is "No".

(Reference with solution was found by J.Glück)

Problem 4. Let T be a C_0-semigroup over a Banach space X. Does the weak attractivity of T (i.e. $\inf_{t\geq 0}\|T(t)x\| = 0$, for all $x \in X$) imply the strong stability (= attractivity) of T, that is $\|T(t)x\| \to 0$ as $t \to \infty$, for all $x \in X$?

Comments. Weak attractivity was introduced (for systems without disturbances) in [1] and which is one of the pillars for the treatment of dynamical systems theory over locally compact metric spaces in [2]. Its variations for systems with controls are widely used also in the control theory. In the language of [5], weak attractivity is precisely the limit property with zero gain. The concept of weak attractivity is intimately related to the classical notion of recurrent sets, see, e.g., [2, Definition 1.1 in Chapter 3].

The uniform variants of this property have played an important role in the stability analysis of nonlinear control systems [3] and in the study of non-coercive Lyapunov functions [4].

Communicated by Andrii Mironchenko.

References:

[1] Nam P Bhatia. Weak attractors in dynamical systems. *Bol. Sot. Mat. Mex.*, 11:56–64, 1966.

[2] Nam Parshad Bhatia and Giorgio P Szegö. *Stability Theory of Dynamical Systems.* Springer Science & Business Media, 2002.

[3] Andrii Mironchenko and Fabian Wirth. Characterizations of input-to-state stability for infinite-dimensional systems. *IEEE Transactions on Automatic Control*, 63(6):1602–1617, 2018.

[4] Andrii Mironchenko and Fabian Wirth. Non-coercive Lyapunov functions for infinite-dimensional systems. *Journal of Differential Equations*, 105:7038–7072, 2019.

[5] E. D. Sontag and Yuan Wang. New characterizations of input-to-state stability. *IEEE Transactions on Automatic Control*, 41(9):1283–1294, 1996.

Problem 5 (Tauberian Theorem for Semigroups of Kernel Operators)**.** Let $E := L^p(\Omega, \mu)$ for $1 \le p \le \infty$ and a σ-finite measure space (Ω, μ) and let $(T_t)_{t\in[0,\infty)} \subseteq \mathcal{L}(E)$ be a strongly continuous semigroup on E such that T_t is a positive *kernel operators* for each $t > 0$, meaning that there exists a measurable function $k_t \colon \Omega \times \Omega \to \mathbb{R}_+$ such that

$$(T_t f)(y) = \int_\Omega k_t(y, x) f(x) \mathrm{d}\mu(x)$$

for almost every $y \in \Omega$.

Assume that $(T_t)_{t\in[0,\infty)}$ is *mean ergodic*, meaning that the limit

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T T_t f \mathrm{d}t$$

exists in E for all $f \in E$. Does it follow that (T_t) is *strongly convergent*, i.e. $\lim_{t\to\infty} T_t f$ exists for all $f \in E$?

Comments. It is well known that positive semigroups of kernel operators are strongly convergent provided that the semigroups possesses a fixed point $f \in E$ satisfying $f(x) > 0$ for almost every $x \in \Omega$. This has been proven in a very general setting in [1, Theorem 3.5]. The existence of such an "quasi-interior" fixed point is crucial and cannot be omitted: For instance, the Gaussian semigroup on $L^1(\mathbb{R})$ is not strongly convergent and fulfils all assumptions of [1, Theorem 3.5] except that it does not possess a quasi-interior fixed point. On the other hand, the Gaussian semigroup is not mean ergodic on $L^1(\mathbb{R})$. In fact, using [1, Theorem 3.5] it is not difficult to show the following Tauberian theorem:

Let $(T_t)_{t\in[0,\infty)}$ be a positive, bounded and mean ergodic C_0-semigroup on $L^1(\Omega)$ for any measure space Ω. If T_{t_0} is kernel operator for some $t_0 > 0$, then $(T_t)_{t\in[0,\infty)}$ is strongly convergent.

The theorem above and similar results for semigroups on spaces of measures can be found in [2, Theorems 2.1, 4.1, 5.4]. This gives rise to the conjecture that the existence of a quasi-interior fixed point in [1, Theorem 3.5] can always, i.e. for every $1 \le p \le \infty$ or – more generally – for every Banach lattice E, be omitted in case of a mean ergodic C_0-semigroup $(T_t)_{t\in(0,\infty)}$. In support of this conjecture one may note that the Gaussian semigroup on $L^p(\mathbb{R})$ for $p \in (1, \infty)$ is mean ergodic and converges strongly to 0.

Communicated by Moritz Gerlach.

References:

[1] M. Gerlach and J. Glück. Convergence of Positive Operator Semigroups. *Trans. Amer. Math. Soc.*, 372(9):6603–6627, 2019.

[2] M. Gerlach, J. Glück, and M. Kunze. Stability of transition semigroups and applications to parabolic equations. preprint, 2020

Positivity

Problem 6 (Infinite speed of propagation)**.** Let $E \coloneqq L^p(\Omega, \mu)$ for $1 \leq p < \infty$ and a σ-finite measure space (Ω, μ) and let $\coloneqq (T_t)_{t \in [0,\infty)} \subseteq \mathcal{L}(E)$ be a strongly continuous semigroup of positive kernel operators on E, meaning that for each $t > 0$ there exists a measurable function $k_t \colon \Omega \times \Omega \to \mathbb{R}_+$ such that

$$(T_t f)(y) = \int_\Omega k_t(y,x) f(x) \mathrm{d}\mu(x)$$

for almost every $y \in \Omega$.

Assume that (T_t) is *irreducible*, i.e. for all non-zero and positive $f, g \in E_+$ there exists $t \geq 0$ such that $T_t f \wedge g \neq 0$ (where $\wedge$ denotes the infimum in the Banach lattice E). Does it follow that (T_t) is *expanding*, i.e. $T_t f > 0$ almost everywhere for all $t > 0$ and all non-zero positive $f \in E_+$?

Comments. The notion of irreducibility is of great importance – for instance in the spectral theory of positive semigroups – and obviously, every expanding semigroup is irreducible. On the other hand, the rotation semigroup on the unit circle is an easy example of an irreducible semigroup that fails to be expanding.

However, under certain circumstances the two properties, irreducible and expanding, are equivalent. For instance, every holomorphic positive semigroup is known to be expanding if it is irreducible [2, Theorem C-III 3.2]. A surprisingly little known fact is that the same holds for all positive semigroups on atomic Banach lattices like ℓ^p for $1 \leq p < \infty$. This is due to a fact which is referred to as "Lévy's Theorem" in literature: for every stochastic transition matrix $(p_{i,j})$ one either has $p_{i,j}(t) = 0$ or $p_{i,j}(t) > 0$ for all $t > 0$. The proof given by K. L. Chung in the appendix of [3] for this statement can easily be transferred to the setting of semigroups on atomic spaces. Since operators on atomic spaces often serve as a prototype for general kernel operators, it is natural to ask whether the same implication is true for semigroups of kernel operators.

The notation *expanding* is not used in a uniform matter in the literature; the property is, for instance, also referred to as "strongly positive" or "positivity improving". We would like to advertise the more systematic naming convention of [1, Section 9.1].

Communicated by Moritz Gerlach.

References:

[1] Y. A. Abramovich and C. D. Aliprantis. *An invitation to operator theory*, volume 50 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2002.

[2] W. Arendt, A. Grabosch, G. Greiner, U. Groh, H. P. Lotz, U. Moustakas, R. Nagel, F. Neubrander, and U. Schlotterbeck. *One-parameter semigroups of positive operators*, volume 1184 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1986.

[3] K. L. Chung. On the boundary theory for Markov chains. *Acta Mathematica*, 110:19 – 77, 1963.

Problem 7 (Positive commutator problem)**.** Let E be a Banach lattice and let $C\colon E \to E$ be a positive quasinilpotent compact operator. Do there exist positive operators $A, B\colon E \to E$ such that $C = AB - BA$ with one of A and B compact?

Comments. Given an associative algebra $\mathcal{A}$, the natural question is to determine all commutators of $\mathcal{A}$. Shoda [11] proved that a matrix $C \in \mathbb{M}_n(F)$ is a commutator if and only if the trace of C is zero. Wintner [12] proved that the identity in a unital Banach algebra is not a commutator. By passing to the Calkin algebra, Wintner's result immediately implies that a bounded operator on a Banach space which is of the form $\lambda I + K$ for some nonzero scalar λ and a compact operator K is not a commutator. Henceforth, researchers tried to characterize which operators on a given Banach space are commutators. The complete characterization of commutators in the Banach algebra $\mathcal{B}(\mathcal{H})$ of all bounded operators on an infinite-dimensional Hilbert space $\mathcal{H}$ is due to Brown and Pearcy [4]. They proved that a bounded operator C on $\mathcal{H}$ is a commutator if and only if it is not of the form $\lambda I + K$ for some nonzero scalar λ and some operator K from the unique maximal ideal in $\mathcal{B}(\mathcal{H})$. Apostol ([1, 2]) proved that a bounded operator on either ℓ^p $(1 < p < \infty)$ or c_0 is a commutator if and only if it is not of the form $\lambda I + K$ where $\lambda \neq 0$ and K is compact. In the case of the Banach space ℓ^1 the same characterization was obtained by Dosev in [5]. In the case of the Banach space ℓ^∞ Dosev and Johnson [6] proved that a bounded operator is a commutator if and only if it is not of the form $\lambda I + K$ where $\lambda \neq 0$ and K is strictly singular.

The study of positive commutators of positive operators on a given Banach lattice was initiated in [3]. The assumption on positivity of A, B and $C := AB - BA$ may lead to some restrictions on the commutator. Namely, the authors proved that the positive commutator of positive compact operators is quasinilpotent. They also posed a question whether the same is true under the assumption that one of operators is compact. This question was affirmatively and independently solved by R. Drnovšek [7] and N. Gao [9]. Inspired by a result of Schneeberger [10] asserting that a compact operator acting on a separable L^p space $(1 \leq p < \infty)$ is a commutator, in [8] authors prove that a positive compact operator acting on a separable L^p space $(1 \leq p < \infty)$ is a commutator of positive operators. In [8] authors provide a technical condition under which the answer to the proposed problem is affirmative for positive operators on ℓ^p space $(1 \leq p \leq \infty)$ satisfying this technical condition.

Communicated by Marko Kandić.

References:

[1] C. Apostol. Commutators on ℓ_p spaces. *Rev. Roumaine Math. Appl.* 17, 1513–1534, 1972.

[2] C. Apostol. Commutators on c_0-spaces and on ℓ_∞-spaces. *Rev. Roumaine Math. Appl.* 18, 1025–1032, 1973.

[3] J. Bračič, R. Drnovšek, Y. B. Farforovskaya, E. L. Rabkin and J. Zemánek. On positive commutators. *Positivity*, 14(3), 431–439, 2010.

[4] A. Brown and C. Pearcy. Structure of commutators of operators. *Ann. of Math.* 82, 112–127, 1965.

[5] D. T. Dosev. Commutators on ℓ_1. *J. Funct. Anal.* 256, 3490–3509, 2009.

[6] D. Dosev and W. B. Johnson. Commutators on l^∞., *Bull. Lond. Math. Soc.* 42(1), 155–169, 2010.

[7] R. Drnovšek. Once more on positive commutators. *Studia Math.*, 211(3), 241–245, 2012.

[8] R. Drnovšek and M. Kandić. Positive operators as commutators of positive operators. *Studia Math.* 245 (2), 185–200, 2019.

[9] N. Gao. On commuting and semi-commuting positive operators. *Proc. Amer. Math. Soc*, 142(8), 2733–2745, 2014.

[10] C. Schneeberger. Commutators on a separable L^p-space. *Proc. Amer. Math. Soc.* 28, 464–472, 1971.

[11] K. Shoda. Einige Satze Uber Matrizen. *Japanese J. Math.* 13, 361–365, 1937.

[12] A. Wintner, The unboudnedness of quantum-mechanical matrices. *Phys. Rev.* 71, 738–739, 1947.

L^∞-bounds

Problem 8 (L^∞-boundedness problem)**.** Let (Ω,μ) be a σ-finite measure space. Let V be a dense subspace of $L^2(\Omega)$ (complex-valued functions) and suppose that $\mathfrak{a} : V \times V \to \mathbb{C}$ is a closed sectorial sesquilinar form, i.e. linear in the first and antilinear in the second argument. Here, sectoriality means that, for some $\theta \in [0, \frac{\pi}{2})$,

$$\forall u \in V : \mathfrak{a}(u,u) \in \Sigma_\theta := \{z \in \mathbb{C} \setminus \{0\} : |\arg z| \leq \theta\} \cup \{0\},$$

and closedness of $\mathfrak{a}$ means that V is complete for the norm

$$\|u\|_{\mathfrak{a}} := \Big(\operatorname{Re}\mathfrak{a}(u,u) + \|u\|^2_{L^2(\Omega)}\Big)^{1/2}.$$

Let the linear operator A in $L^2(\Omega)$ be associated with a $\mathfrak{a}$ in the sense that, for $u, h \in L^2(\Omega)$,

$$u \in D(A) \text{ and } Au = h \quad \Longleftrightarrow \quad u \in V \text{ and } \forall v \in V : \mathfrak{a}(u,v) = \langle u, h\rangle,$$

where $\langle u, h\rangle := \int_\Omega f\overline{h}\,d\mu$ denotes the usual scalar product in $L^2(\Omega)$. Then $-A$ is negative generator of a bounded analytic semigroup $(T(\cdot))$ in $L^2(\Omega)$, which is contractive on the sector $\Sigma_{\frac{\pi}{2}-\theta}$.

The semigroup $(T(t))_{t\geq 0}$ is called L^∞-*bounded*, if there exists $M > 0$ such that

$$\|T(t)f\|_\infty \leq M\|f\|_\infty, \qquad \text{for all } f \in L^2(\Omega) \cap L^\infty(\Omega). \tag{1}$$

Can L^∞-boundedness of $(T(t))_{t\geq 0}$ be characterized in terms of the sesquilinear form $\mathfrak{a}$?

Comments. The problem came up in discussions with Sönke Blunck (at the end of the 1990s). The case $M = 1$ in (1), i.e., L^∞-*contractivity* of $(T(t))_{t\geq 0}$, is characterized in terms of the sesquilinear form $\mathfrak{a}$ by the well-know Beurling-Deny criterion (see, e.g., [3, Theorem 2.7] or [5, Section 2.2]), namely by the condition

$$\forall u \in V : \operatorname{sgn} u(|u|-1)_+ \in V \text{ and } \operatorname{Re}\mathfrak{a}(u, \operatorname{sgn} u(|u|-1)_+) \geq 0. \tag{2}$$

Here $v_+ := -((-v) \wedge 0)$, where $\wedge$ denotes the pointwise minimum, and $\operatorname{sgn} u := \frac{u}{|u|}1_{\{u\neq 0\}}$ denotes the sign of the function u.

A characterization of L^∞-*boundedness* of $(T(t))_{t\geq 0}$ could certainly be very useful (for the restrictions that L^∞-contractivity imposes on the coefficients of second order elliptic operators on domains we refer to [5, Section 4.3]).

It is clear that $(T(t))_{t\geq 0}$ is L^∞-bounded if there exists a function $g \in L^\infty(\Omega)$ with $g > 0$ μ-a.e. and $1/g \in L^\infty(\Omega)$ satisfying

$$\|gT(t)f\|_\infty \leq \|gf\|_\infty \qquad \text{for all } f \in L^\infty(\Omega) \cap L^2(\Omega), \tag{3}$$

since the norm $f \mapsto \|gf\|_\infty$ is equivalent to $\|\cdot\|_\infty$. A modification of condition (2) characterizes (3), namely

$$\forall u \in V : \operatorname{sgn} u(|u| - g)_+ \in V \text{ and } \operatorname{Re} \mathfrak{a}(u, \operatorname{sgn} u(|u| - g)_+) \geq 0. \tag{4}$$

The proof can be done similar to the proof of equivalence of (2) and L^∞-contractivity. One can also resort to invariance results in $L^2(\Omega)$ and consider the closed convex set

$$K_g := \{f \in L^2(\Omega) : |f| \leq g \ \mu\text{-a.e.}\}.$$

Then it is not hard to check that the projection P_g of $L^2(\Omega)$ onto K is given by $P_g f := \operatorname{sgn} f(|f| \wedge g)$ ($P_g(f)$ is the best approximation of f in K). Then observe $u - P_g(u) = \operatorname{sgn} u(|u| - g)_+$ and apply [4, Theorem 2.1] (we also refer to [5, Section 2.1]).

In view of the comments to Problem 8 the following seems natural to ask.

Problem 9 (L^∞-contraction for a weight problem)**.** In the setting of Problem 8 assume that the semigroup $(T(t))_{t\geq 0}$ is L^∞-bounded, i.e., satisfies (1) for some $M \geq 1$. Does there exist a function $g \in L^\infty(\Omega)$ satisfying $g > 0$ μ-a.e., $1/g \in L^\infty(\Omega)$, and (3)?

Comments. In general this might be more than one can hope for.

Hence we are lead to the following.

Problem 10 (Characterization of L^∞-contraction for a weight)**.** In the setting of Problem 8, can one characterize those L^∞-bounded semigroups $(T(t))_{t\geq 0}$, for which a function $g \in L^\infty(\Omega)$ satisfying $g > 0$ μ-a.e., $1/g \in L^\infty(\Omega)$, and (3) exists?

Comments. The question is whether $(T(t))_{t\geq 0}$ can be made to be a contractive semigroup on the $\|\cdot\|_\infty$-closure of $L^2(\Omega) \cap L^\infty(\Omega)$ in $L^\infty(\Omega)$ for an equivalent norm of the special form $f \mapsto \|gf\|_\infty$. There is, of course and well-known from semigroup theory, a norm $|||\cdot|||$ on $L^2(\Omega) \cap L^\infty(\Omega)$, equivalent to $\|\cdot\|_\infty$, such that

$$|||T(t)f||| \leq |||f||| \qquad \text{for all } f \in L^\infty(\Omega) \cap L^2(\Omega).$$

One can take $|||f||| := \sup_{t\geq 0} \|T(t)f\|_\infty$, which satisfies

$$\|f\|_\infty \leq |||f||| \leq M\|f\|_\infty$$

by assumption (1). So the problem might be seen as an L^∞-counterpart to the question if every bounded C_0-semigroup on a Hilbert space can be made contractive for an equivalent scalar product. The answer to this question is known to be negative and there is a nice characterization of those bounded analytic C_0-semigroups that are contractive for an equivalent scalar product in terms of bounded imaginary powers (and bounded H^∞-calculus) for the negative generator (see [2]).

Still another question seems natural.

Problem 11 (Weight construction for L^∞-contraction)**.** In the setting of Problem 8, assume that a function $g \in L^\infty(\Omega)$ satisfying $g > 0$ μ-a.e., $1/g \in L^\infty(\Omega)$, and (3) exists. How can we find or construct such a function g?

Comments. In the general situation of Problems 8, 9, 10, and 11 it might well be that positivity of the semigroup $(T(t))_{t\geq 0}$ can help, i.e. the assumption that $f \geq 0$ a.e. on Ω implies $T(t)f \geq 0$ a.e. on Ω for all $t > 0$. Recall that positivity of the semigroup can be characterized in terms of the sesquilinear form $\mathfrak{a}$ (see [3], [5]).

Assume for the following that the semigroup $(T(t))_{t\geq 0}$ is positive and that the measure space (Ω, μ) is *finite*. Then $L^\infty(\Omega) \subseteq L^2(\Omega)$ and hence $T(t)f \in L^2(\Omega)$ is defined for any $f \in L^\infty(\Omega)$. In this situation, L^∞-contractivity of $(T(t))_{t\geq 0}$ is characterized by $T(t)1_\Omega \leq 1_\Omega$ μ-a.e. for all $t > 0$ where 1_Ω denotes the characteristic function of Ω.

Now let $g \in L^\infty(\Omega)$ such that $g > 0$ μ-a.e. on and $1/g \in L^\infty(\Omega)$. Then, consequently, (3) is characterized by $T(t)g^{-1} \leq g^{-1}$ μ-a.e. for all $t > 0$ (since this is equivalent to L^∞-contractivity of the positive semigroup $(gT(t)g^{-1})_{t\geq 0}$).

In particular, one has (3) for $g \in L^\infty(\Omega)$ with $g > 0$ μ-a.e. if $h := 1/g \in L^\infty(\Omega)$ is an *eigenfunction* for an eigenvalue $\lambda \geq 0$ of A: Recalling that $-A$ is the generator of $(T(t))_{t\geq 0}$ we obtain $T(t)h = e^{-\lambda t}h \leq h$ μ-a.e. for all $t > 0$.

Specializing further, take $\Omega \subset \mathbb{R}^d$ a bounded domain (with μ the Lebesgue measure) and the usual Dirichlet form $\mathfrak{a}(u, v) := \int_\Omega \nabla u \cdot \overline{\nabla v}\, dx$ with form domain $V := V_N := H^1(\Omega)$ (Neumann boundary conditions) or $V := V_D := H^1_0(\Omega)$ (Dirichlet boundary conditions), and denote the associated operators by A_N and A_D, respectively (the negative Laplacian on Ω with Neumann/Dirichlet boundary conditions).

Both semigroups are well-known to be positive and L^∞-contractive, i.e. they satisfy (3) for $g = 1_\Omega$. Now 1_Ω is an eigenfunction of A_N for the eigenvalue 0. However, for A_D we do not even have $1_\Omega \in V_D$. We do have an eigenfunction $h \in L^\infty(\Omega)$, $h > 0$ μ-a.e. on Ω, for the first eigenvalue $\lambda_0 > 0$ of A_D, but $1/h \notin L^\infty(\Omega)$ due to $h \in V_D = H^1_0(\Omega)$. Hence, considering (positive) eigenfunctions is not sufficient, in general.

It might be more adequate to consider *positive subeigenfunctions*, we refer to [1, II-C Section 3] for the notion of positive subeigenvectors and their role in the characterization of positivity of semigroups on Banach lattices.

Communicated by Peer Kunstmann.

References:

[1] W. Arendt et al. *One-parameter Semigroups of Positive Operators.* Edited by R. Nagel. Springer Lecture Notes Math. 1184, Springer, Berlin, Heidelberg, 1986.

[2] Christian Le Merdy. The similarity problem for bounded analytic semigroups on Hilbert space. *Semigroup Forum* 56 (1998) 205–224.

[3] El-Maati Ouhabaz. L^∞-contractivity of semigroups generated by sectorial forms. *J. London Math. Soc.* (2) 46 (1992) 529–542.

[4] El-Maati Ouhabaz. Invariance of Closed Convex Sets and Domination Criteria for Semigroups. *PotentialAnalysis* 5: 611–625, 1996.

[5] El-Maati Ouhabaz. *Analysis of Heat Equations on Domains.* Princeton University Press, Princeton, 2005.