Frequency-domain conditions for the exponential stability of compound cocycles generated by delay equations and effective dimension estimates of global attractors

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Introduction.

Usually, dimension estimates for global attractors are obtained via the Liouville trace formula (see S. Zelik [9]), possibly with the use of adapted metrics (see N.V. Kuznetsov and V. Reitmann [7]). In our paper [3], we showed that this approach provides rough bounds and does not allow to obtain effective estimates in the case of delay equations. This is caused not only by the non-self-adjointness of arising operators, but also by the specificity of delay equations where we deal with boundary perturbations.

It is also shown in [3] that a resolution to the problem can be given in the case of scalar equations with monotone feedback by combining results of J. Mallet-Paret and R.D. Nussbaum [8] with the ergodic variational principle and Poincaré-Bendixson trichotomy satisfied in these equations. However, there are many scalar equations, not to mention systems of equations, that exhibit chaotic behavior such as the Mackey-Glass equation or periodically forced Suarez-Schopf oscillator [2]. These models go beyond the described approach.

Main results. We are going to discuss our recent result [1] concerned with cocycles generated by the following class of nonautonomous delay equations in $\mathbb{R}^n$ over a semiflow π on a complete metric space $\mathcal{P}$:

$$\dot{x}(t) = \widetilde{A}x_t + \widetilde{B}F'(\pi^t(\mathfrak{p}))Cx_t, \tag{1}$$

where $\widetilde{A}\colon C([-\tau,0];\mathbb{R}^n) \to \mathbb{R}^n$, $C\colon C([-\tau,0];\mathbb{R}^n) \to \mathbb{M}$ are bounded linear operators; $\widetilde{B}\colon \mathbb{U} \to \mathbb{R}^n$ is a linear operator and $F'\colon \mathcal{P} \to \mathcal{L}(\mathbb{M};\mathbb{U})$ is a continuous mapping such that for some $\Lambda > 0$ we have

$$\|F'(\mathfrak{p})\|_{\mathcal{L}(\mathbb{M};\mathbb{U})} \leq \Lambda \text{ for all } \mathfrak{p} \in \mathcal{P}. \tag{2}$$

Here $\mathbb{U}$ and $\mathbb{M}$ are finite-dimensional Euclidean spaces.

Equations such as (1) arise as linearizations of nonlinear delay equations.

It can be shown that (1) generates a cocycle Ξ over $(\mathcal{P},\pi)$ in the Hilbert space $\mathbb{H} = L_2([-\tau,0];\mu;\mathbb{R}^n)$, where μ is the sum of the Lebesgue measure on $[-\tau,0]$ and the δ-measure concentrated at 0 (see [3]). We study the m-fold compound cocycle Ξ_m given by the multiplicative extension of Ξ to the m-fold exterior power $\mathbb{H}^{\wedge m}$ of $\mathbb{H}$.

To the operator $\widetilde{A}$ from (1) there corresponds an operator A in $\mathbb{H}$ which generates an eventually compact C_0-semigroup G. It can be shown that the m-fold multiplicative extension $G^{\wedge m}$ of G onto $\mathbb{H}^{\wedge m}$ is an eventually compact C_0-semigroup in $\mathbb{H}^{\wedge m}$. Its generator $A^{[\wedge m]}$ is called the *additive (antisymmetric) compound* of A. Using the Spectral Mapping Theorem for Semigroups, one can describe the spectrum of $A^{[\wedge m]}$ through the spectrum of A.

We provide conditions for the uniform exponential stability or, more generally, for the existence of gaps in the Sacker-Sell spectrum of Ξ_m by considering it as a perturbation of $G^{\wedge m}$. On the infinitesimal level, we have that the generator of Ξ_m is given by a nonautonomous boundary perturbation of $A^{[\wedge m]}$. Such perturbations can be described via unbounded in $\mathbb{H}$ quadratic constraints leading to the associated infinite-horizon quadratic regulator problem posed for a proper control system. The latter is resolved via the Frequency Theorem developed in our adjacent work [5] (see also [6]). As a consequence, we obtain frequency-domain conditions which

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guarantee the existence of a proper (indefinite) bounded quadratic Lyapunov-like functional. Such functionals can be used to obtain the desired dichotomy properties for Ξ_m (see [4]).

Our frequency-domain conditions are given by strict frequency inequalities involving resolvents of additive compound operators $A^{[\wedge m]}$. Computing such operators requires solving a first order PDE with boundary conditions containing both partial derivatives and delays that makes it hard to deal with it analytically. However, verification of frequency inequalities reduces to the optimization of a Rayleigh quotient that is feasible to numerical investigation and reflects the computational complexity of the problem.

References:

[1] M.M. Anikushin. Spectral comparison of compound cocycles generated by delay equations in Hilbert spaces // arXiv preprint, arXiv:2302.02537 (2023)

[2] M.M. Anikushin, A.O. Romanov. Hidden and unstable periodic orbits as a result of homoclinic bifurcations in the Suarez-Schopf delayed oscillator and the irregularity of ENSO // Phys. D: Nonlinear Phenom. 445 (2023) 133653

[3] M.M. Anikushin Nonlinear semigroups for delay equations in Hilbert spaces, inertial manifolds and dimension estimates // Differ. Uravn. Protsessy Upravl. 4 (2022)

[4] M.M. Anikushin. Inertial manifolds and foliations for asymptotically compact cocycles in Banach spaces // arXiv preprint, arXiv:2012.03821v3 (2022)

[5] M.M. Anikushin. Frequency theorem and inertial manifolds for neutral delay equations // arXiv preprint, arXiv:2003.12499v4 (2022)

[6] M.M. Anikushin. Frequency theorem for parabolic equations and its relation to inertial manifolds theory // J. Math. Anal. Appl. 505:1 (2022) 125454

[7] N.V. Kuznetsov, V. Reitmann. Attractor Dimension Estimates for Dynamical Systems: Theory and Computation. Switzerland: Springer International Publishing AG (2020)

[8] J. Mallet-Paret, R.D. Nussbaum. Tensor products, positive linear operators, and delay-differential equations // J. Dyn. Diff. Equat., 25 (2013) 843–905

[9] S. Zelik. Attractors. Then and now // arXiv preprint, arXiv:2208.12101v1 (2022)