Inversion of the Pompeiu transform associated to spherical means

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Keywords: distributions; convolution equations; Pompeiu transform; inversion formulas.

MSC2010 codes: 44A35, 46F12, 53C35, 45E10

Let $n \geq 2$, $\mathcal{D}'(\mathbb{R}^n)$ be the space of distributions on $\mathbb{R}^n$, σ_r be the surface delta function concentrated on the sphere $S_r = \{x \in \mathbb{R}^n : |x| = r\}$. The problem of L. Zalcman on reconstruction of a distribution $f \in \mathcal{D}'(\mathbb{R}^n)$ by known convolutions $f * \sigma_{r_1}$ and $f * \sigma_{r_2}$ is studied (see [1], Sect. 8). The result obtained (see Theorem 2 below) significantly simplifies known procedures for recovering a function f from given spherical means $f * \sigma_{r_1}$ and $f * \sigma_{r_2}$.

Let $r > 0$ be fixed and λr be an arbitrary positive zero of the Bessel function J_0. Then, for any $k \in \mathbb{Z}$, the function $J_k(\lambda\rho)e^{ik\varphi}$ (ρ, φ are the polar coordinates in $\mathbb{R}^2$) has zero integrals over all circles of radius r in $\mathbb{R}^2$ (see [2], Sect. C). Similar examples related to the zeros of the Bessel function $J_{n/2-1}$ can also be constructed for spherical means in $\mathbb{R}^n$ for $n \geq 2$. This shown that knowing the averages of a function f over all spheres of the same radius is not enough to uniquely reconstruct f. Subsequently, the class of functions $f \in C(\mathbb{R}^n)$ that have zero integrals over all spheres of fixed radius in $\mathbb{R}^n$ was studied by many authors (see [3]–[6] and the references to these works). A well-known result in this direction is the following analogue of Delsarte's famous two-radius theorem for harmonic functions.

Theorem 1 ([1], [3]). Let $r_1, r_2 \in (0, +\infty)$, $\Upsilon_n = \{\gamma_1, \gamma_2, \ldots\}$ be the sequence of all positive zeros of the function $J_{n/2-1}$ numbered in ascending order, M_n be the set of numbers of the form α/β, where $\alpha, \beta \in \Upsilon_n$.

1) If $r_1/r_2 \notin M_n$, $f \in C(\mathbb{R}^n)$ and

$$\int_{|x-y|=r_1} f(x)d\sigma(x) = \int_{|x-y|=r_2} f(x)d\sigma(x) = 0, \quad y \in \mathbb{R}^n \tag{1}$$

($d\sigma$ is the area element), then $f = 0$.

2) If $r_1/r_2 \in M_n$, then there exists a nonzero real analytic function $f : \mathbb{R}^n \to \mathbb{C}$ satisfying the relations in (1).

In terms of convolutions Theorem 1 means that the Pompeiu transform

$$\mathcal{P}f = (f * \sigma_{r_1}, f * \sigma_{r_2}), \quad f \in C(\mathbb{R}^n)$$

is injective if and only if $r_1/r_2 \notin M_n$. Here we present a new inversion formula for the operator $\mathcal{P}$ under the condition $r_1/r_2 \notin M_n$.

Let $\mathcal{E}'(\mathbb{R}^n)$ be the space of compactly supported distributions on $\mathbb{R}^n$, $\mathcal{E}'_\natural(\mathbb{R}^n)$ be the space of radial (invariant under rotations of the space $\mathbb{R}^n$) distributions in $\mathcal{E}'(\mathbb{R}^n)$. If $T_1, T_2 \in \mathcal{D}'(\mathbb{R}^n)$ and at least one of these distributions has compact support then their convolution $T_1 * T_2$ is a distribution in $\mathcal{D}'(\mathbb{R}^n)$ acting according to the rule

$$\langle T_1 * T_2, \varphi\rangle = \langle T_2(y), \langle T_1(x), \varphi(x+y)\rangle\rangle, \quad \varphi \in \mathcal{D}(\mathbb{R}^n),$$

where $\mathcal{D}(\mathbb{R}^n)$ is the space of finite infinitely differentiable functions on $\mathbb{R}^n$. The spherical transform $\widetilde{T}$ of a distribution $T \in \mathcal{E}'_\natural(\mathbb{R}^n)$ is defined by

$$\widetilde{T}(z) = 2^{\frac{n}{2}-1}\Gamma\left(\frac{n}{2}\right)\langle T, \mathbf{I}_{\frac{n}{2}-1}(z|x|)\rangle, \quad z \in \mathbb{C},$$

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where

$$\mathbf{I}_\nu(z) = \frac{J_\nu(z)}{z^\nu}, \quad \nu \in \mathbb{C}.$$

We note that $\widetilde{T}$ is an even entire function of exponential type and the Fourier transform $\widehat{T}$ is expressed in terms of $\widetilde{T}$ by

$$\widehat{T}(\zeta) = \widetilde{T}\big(\sqrt{\zeta_1^2 + \ldots + \zeta_n^2}\big), \quad \zeta \in \mathbb{C}^n.$$

The set of all zeros of the function $\widetilde{T}$ that lie in the half-plane $\operatorname{Re} z \geq 0$ and do not belong to the negative part of the imaginary axis will be denoted by $\mathcal{Z}_+(\widetilde{T})$.

Using the well-known properties of the zeros of the Bessel functions one can obtain the corresponding information about the set $\mathcal{Z}_+(\widetilde{\sigma}_r)$. In particular, all the zeros of $\widetilde{\sigma}_r$ are simple, belong to $\mathbb{R}\backslash\{0\}$ and

$$\mathcal{Z}_+(\widetilde{\sigma}_r) = \left\{\frac{\gamma_1}{r}, \frac{\gamma_2}{r}, \ldots\right\}.$$

In addition, since the functions $J_{\frac{n}{2}-1}$ and $J_{\frac{n}{2}}$ do not have common zeros on $\mathbb{R}\backslash\{0\}$, the function

$$\sigma_r^\lambda(x) = -\frac{1}{r\lambda^2} \frac{\mathbf{I}_{\frac{n}{2}-1}(\lambda|x|)}{\mathbf{I}_{\frac{n}{2}}(\lambda r)} \chi_r(x), \quad \lambda \in \mathcal{Z}_+(\widetilde{\sigma}_r),$$

is well defined, where χ_r is the indicator of the ball $B_r = \{x \in \mathbb{R}^n : |x| < r\}$.

Let

$$P_r(z) = \prod_{j=1}^{[\frac{n+5}{4}]} \left(z - \left(\frac{\gamma_j}{r}\right)^2\right), \quad \Omega_r = P_r(\Delta)\sigma_r,$$

where Δ is the Laplace operator. Then, by virtue of the formula

$$\widetilde{p(\Delta)T}(z) = p(-z^2)\widetilde{T}(z) \quad (p \text{ is an algebraic polynomial}),$$

we have

$$\widetilde{\Omega}_r(z) = P_r(-z^2)\widetilde{\sigma}_r(z),$$

$$\mathcal{Z}_+\big(\widetilde{\Omega}_r\big) = \left\{\frac{\gamma_1}{r}, \frac{\gamma_2}{r}, \ldots\right\} \cup \left\{\frac{i\gamma_1}{r}, \frac{i\gamma_2}{r}, \ldots, \frac{i\gamma_m}{r}\right\},$$

and all zeros of $\widetilde{\Omega}_r$ are simple. In addition,

$$\mathcal{Z}_+\big(\widetilde{\Omega}_{r_1}\big) \cap \mathcal{Z}_+\big(\widetilde{\Omega}_{r_2}\big) = \varnothing \quad \Leftrightarrow \quad \frac{r_1}{r_2} \notin M_n.$$

For $\lambda \in \mathcal{Z}_+\big(\widetilde{\Omega}_r\big)$, we set

$$\Omega_r^\lambda = P_r(\Delta)\sigma_r^\lambda \quad \text{if} \quad \lambda \in \mathcal{Z}_+(\widetilde{\sigma}_r),$$

and

$$\Omega_r^\lambda = Q_{r,\lambda}(\Delta)\sigma_r \quad \text{if} \quad P_r(-\lambda^2) = 0,$$

where

$$Q_{r,\lambda}(z) = -\frac{P_r(z)}{z + \lambda^2}.$$

Theorem 2. Let $\frac{r_1}{r_2} \notin M_n$, $f \in \mathcal{D}'(\mathbb{R}^n)$, $n \geq 2$. Then

$$f = \sum_{\lambda \in \mathcal{Z}_+(\widetilde{\Omega}_{r_1})} \sum_{\mu \in \mathcal{Z}_+(\widetilde{\Omega}_{r_2})} \frac{4\lambda\mu}{(\lambda^2 - \mu^2)\widetilde{\Omega}_{r_1}'(\lambda)\widetilde{\Omega}_{r_2}'(\mu)} \Big(P_{r_2}(\Delta)\big((f * \sigma_{r_2}) * \Omega_{r_1}^\lambda\big) -$$

$$-P_{r_1}(\Delta)\Big((f*\sigma_{r_1})*\Omega^{\mu}_{r_2}\Big)\Big), \tag{2}$$

where the series in (2) converges unconditionally in the space $\mathcal{D}'(\mathbb{R}^n)$.

Equality (2) reconstruct an arbitrary distribution f from its known convolutions $f*\sigma_{r_1}$ and $f*\sigma_{r_2}$ (see formulas above). For other results related to the inversion of the spherical mean operator, see [6], [7].

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