Local Solutions of Quasilinear Equations with Gerasimov — Caputo Derivatives. Sectorial Case.

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Introduction. Over the past few decades, there has been a sharp increase in the interest of researchers in fractional differential equations, primarily due to their increasing importance in modeling various phenomena that arise in physics, chemistry, mathematical biology, and engineering [1,2].

The unique solvability issues for initial problems to some types of equations in Banach spaces with the Gerasimov — Caputo time-fractional derivative were researched in the works [3–7].

In this paper, we study the Cauchy problem $z^{(l)}(t_0) = z_l$, $l = 0, 1, \dots, m-1$, for a differential equation with several fractional derivatives in the linear and nonlinear parts

$$D^\alpha z(t) = \sum_{k=1}^{n} D^{\alpha_k} A_k z(t) + B(t, D^{\gamma_1} z(t), D^{\gamma_2} z(t), \dots, D^{\gamma_r} z(t)). \tag{1}$$

Here D^β is the Gerasimov — Caputo derivative of the order $\beta > 0$, or the Riemann — Liouville integral of the order $-\beta$ in the case $\beta \le 0$, $m-1 < \alpha \le m \in \mathbb{N}$, $n, r \in \mathbb{N} \cup \{0\}$, $\alpha_1 < \alpha_2 < \dots < \alpha_n < \alpha$, $\gamma_1 < \gamma_2 < \dots < \gamma_r < \alpha$, $\mathcal{Z}$ — Banach space, A_k, $k = 1, 2, \dots, n$, are linear closed operators with domains $D_{A_k} \subset \mathcal{Z}$, the non-linear mapping $B : [t_0, T] \times \mathcal{Z}^r \to D := \bigcap\limits_{k=1}^{n} D_{A_k}$ is continuous in the norm $\|\cdot\|_D = \|\cdot\|_\mathcal{Z} + \sum\limits_{k=1}^{n} \|A_k \cdot\|_\mathcal{Z}$. The unique solvability of the Cauchy problem for the linear inhomogeneous equation (1) ($B = f(t)$) in the case when the operators A_k are bounded, $k = 1, 2, \dots, n$, was proved in [7]. In the case when the set of unbounded operators $(A_1, A_2, \dots, A_n)$ belongs to the class $\mathcal{A}^n_{\alpha,G}$, the unique solvability of the Cauchy problem for linear inhomogeneous equation (1) was studied in [8, 9]. Under the condition that the nonlinear operator B is locally Lipschitz, we obtain a theorem on the local unique solvability of the Cauchy problem for quasilinear equation (1). For this, the fixed point theorem in a specially constructed metric space is used.

Local solution. Denote $D := \bigcap_{k=1}^{n} D_{A_k}$, $R_\lambda := \left(\lambda^\alpha I - \sum\limits_{k=1}^{n} \lambda^{\alpha_k} A_k\right)^{-1} : \mathcal{Z} \to D$. We endow the set D with the norm $\|\cdot\|_D = \|\cdot\|_\mathcal{Z} + \sum\limits_{k=1}^{n} \|A_k \cdot\|_\mathcal{Z}$, with respect to which D is a Banach space, since it is the intersection of the Banach spaces $D_{A_1}, D_{A_2}, \dots, D_{A_n}$ with the corresponding graph norms.

Denote $n_l := \min\{k \in \{1, 2, \dots, n\} : l \le m_k - 1\}$ for $l = 0, 1, \dots, m-1$. If the set $\{k \in \{1, 2, \dots, n\} : l \le m_k - 1\}$ is empty for some $l \in \{0, 1, \dots, m-1\}$ (this holds exactly when $\alpha_n \le m-1$), then we set $n_l := n+1$.

Definition 1. The set of operators $(A_1, A_2, \dots, A_n)$ belongs to the class $\mathcal{A}^n_{\alpha,G}(\theta_0, a_0)$ for some $\theta_0 \in (\pi/2, \pi)$, $a_0 \ge 0$, if

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(i) D is dense in $\mathcal{Z}$;

(ii) for all $\lambda \in S_{\theta_0,a_0} := \{\mu \in \mathbb{C} : |\arg(\mu - a_0)| < \theta_0,\ a \neq a_0\}$, $l = 0, 1, \dots m-1$ we have

$$R_\lambda \cdot \left(I - \sum_{k=n_l}^{n} \lambda^{\alpha_k - \alpha} A_k \right) \in \mathcal{L}(\mathcal{Z});$$

(iii) for any $\theta \in (\pi/2, \theta_0)$, $a > a_0$ there exists $K(\theta, a) > 0$, such that for all $\lambda \in S_{\theta,a}$, $l = 0, 1, \dots m-1$

$$\|R_\lambda\|_{\mathcal{L}(\mathcal{Z})} \le \frac{K(\theta,a)}{|\lambda - a||\lambda|^{\alpha-1}}, \quad \left\| R_\lambda \left(I - \sum_{k=n_l}^{n} \lambda^{\alpha_k - \alpha} A_k \right) \right\|_{\mathcal{L}(\mathcal{Z})} \le \frac{K(\theta,a)}{|\lambda - a||\lambda|^{\alpha-1}}.$$

Let $\gamma_1 < \gamma_2 < \dots < \gamma_r < \alpha$, $r_i - 1 < \gamma_i \le r_i \in \mathbb{Z}$, $i = 1, 2, \dots, r$, U be an open set in $\mathbb{R} \times \mathcal{Z}^r$, $B : U \to \mathcal{Z}$. Consider the Cauchy problem

$$z^{(l)}(t_0) = z_l, \quad l = 0, 1, \dots, m-1, \tag{2}$$

$$D^\alpha z(t) = \sum_{k=1}^{n} D^{\alpha_k} A_k z(t) + B(t, D^{\gamma_1} z(t), D^{\gamma_2} z(t), \dots, D^{\gamma_r} z(t)). \tag{3}$$

A solution of problem (2), (3) on a segment $[t_0, t_1]$ is a function $z \in C((t_0, t_1]; D) \cap C^{m-1}([t_0, t_1]; \mathcal{Z})$ for which $D^\alpha z \in C((t_0, t_1]; \mathcal{Z})$, $D^{\alpha_k} A_k z \in C((t_0, t_1]; \mathcal{Z})$, $k = 1, 2, \dots, n$, $D^{\gamma_i} z \in C([t_0, t_1]; \mathcal{Z})$, $i = 1, 2, \dots, r$, the inclusion $(t, D^{\gamma_1} z(t), D^{\gamma_2} z(t), \dots, D^{\gamma_r} z(t)) \in U$ for $t \in [t_0, t_1]$ and the equality (2) for all $t \in (t_0, t_1]$, as well as the conditions (3) are satisfied.

Denote $\bar{x} := (x_1, x_2, \dots, x_r) \in \mathcal{Z}^r$, $S_\delta(\bar{x}) = \{\bar{y} \in \mathcal{Z}^r : \|y_l - x_l\|_{\mathcal{Z}} \le \delta, l = 1, 2, \dots, r\}$. A mapping $B : U \to \mathcal{Z}$ is called locally Lipschitz in $\bar{x}$, if for any $(t, \bar{x}) \in U$ there exist $\delta > 0, q > 0$ such that $[t - \delta, t + \delta] \times S_\delta(\bar{x}) \subset U$ and for any $(s, \bar{y}), (s, \bar{v}) \in [t - \delta, t + \delta] \times S_\delta(\bar{x})$ the inequality $\|B(s, \bar{y}) - B(s, \bar{v})\|_{\mathcal{Z}} \le q \sum\limits_{i=1}^{r} \|y_i - v_i\|_{\mathcal{Z}}$ holds.

Using the initial data $z_0, z_1, \dots, z_{m-1}$, we define the polynomial

$$\tilde{z}(t) = z_0 + (t - t_0) z_1 + \frac{(t - t_0)^2}{2!} z_2 + \dots + \frac{(t - t_0)^{m-1}}{(m-1)!} z_{m-1}$$

and vectors $\tilde{z}_i = D^{\gamma_i}|_{t=t_0} \tilde{z}(t)$, $i = 1, 2, \dots, r$. Note that $\tilde{z}_i = 0$ if $\gamma_i \notin \{0, 1, \dots, m-1\}$. In the case $\gamma_i \in \{0, 1, \dots, m-1\}$ we have $\tilde{z}_i = z_{\gamma_i}$. Thus, the value of the argument of the nonlinear operator B at the initial moment of time is $(t_0, \tilde{z}_1, \tilde{z}_2, \dots, \tilde{z}_r)$.

Lemma 1. [10]. Let $l - 1 < \beta \le l \in \mathbb{N}$. Then

$$\exists C > 0 \quad \forall h \in C^l([t_0, t_1]; \mathcal{Z}) \quad \|D_t^\beta h\|_{C([t_0,t_1];\mathcal{Z})} \le C \|h\|_{C^l([t_0,t_1],\mathcal{Z})}.$$

Lemma 2. Let $\alpha_1 < \alpha_2 < \dots < \alpha_n < \alpha$, $\gamma_1 < \dots < \gamma_r < \alpha$, $m - 1 < \alpha \le m \in \mathbb{N}$, $(A_1, A_2, \dots, A_n) \in \mathcal{A}^n_{\alpha,G}(\theta_0, a_0)$ for some $\theta_0 \in (\pi/2, \pi)$, $a_0 \ge 0$, $z_l \in D$, $l = 0, 1, \dots, m-1$, U be an open set in $\mathbb{R} \times \mathcal{Z}^r$, $B \in C(U; D)$, $(t_0, \tilde{z}_1, \tilde{z}_2, \dots, \tilde{z}_r) \in U$. Then the function z is a solution to the problem (2), (3) on the segment $[t_0, t_1]$ if and only if $z \in C^{m-1}([t_0, t_1]; \mathcal{Z})$, $D^{\gamma_i} z \in C([t_0, t_1]; \mathcal{Z})$, $i = 1, 2, \dots, r$, and for all $t \in [t_0, t_1]$ the inclusion $(t, D^{\gamma_1} z(t), D^{\gamma_2} z(t), \dots, D^{\gamma_r} z(t)) \in U$ and equality

$$z(t) = \sum_{l=0}^{m-1} Z_l(t - t_0) z_l + \int_{t_0}^{t} Z(t - s) B(s, D^{\gamma_1} z(s), D^{\gamma_2} z(s), \dots, D^{\gamma_r} z(s)) ds \tag{4}$$

are valid.

Denote $i_* := \min\{i \in \{1,2,\dots,r\} : \gamma_i > m-1\}$ if the set $\{i \in \{1,2,\dots,r\} : \gamma_i > m-1\}$ is not empty, otherwise $i_* := r+1$. For $t_1 > t_0$ we define the space $C^{m-1,\{\gamma_i\}}([t_0,t_1];\mathcal{Z}) := \{z \in C^{m-1}([t_0,t_1];\mathcal{Z}) : D^{\gamma_i} z \in C([t_0,t_1];\mathcal{Z}),\ i = i_*, i_*+1,\dots,r\}$ and equip this space with the norm

$$\|z\|_{C^{m-1,\{\gamma_i\}}([t_0,t_1];\mathcal{Z})} = \|z\|_{C^{m-1}([t_0,t_1];\mathcal{Z})} + \sum_{i=i_*}^{r} \|D^{\gamma_i} z\|_{C([t_0,t_1];\mathcal{Z})}.$$

Remark 1. For the function $z \in C^{m-1}([t_0,t_1];\mathcal{Z})$, by Lemma 1 $D^{\gamma_i} z \in C([t_0,t_1];\mathcal{Z})$, $i = 1,2,\dots,i_*-1$. Therefore, functions from $C^{m-1}([t_0,t_1];\mathcal{Z})$ for which $D^{\gamma_i} z \in C([t_0,t_1];\mathcal{Z})$, $i = 1,2,\dots,r$ referred to in the Lemma 2 are exactly functions from $C^{m-1,\{\gamma_i\}}([t_0,t_1];\mathcal{Z})$.

Lemma 3. $C^{m-1,\{\gamma_i\}}([t_0,t_1];\mathcal{Z})$ is a Banach space.

Theorem 1. Let $\alpha_1 < \alpha_2 < \dots < \alpha_n < \alpha$, $\gamma_1 < \dots < \gamma_r < \alpha$, $m-1 < \alpha \le m \in \mathbb{N}$, $(A_1, A_2, \dots, A_n) \in \mathcal{A}^n_{\alpha,G}(\theta_0, a_0)$ for some $\theta_0 \in (\pi/2, \pi)$, $a_0 \ge 0$, $z_l \in D$, $l = 0,1,\dots,m-1$, U be an open set in $\mathbb{R} \times \mathcal{Z}^r$, $B \in C(U;D)$ be locally Lipschitz in $\bar{x}$, $(t_0, \tilde{z}_1, \tilde{z}_2, \dots, \tilde{z}_r) \in U$. Then for some $t_1 > t_0$ problem (2), (3) has a unique solution on the interval $[t_0, t_1]$.

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