Problem 1 (Commutativity with the semigroup problem)**.** Given C_0-semigroup $(e^{tA})_{t\geq 0}$ with (probably unbounded) generator $(A, D(A))$ in Banach space V, and (probably unbounded) linear operator $(B, D(B))$ in V, provide conditions that are sufficient and/or nesessary to have

$$e^{tA}B = Be^{tA}$$

in some reasonable sense.

This problem can be also stated as follows. Suppose that $R(\lambda, A)$ is the resolvent operator for $(A, D(A))$, and $R(\lambda, e^{tA})$ is the resolvent operator for e^{tA}. Find the all the logical connections between the following four condtions, understanding them in some reasonable sense and ussuming that clarifying this sense is a part of the problem:

- $Be^{tA} = e^{tA}B$,
- $AB = BA$,
- $R(\lambda, A)B = BR(\lambda, A)$,
- $R(\lambda, e^{tA})B = BR(\lambda, e^{tA})$.

What is the most general case when all those four conditions are equivalent?

Comments.

1. This is not a separate problem but a family of problems which, when properly stated and solved, can become a beautiful theory in the framework of operator semigroups.

2. The problem may be known for ages probably, I do not know the history of this problem, it just appeared in my work very recently. **Quiestion to the community: should we include it to the list?**

3. A simple variant of the problem is the case when operators A and B are both bounded. The simplest variant is when A and B are 2x2 matrices. Even at this setting I do not know the answer (however, I met this problem really recently).

4. J. Glück provided the following reference: [1], Appendix A, pages 284-285. Discussion in this reference is not covering the problem in full generality, however it can be considering as a starting of the discussion of the problem.

References:

[1] Markus Haase. The Functional Calculus for Sectorial Operators. *Operator Theory: Advances and Applications, 169. Birkhäuser, 2006*

Communicated by Ivan Remizov.