A Class of Fractional Quasilinear Equations in the Sectorial Case

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Introduction. In the operator semigroup theory [1] the introduction of fractional powers A^γ for a continuously invertible generator $-A$ of an analytic resolving semigroup and of spaces $\mathcal{Z}_\gamma$ as the domains of A^γ with the graph norm allows to study the solvability issues of partial differential equations with nonlinearity, which depends on lower order derivatives with respect to spatial variables. In this work we consider complex powers of an operator A, such that $-A$ generates an analytic resolving family of operators of a fractional order equation $D^\alpha z(t) + Az(t) = 0$, and use them for a quasilinear equation

$$D^\alpha z(t) + Az(t) = B\left(D^{\alpha_1} z(t), D^{\alpha_2} z(t), \dots, D^{\alpha_n} z(t), D^{\alpha-m-r} z(t), \dots, D^{\alpha-1} z(t)\right),$$

where $m-1 < \alpha \leq m \in \mathbb{N}$, $r \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, $n \in \mathbb{N}$, $\alpha_1 < \alpha_2 < \dots < \alpha_n < \alpha - 1$, $m_k - 1 < \alpha_k \leq m_k \in \mathbb{Z}$, $\alpha_k - m_k \neq \alpha - m$, $k = 1, 2, \dots, n$, D^β is the fractional Riemann — Liouville derivative of an order $\beta > 0$, or the fractional Riemann — Liouville integral of an order $-\beta$, if $\beta \leq 0$, operator B is locally Lipschitzian with respect to the norm in $\mathcal{Z}_\gamma$, $\gamma \in (0,1)$. Abstract result we apply to the study of an initial boundary value problem with a nonlinear part, containing partial derivatives in spatial variables in the nonlinear part.

Fractional sectorial operators and their complex powers. Denote by $\rho(A)$ the resolvent set of an operator A, $R_\lambda(A) := (\lambda I - A)^{-1}$, $S_{\theta_0,a_0} := \{\lambda \in \mathbb{C} : |\arg(\lambda - a_0)| < \theta_0,\ \lambda \neq a_0\}$, $\Sigma_\varphi := \{\tau \in \mathbb{C} : |\arg \tau| < \varphi,\ \tau \neq 0\}$.

Let $\theta_0 \in (\pi/2, \pi)$, $a_0 \geq 0$, denote by $\mathcal{A}_\alpha(\theta_0, a_0)$ a class of linear, closed and densely defined in $\mathcal{Z}$ operators A, acting into $\mathcal{Z}$, such that the following conditions are satisfied [2]:

(i) for every $\lambda \in S_{\theta_0,a_0}$ the inclusion $\lambda^\alpha \in \rho(A)$ is valid;

(ii) for any $\theta \in (\pi/2, \theta_0)$, $a \geq a_0$ there exists $K = K(\theta, a) > 0$, such that

$$\forall \lambda \in S_{\theta,a} \quad \|R_{\lambda^\alpha}(A)\|_{\mathcal{L}(\mathcal{Z})} \leq \frac{K(\theta, a)}{|\lambda^{\alpha-1}(\lambda - a)|}.$$

If $\alpha > 0$, $-A \in \mathcal{A}_\alpha(\theta_0, a_0)$, $\beta \in \mathbb{R}$, then the operators

$$Z_\beta(t) := \frac{1}{2\pi i} \int\limits_\Gamma \mu^{\alpha-1+\beta} R_{\mu^\alpha}(-A) e^{\mu t} d\mu, \quad t \in \mathbb{R}_+,$$

are defined and analytically extendable on $\Sigma_{\theta_0 - \pi/2}$, where $\Gamma := \Gamma_+ \cup \Gamma_- \cup \Gamma_0$, $\Gamma_\pm := \{\mu \in \mathbb{C} : \mu = a + re^{\pm i\theta},\ r \in (\delta, \infty)\}$, $\Gamma_0 := \{\mu \in \mathbb{C} : \mu = a + \delta e^{i\varphi},\ \varphi \in (-\theta, \theta)\}$ for $\delta > 0$, $a > a_0$, $\theta \in (\pi/2, \theta_0)$ (see [3]). The estimates

$$\|Z_\beta(t)\|_{\mathcal{L}(\mathcal{Z})} \leq C_\beta(\theta, a) e^{at} (t^{-1} + a)^\beta, \quad t > 0, \quad \beta \geq 0,$$

$$\|Z_\beta(t)\|_{\mathcal{L}(\mathcal{Z})} \leq C_\beta(\theta, a) e^{at} t^{-\beta}, \quad t > 0, \quad \beta < 0,$$

hold for every $a > a_0$ [3].

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Theorem 1. Let $\alpha > 0$, $-A \in \mathcal{A}_\alpha(\theta_0, a_0)$. Then for all $\beta < 1$, $\delta < 1$, $t, s > 0$

$$Z_\beta(s)Z_\delta(t) = -\frac{1}{\alpha}Z_{\beta+\delta}(s+t) + \frac{t^{-\delta}}{2\pi i}\int\limits_\Gamma \mu^{\alpha-1+\beta}R_{\mu^\alpha}(-A)E_{\alpha,1-\delta}(\mu^\alpha t^\alpha)e^{\mu s}d\mu +$$

$$+\frac{s^{-\beta}}{2\pi i}\int\limits_\Gamma \mu^{\alpha-1+\delta}R_{\mu^\alpha}(-A)E_{\alpha,1-\beta}(\mu^\alpha s^\alpha)e^{\mu t}d\mu.$$

It is known that for $\alpha = 1$ $\{Z_0(t) \in \mathcal{L}(\mathcal{Z}) : t \in \mathbb{R}_+\}$ is an analytic semigroup of operators [1]. Take in Theorem 1 $\alpha = 1$, $\beta = \delta = 0$ and obtain the semigroup property $Z_0(t)Z_0(s) = Z_0(t+s)$, $t, s > 0$. Thus, Theorem 1 gives some generalization of the semigroup property for resolving families of operators, which are generated by an operator from the class $\mathcal{A}_\alpha(\theta_0, a_0)$.

As in [1] complex powers A^γ, $\gamma \in \mathbb{C}$, of such A can be defined.

Theorem 2. Let $\alpha > 0$, $-A \in \mathcal{A}_\alpha(\theta_0, 0)$, $0 \in \rho(A)$. Then
(i) for $\gamma \in \mathbb{C}$ A^γ is a closed operator;
(ii) if $\mathrm{Re}\gamma > \mathrm{Re}\beta \geq 0$, then $D_{A^\gamma} \subset D_{A^\beta}$;
(iii) $\overline{D}_{A^\gamma} = \mathcal{Z}$ for every $\mathrm{Re}\gamma \geq 0$;
(iv) if $\gamma, \beta \in \mathbb{C}$, then $A^{\gamma+\beta}z = A^\gamma A^\beta z$ for every $z \in D_{A^\gamma} \cap D_{A^\beta} \cap D_{A^{\gamma+\beta}}$;
(v) $Z_\beta(t) : \mathcal{Z} \to D(A^\gamma)$ for all $\beta \in \mathbb{R}$, $\mathrm{Re}\gamma \in [0, 1)$, $t > 0$;
(vi) $Z_\beta(t)A^\gamma z = A^\gamma Z_\beta(t)z$ for $\beta \in \mathbb{R}$, $\gamma \in \mathbb{C}$, $z \in D(A^\gamma)$;
(vii) for $\beta \in \mathbb{R}$, $\mathrm{Re}\gamma < 1$, $t > 0$ the operator $A^\gamma Z_\beta(t)$ is bounded;
(viii) for $\beta < 1$, $\mathrm{Re}\gamma \in (0, 1)$

$$A^{-\gamma} = \frac{\alpha \sin \pi\gamma}{\sin(\pi(\alpha + \gamma\beta))\Gamma(\alpha\gamma + \beta)}\int\limits_0^\infty t^{\alpha\gamma+\beta-1}Z_\beta(t)dt;$$

(ix) for $\beta \in \mathbb{R}$, $t > 0$ $\|AZ_\beta(t)\|_{\mathcal{L}(\mathcal{Z})} \leq Ct^{-\alpha-\beta}$;
(x) for $\beta \in (-\alpha\mathrm{Re}\gamma, 1)$, $\mathrm{Re}\gamma \in (0, 1)$, $t > 0$ $\|A^\gamma Z_\beta(t)\|_{\mathcal{L}(\mathcal{Z})} \leq C_\gamma t^{-\alpha\mathrm{Re}\gamma-\beta}$;
(xi) for $\beta < 1$, $\mathrm{Re}\gamma \in (0, 1)$, $z \in D(A^\gamma)$

$$\|D^{-\beta}Z_\beta(t)z - z\|_\mathcal{Z} \leq C_\gamma t^{\alpha\mathrm{Re}\gamma}\|A^\gamma z\|_\mathcal{Z}.$$

Incomplete Cauchy type problem for a quasilinear equation. Consider a quasilinear equation

$$D^\alpha z(t) + Az(t) = B\left(D^{\alpha_1}z(t), D^{\alpha_2}z(t), \ldots, D^{\alpha_n}z(t), D^{\alpha-m-r}z(t), \ldots, D^{\alpha-1}z(t)\right), \quad (1)$$

where $m-1 < \alpha \leq m \in \mathbb{N}$, $r \in \mathbb{N}_0$, $n \in \mathbb{N}$, $\alpha_1 < \alpha_2 < \cdots < \alpha_n < \alpha-1$, $m_k-1 < \alpha_k \leq m_k \in \mathbb{Z}$, $\alpha_k - m_k \neq \alpha - m$, $k = 1, 2, \ldots, n$. Some of α_k may be negative. As in [4] denote $\underline{\alpha} := \max\{\alpha_k : \alpha_k - m_k < \alpha - m,\ k = 1, 2, \ldots, n\}$, $\underline{m} := \lceil\underline{\alpha}\rceil$, $\overline{\alpha} := \max\{\alpha_k : \alpha_k - m_k > \alpha - m,\ k = 1, 2, \ldots, n\}$, $\overline{m} := \lceil\overline{\alpha}\rceil$, $m^* := \max\{\underline{m} - 1, \overline{m}\}$, $m^{**} := \max\{m^* + 1, 0\}$. For the study of an initial problem to (1) we need the existence of finite limits $\lim\limits_{t\to t_0} D^{\alpha_l}z(t) := D^{\alpha_l}z(t_0)$, $l = 1, 2, \ldots, n$, therefore, as it follows from results of [4], problem

$$D^{\alpha-m+k}z(t_0) = z_k, \quad k = m^{**}, m^{**} + 1, \ldots, m - 1, \quad (2)$$

will be considered with the necessary condition $D^{\alpha-m+k}z(t_0) = 0$, $k = 0, 1, \ldots, m^{**}$. Since $\alpha_n < \alpha - 1$, we have $m^* \leq m - 2$, $m^{**} \leq m - 1$, therefore, (2) contains one condition at least.

Let $\gamma \in (0,1)$, $\mathcal{Z}_\gamma := D_{A^\gamma}$ is a Banach space with the norm $\|\cdot\|_\gamma := \|A^\gamma \cdot\|_{\mathcal{Z}}$, since A^γ is a continuously invertible closed operator. Let U be an open subset of $\mathbb{R} \times \mathcal{Z}_\gamma^{n+m+r}$, a mapping $B : U \to \mathcal{Z}$ is given, for every $(t, x_1, x_2, \dots, x_{n+m+r}) \in U$ there exists a neighbourhood $V \subset U$, $C > 0$, $\delta \in (0,1]$ such that for all $(t, y_1, y_2, \dots, y_{n+m+r}), (s, v_1, v_2, \dots, v_{n+m+r}) \in V$

$$\|B(t, y_1, y_2, \dots, y_{n+m+r}) - B(s, v_1, v_2, \dots, v_{n+m+r})\|_{\mathcal{Z}} \le C\left(|t-s|^\delta + \sum_{k=1}^{n+m+r} \|y_k - v_k\|_\gamma\right). \quad (3)$$

A function $z \in C((t_0,t_1]; D_A)$, such that $J^{\alpha-m}z \in C^{m-1}([t_0,t_1];\mathcal{Z}) \cap C^m((t_0,t_1];\mathcal{Z})$, $D^{\alpha_1}z$, $D^{\alpha_2}z, \dots, D^{\alpha_n}z \in C([t_0,t_1];\mathcal{Z})$, is called a solution of Cauchy type problem (1), (2) on a segment $[t_0,t_1]$, if it satisfies conditions (2), for all $t \in (t_0,t_1]$ $(D^{\alpha_1}z(t), D^{\alpha_2}z(t), \dots, D^{\alpha-1}z(t)) \in U$ and (1) holds.

Theorem 3. Let $\alpha > 0$, $\alpha_1 < \alpha_2 < \dots < \alpha_n < \alpha - 1$, $-A \in \mathcal{A}_\alpha(\theta_0, 0)$, $0 \in \rho(A)$, a map $B : U \to \mathcal{Z}$ satisfy condition (3), $\gamma > 1 - 1/\alpha$, $(t_0, 0, \dots, 0, z_{m^{**}}, z_{m^{**}+1}, \dots, z_{m-1}) \in U$, $z_k \in \mathcal{Z}_{1+\gamma}$, $k = m^{**}, m^{**}+1, \dots, m-1$. Then for some $t_1 > t_0$ there exists a unique solution of problem (1), (2) on $[t_0,t_1]$.

Application. Let $\Omega \subset \mathbb{R}^3$ be a bounded region with a smooth boundary $\partial\Omega$, $\alpha \in (1,2)$, then $m^{**} = 0$, or $m^{**} = 1$. Consider the initial boundary value problem

$$D^{\alpha-m+k}v(\xi, t_0) = v_k(\xi), \quad k = m^{**}, 1, \quad \xi \in \Omega, \quad (4)$$

$$v(\xi, t) = 0, \quad \xi \in \partial\Omega,\ t > t_0, \quad (5)$$

for an equation

$$D_t^\alpha v(\xi,t) = \Delta v(\xi,t) + \sum_{l=1}^{n} D_t^{\alpha_l} v(\xi,t) \sum_{i=1}^{3} \frac{\partial}{\partial \xi_i} D_t^{\alpha_l} v(\xi,t) +$$

$$+ \sum_{k=-r}^{m-1} D_t^{\alpha-m+k} v(\xi,t) \sum_{i=1}^{3} \frac{\partial}{\partial \xi_i} D_t^{\alpha-m+k} v(\xi,t), \quad \xi \in \Omega,\ t > t_0, \quad (6)$$

where $D_t^\beta v$ are the partial fractional derivatives for $\beta > 0$ or integrals for $\beta \le 0$ with respect to t. Take $\mathcal{Z} = L_2(\Omega)$, $A = -\Delta$, $D_A = H^2(\Omega) \cap H_0^1(\Omega)$, then $-A \in \mathcal{A}_\alpha(\theta_0, 0)$ for any $\theta_0 \in (\pi/2, \pi)$, since $\alpha \in (1,2)$ (see Theorem 4 in [5] for $n = 0$, $P_0 \equiv 1$, $p = 1$, $Q_1(\lambda) = \lambda$). Reasoning as in [1, §8.8.3], obtain that the nonlinear operators of the form $f_l(v) = D_t^{\alpha_l} v \sum\limits_{i=1}^{3} \frac{\partial}{\partial \xi_i} D_t^{\alpha_l} v$ satisfy the conditions of Theorem 3 at $\gamma > 3/4$. Therefore, for all $v_k \in D_{A^{1+\gamma}}$, $k = m^{**}, 1$, there exists a unique solution of problem (4)–(6) in $\Omega \times [t_0, t_1]$ with some $t_1 > t_0$.

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