Knot as a complete invariant of diffeomorphisms of surfaces with th

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It is known that Morse-Smale diffeomorphisms with two hyperbolic periodic orbits exist only on the sphere and all of them are topologically conjugate to each other. However, if we admit the existence of three orbits, then the range of manifolds that admit them expands considerably. In particular, such orientation-preserving diffeomorphisms admit surfaces of any kind. In this paper, we find a complete invariant of topological conjugacy of Morse-Smale diffeomorphisms with three periodic orbits. It is completely determined by the homotopy type (a pair of coprime numbers) of a knot on the torus, which is the space of orbits of an unstable saddle separatrix in the space of orbits of the sink basin. With the help of the obtained result, it is possible to calculate the exact number of topological conjugacy classes of the considered diffeomorphisms on a given surface, as well as the relation of the genus of this surface to the homotopy type of the knot.

Introduction

Let S_p be a closed orientable surface of genus p with metric d, and let $f : S_p \to S_p$ be an orientation-preserving diffeomorphism with a finite hyperbolic nonwandering set f. Denote by G the set of orientation-preserving Morse-Smale diffeomorphisms $f : S_p \to S_p$ whose nonwandering set consists of exactly three periodic orbits.

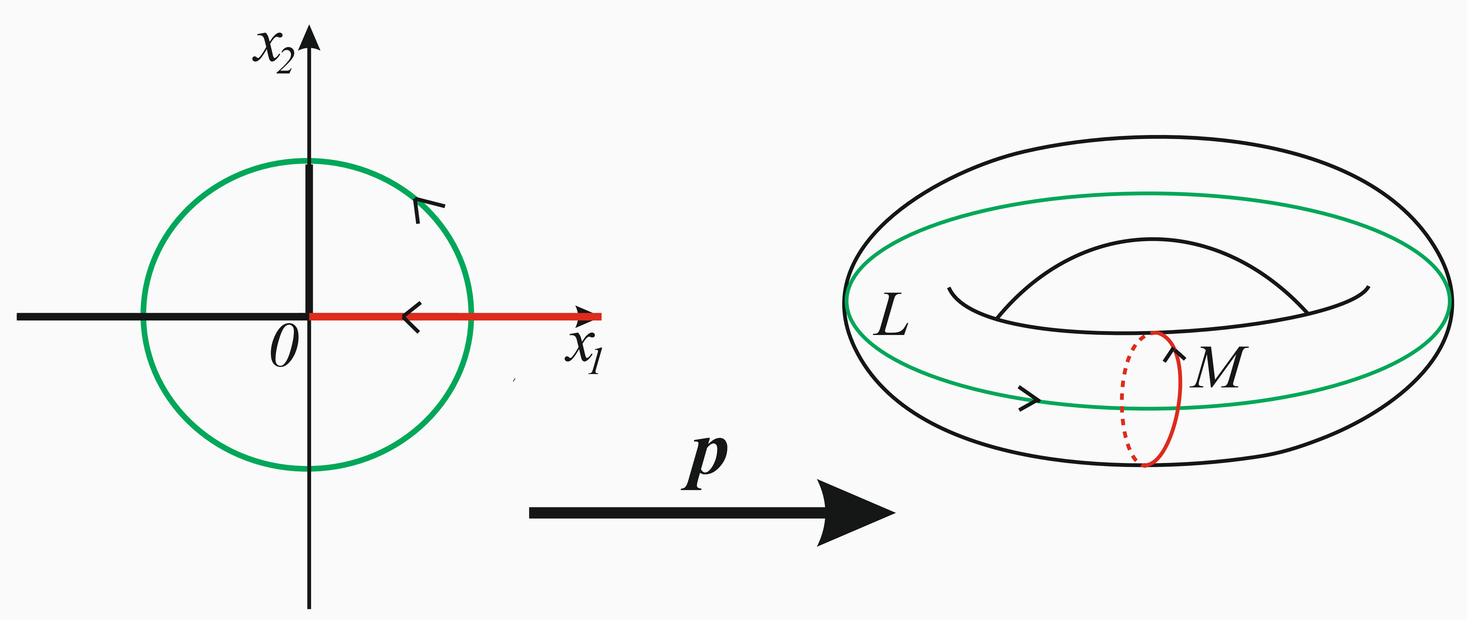
Statement

The nonwandering set of any diffeomorphism $f \in G$ consists of the sink orbit $\mathcal{O}_\omega$, the source orbit $\mathcal{O}_\alpha$, and the saddle orbit $\mathcal{O}_\sigma$. In this case, the saddle orbit has a negative orientation type and at least one of the nodal orbits of the diffeomorphism has period 1.

Natural projection

Let $T^2 = (R^2 \setminus (0,0))/A$ and denote by $p : R^2 \setminus \to T^2$ the natural projection. We introduce generators on the torus as follows. *Parallel* L on the torus T^2 is the image of the unit circle $S^1 = \{(x_1, x_2) \in R^2 : x_1^2 + x_2^2 = 1\}$ ($L = p(S^1)$) with counterclockwise orientation, the parallel has homotopy type $\langle 1, 0\rangle$. *Meridian* M is the image of the positive semiaxis $Ox_1^+ = \{(x_1, x_2) \in R^2 : x_1 > 0,\ x_2 = 0\}$ of the Ox_1axis ($M = p(Ox_1^+)$) with orientation in the direction of decreasing x_1, the meridian has homotopy type $\langle 0, 1\rangle$.

Figure



Homotopic knot type

Let $p_f = p\psi_f : V_f \to T^2$ and $\gamma_f = p_f(W^u_{\mathcal{O}_\sigma})$. The set γ_f is an essential node on the torus T^2 of homotopy type $\langle \lambda_f, \mu_f\rangle$, where $\mu_f > 0$ and $\gcd(\lambda_f, \mu_f) = 1$. The homotopy type of the knot γ_f depends on the choice of the homeomorphism ψ_f so that if $(\tilde\lambda_f, \tilde\mu_f)$ is the homotopy type of the knot γ_f for some homeomorphism $\tilde\psi_f$, different from ψ_f, then $\tilde\mu_f = \mu_f$, $\tilde\lambda_f \equiv \lambda_f \pmod{\mu_f}$. Thus, without loss of generality, we assume that the homeomorphism ψ_f is chosen so that the node γ_f has the homotopy type

$$\langle \lambda_f, \mu_f\rangle :\ \mu_f > 0,\ \gcd(\lambda_f, \mu_f) = 1,\ 0\lambda_f < \mu_f. \qquad (*)$$

Theorem 1. Classification of diffeomorphisms of the class G

The topological conjugacy class of the diffeomorphism $f \in G$ is uniquely determined by the homotopy type $\langle \lambda_f, \mu_f\rangle$ of the knot γ_f. That is, diffeomorphisms $f, f' \in G$ are topologically conjugate if and only if $\lambda_f = \lambda_{f'}$ and $\mu_f = \mu_{f'}$.

Theorem 2. Relationship between the genus of the carrier surface and the homotopy type node γ_f

On a surface S_p of genus $p0$, a diffeomorphism $f \in G$ with knot γ_f of homotopy type $\langle \lambda_f, \mu_f\rangle$ exists if and only if

$$\mu_f = 4p \text{ or } \mu_f = 4p + 2.$$

The number N_p of topological conjugacy classes of $f \in G$ diffeomorphisms defined on the surface S_p is calculated by the formula

$$N_p = \varphi(4p) + \varphi(4p + 2),$$

where $\varphi(n)$ is the Euler function, that is, the number of coprime numbers with n not exceeding n.

Periodic data of the diffeomorphism $f \in G$

Statement. Any orientation-preserving gradient-like diffeomorphism $f : S_p \to S_p$ can be represented as a composition $f = \varphi \circ \xi^1$, where ξ^1 is the shift per unit time along the trajectories gradient flow ξ^t of some Morse function[1], and φ is a periodic homeomorphism. Wherein:

- points of smaller period of the homeomorphism φ are also periodic points of the diffeomorphism f, and their periods coincide;
- the period of the separatrix of any saddle point of the diffeomorphism f coincides with the period of the homeomorphism φ.

Periodic data of the diffeomorphism $f \in G$

Lemma. Let $f = \varphi \circ \xi^1 \in G$. Then if the mapping φ has exactly three points of smaller period, then it has one of the following full characteristics:

1) $(n = 4p, g = 0, p > 0, n_1 = 2p, n_2 = 1, n_3 = 1, d_1 = 1, d_2, d_3 = 2p - d_2)$, $0 < d_2 < 2p$, $\gcd(d_2, 2p) = 1$;
2) $(n = 4p, g = 0, p > 0, n_1 = 2p, n_2 = 1, n_3 = 1, d_1 = 1, d_2, d_3 = 6p - d_2)$, $2p < d_2 < 4p$, $\gcd(d_2, 2p) = 1$;
3) $(n = 4p + 2, g = 0, p > 0, n_1 = 2p + 1, n_2 = 2, n_3 = 1, d_1 = 1, d_2, d_3 = 2p + 1 - 2d_2)$, $0 < d_2p$, $\gcd(d_2, 2p + 1) = 1$;
4) $(n = 4p + 2, g = 0, p > 0, n_1 = 2p + 1, n_2 = 2, n_3 = 1, d_1 = 1, d_2, d_3 = 6p + 3 - 2d_2)$, $p < d_22p$, $\gcd(d_2, 2p + 1) = 1$.

The number of diffeomorphisms depending on the genus of the surface