Resonance in oscillators with bounded nonlinearities

E. N. Pelinovsky [1], I. E. Melnikov [2]

Keywords: Nonlinear resonance, self-consistent source, oscillatory systems with bounded nonlinearity

MSC2020 codes: 34A34

Abstract. The present paper discusses a method for finding self-consistent external influences on a nonlinear oscillator that lead to the phenomenon of resonance as in the linear case. It is shown that for bounded nonlinear systems it is possible to find such a self-consistent external force. To illustrate the search for self-consistent external influences, the simplest nonlinear systems are selected.

Introduction. Resonant phenomena in linear oscillatory systems are well studied and described in all books on general physics and oscillation theory. If the external force frequency coincides with one of the partial natural frequencies of the linear system, the oscillation amplitude in the absence of attenuation increases according to the linear law and can reach significant values, thus, leading to the structure destruction. In the case of nonlinear systems, the monochromatic effect does not lead to a significant increase in the oscillation amplitude, since their frequency depends on the amplitude and, consequently, the equality of the frequencies of the external force and natural oscillations is violated. This problem was encountered during the construction of the first cyclotrons [1], [2]. As a result, the resonance curve becomes limited and asymmetric with respect to the linear oscillation frequency. This process in weakly nonlinear systems is also well described in literature; see, for example, [3], [4], [5], [6].

One of the ways to overcome the movement of the natural frequency of a nonlinear system from resonance is to control the external force by adjusting its frequency to the local natural frequency. Such a mechanism is called autoresonance, it has become widespread in discrete [7] and distributed systems [8], [9].

The purpose of this study is to search for self-consistent external influences that make it possible to swing oscillations in nonlinear systems. Here we will limit ourselves to the simplest nonlinear oscillator models and show that it is possible to select limited external influences that lead to the resonant phenomena similar to those existing in the linear system.

Self-consistent source in a bounded nonlinear oscillator. Let us consider the following bounded nonlinear system which are described by the equation:

$$\frac{d^2u}{dt^2} + u + F(u) = 0 \tag{1}$$

where $F(u)$ is some continuous nonlinear function. We will consider the nonlinearity bounded by $|F(u)| < F_0$, $F_0 \in \mathbb{R}$, $F(0) = 0$ and $F(u) \to \mu u^2$ for $u \to 0$, $\mu \in \mathbb{R}$, for the nonlinearity to be infinitely small of a higher order than u, in order for a linear resonance to be obtained in this neighborhood with a sinusoidal effect with a unit frequency.

Then, instead of solving this equation, we can assume that we will be able to obtain a resonant solution, as in the linear case, due to an some external force $f(t)$.

$$\frac{d^2u}{dt^2} + u + F(u) = 2\cos t + f(t) \tag{2}$$

[1]Institute of Applied Physics, Nizhny Novgorod, Russia. National Research University-Higher School of Economics, Moscow, Russia. Email: pelinovsky@appl.sci-nnov.ru

[2]Institute of Applied Physics, Nizhny Novgorod, Russia. National Research University-Higher School of Economics, Nizhny Novgorod, Russia. Email: melnicovioann@gmail.com

Assuming that the solution of this equation is $u(t) = t \sin t$, we can find out expression for the function $f(t)$:

$$f(t) = F(t \sin t)$$

The resulting external force is continuous and limited, but not monochromatic, as in the linear case, but with a wide spectrum.

Examples of finding a self-consistent source. As an illustration of this approach, we chose a system with sinusoidal nonlinearity (3), as well as with saturation-type nonlinearity (4).

$$\frac{d^2u}{dt^2} + u + a \sin^2 bu = 0 \tag{3}$$

The phase portrait of system (3) consists of a finite number of alternating centers and saddles, the number of which depends on the parameters of this system. In this case, the external effect will look like this

$$f(t) = a \sin^2(bt \sin t) \tag{5}$$

It is shown in [10] that the amount of energy to maintain this system in a state of resonance increases linearly over time, and the external force becomes more and more high-frequency.

Along with sinusoidal nonlinearity, we also analyzed an example of saturation type nonlinearity. The saturation nonlinearity systems are very common in technical applications [11], [12].

$$\frac{d^2u}{dt^2} + u + \frac{au^2}{1 + b^2u^2} = 0 \tag{4}$$

The phase portrait in this case has only 2 possible positions - it is only the center or 2 centers and the saddle. And in this case external force will be

$$f(t) = a\frac{t^2 \sin^2 t}{1 + b^2t^2 \sin^2 t} \tag{6}$$

We have numerically shown that with a small deviation of the parameters of the external force from the parameters of the original system, it will still cause resonance. It is shown that as the amplitude of the nonlinear saturation function increases, the system becomes more sensitive to changes in the amplitude of a self-consistent external force.

In our opinion, the search and study of resonance in nonlinear isochronous systems, which are not rare or exceptional examples of nonlinear systems, is an interesting subject for subsequent research.

Acknowledgments. The work was supported by the Laboratory of Dynamic Systems and Applications of the Higher School of Economics, grant of the Ministry of Science and Higher Education of the Russian Federation Agreement No. 075-15-2022-1101 and the RSF grant 19-12-00253.

References

[1] V.I. Veksler. *A new method of acceleration of relativistic particles.* // J. Phys. USSR. 1945. Vol. 9, P. 153.

[2] E.M. McMillan. *The synchrotron - A proposed high energy particle accelerator.* // Phys. Rev. 1945. Vol. 68, P. 143.

[3] A.A. Andronov, A.A. Witt, S.E. Khaykin. *Theory of Oscillators.* Cambridge University Press, 1966.

[4] L.D. Landau, E.M. Lifshitz. *Mechanics.* Third Edition: Vol. 1 (Course of Theoretical Physics), Butterworth-Heinemann, 1976.

[5] E. Kartashova. *Nonlinear Resonance Analysis.* // Cambridge University Press, 2010.

[6] S. Rajasekar, M.A.F. Sanjuan. *Nonlinear resonances.* // Switzerland: Springer International Publishing, 2016.
[7] J. Fajans, L. Friedland. *Autoresonant (nonstationary) excitation of pendulums, Plutinos, plasmas, and other nonlinear oscillators.* // Am. J. Phys. 2001. Vol. 69 P. 1096
[8] I. Aranson, B. Meerson, T. Tajima. *Excitation of solitons by an external resonant wave with a slowly varying phase velocity.* // Phys. Rev. A. 1992. Vol. 45 P. 7500.
[9] L. Friedland. *Autoresonance in nonlinear systems.* // Scholarpedia. 2009. Vol. 4 No. 1. P. 5473.
[10] E.N. Pelinovsky, I.E. Melnikov. *Resonance in bounded nonlinear pendulum-type systems.* // Zhurnal Srednevolzhskogo matematicheskogo obshchestva [Middle Volga Mathematical Society Journal]. 2022. Vol. 24 No. 3 P. 289-PIP‚вЂњ296, doi: https://doi.org/10.15507/2079-6900.24.202203.289-296 (In Russian)
[11] P. Mellodge. *A practical approach to dynamical systems for engineers.* // Woodhead Publishing. 2015.
[12] N.G. Vakhitov, A.A. Kolokolov. *Stationary solutions of the wave equation in the medium with nonlinearity saturation.* // Radiophysics and Quantum Electronics. 1973. Vol. 16 No. 7 P. 783–789. doi: https://doi.org/10.1007/BF01031343