Unique solvability of Equations with a Distributed Fractional Derivative Given by the Stieltjes Integral

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Let $\mathcal{L}(\mathcal{Z})$ be the Banach space of all linear continuous operators on a Banach space $\mathcal{Z}$, denote by $\mathcal{C}l(\mathcal{Z})$ the set of all linear closed operators, densely defined in $\mathcal{Z}$, acting in the space $\mathcal{Z}$. Introduce the notations $S_{\theta,a} := \{\mu \in \mathbb{C} : |\arg(\mu - a)| < \theta,\ \mu \neq a\}$ for $\theta \in [\pi/2, \pi]$, $a \in \mathbb{R}$.

Let $b, c \in \mathbb{R}$, $b < c$, $\mu : [b, c] \to \mathbb{C}$ is a function with a bounded variation. Introduce the notations of the complex-valued function $W(\lambda) := \int\limits_b^c \lambda^\alpha d\mu(\alpha)$. Here the integral is understood in the sense of Riemann — Stieltjes.

We define a class $\mathcal{A}_W(\theta_0, a_0)$ as the set of all operators $A \in \mathcal{C}l(\mathcal{Z})$ satisfying the following conditions:

(i) there exist $\theta_0 \in (\pi/2, \pi]$, $a_0 \geq 0$, such that $W(\lambda) \in \rho(A)$ for every $\lambda \in S_{\theta_0,a_0}$;

(ii) for every $\theta \in (\pi/2, \theta_0)$, $a > a_0$ there exists $K(\theta, a) > 0$, such that for all $\lambda \in S_{\theta,a}$

$$\|(W(\lambda)I - A)^{-1}\|_{\mathcal{L}(\mathcal{Z})} \leq \frac{|\lambda| K(\theta, a)}{|W(\lambda)||\lambda - a|}.$$

Theorem 1. [1]. Let $b, c \in \mathbb{R}$, $b < c$, $m - 1 < c \leq m \in \mathbb{N}$, $\mu : [b, c] \to \mathbb{C}$ is a function with a bounded variation, c be a variation point of the measure $d\mu(t)$, $\theta_0 \in (\pi/2, \pi]$, $a_0 \geq 0$, $A \in \mathcal{A}_W(\theta_0, a_0)$, $g \in C([0, T]; D_A) \cup C^\gamma([0, T]; \mathcal{Z})$, $\gamma \in (0, 1]$, $z_k \in D_A$, $k = 0, 1, \ldots, m - 1$. Then there exists a unique solution of problem

$$z^{(k)}(0) = z_k, \quad k = 0, 1, \ldots, m - 1, \tag{1}$$

for the inhomogeneous equation

$$\int\limits_b^c D^\alpha z(t) d\mu(\alpha) = Az(t) + g(t), \ \ t \in (0, T]. \tag{2}$$

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References

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