Resolving Families of Operators and Fractional Multi-Term Quasilinear Equations

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Let $A_1, A_2, \dots, A_{m-1}, B_1, B_2, \dots, B_n, C_1, C_2, \dots, C_r$ be closed linear operators in a Banach space $\mathcal{Z}$ with domains $D_{A_1}, D_{A_2}, \dots, D_{A_{m-1}}$, $D_{B_1}, D_{B_2}, \dots, D_{B_n}$, $D_{C_1}, D_{C_2}, \dots, D_{C_r}$ respectively, $m-1 < \alpha \leq m \in \mathbb{N}$, $n, r, \varrho, q \in \mathbb{N} \cup \{0\}$, Z be an open subset in $\mathbb{R} \times \mathcal{Z}^{m+\varrho+q}$, $B \in C(Z; \mathcal{Z})$. Consider the quasilinear multi-term fractional equation

$$D^\alpha z(t) = \sum_{j=1}^{m-1} A_j D^{\alpha-m+j} z(t) + \sum_{l=1}^{n} B_l D^{\alpha_l} z(t) + \sum_{s=1}^{r} C_s J^{\beta_s} z(t) + \\ + F(t, D^{\alpha-m-\varrho} z(t), \dots, D^{\alpha-1} z(t), D^{\gamma_1} z(t), D^{\gamma_2} z(t), \dots, D^{\gamma_q} z(t)). \tag{1}$$

Here D_t^δ is the Riemann — Liouville derivative with $\delta > 0$ and the Riemann — Liouville integral with $\gamma < 0$, $0 < \alpha_1 < \alpha_2 < \dots < \alpha_n < \alpha$, $m_l - 1 < \alpha_l \leq m_l \in \mathbb{Z}$, $\alpha_l - m_l \neq \alpha - m$, $l = 1, 2, \dots, n$, $\gamma_1 < \gamma_2 < \dots < \gamma_q < \alpha$, $n_i - 1 < \gamma_i \leq n_i \in \mathbb{Z}$, $\gamma_i - n_i \neq \alpha - m$, $i = 1, 2, \dots, q$. Some γ_i may be negative. Let us define $\mu^* := m^*(\alpha, \alpha_1, \alpha_2, \dots, \alpha_n, \gamma_1 + 1, \gamma_2 + 1, \dots, \gamma_q + 1)$ (see [1]), $\mu_0^* := \max\{\mu^*, 0\}$, so for solving the Cauchy type problem

$$D^{\alpha-m+k} z(t_0) = z_k, \quad k = 0, 1, \dots, m-1, \tag{2}$$

for equation (1) conditions are met

$$D^{\alpha-m+k} z(t_0) = 0, \quad k = -r, -r+1, \dots, \mu_0^* - 1;$$

$$D^{\alpha_l - m_l + k} z(t_0) = 0, \quad k = 0, 1, \dots, m_l - 1, \ l = 1, 2, \dots, n;$$

$$D^{\gamma_i - n_i + k} z(t_0) = 0, \quad k = 0, 1, \dots, n_i, \ i = 1, 2, \dots, q.$$

Define by $\mathcal{L}(\mathcal{Z})$ the Banach space of all linear bounded operators on $\mathcal{Z}$,

$$\mathcal{D} := \bigcap_{j=1}^{m-1} D_{A_j} \cap \bigcap_{l=1}^{n} D_{B_l} \cap \bigcap_{s=1}^{r} D_{C_s}, \quad \|\cdot\|_{\mathcal{D}} = \sum_{j=1}^{m-1} \|\cdot\|_{D_{A_j}} + \sum_{l=1}^{n} \|\cdot\|_{D_{B_l}} + \sum_{s=r}^{m-1} \|\cdot\|_{D_{C_s}}.$$

A solution to problem (1), (2) on $(t_0, t_1]$ is a function $z : (t_0, t_1] \to \mathcal{D}$, such that $J^{m-\alpha} z \in C^m((t_0, t_1]; \mathcal{Z}) \cap C^{m-1}([t_0, t_1]; \mathcal{Z})$, $D^{\alpha-m+j} z \in C((t_0, t_1]; D_{A_j})$, $j = 1, 2, \dots, m-1$, $D^{\alpha_l} z \in C((t_0, t_1]; D_{B_l})$, $l = 1, 2, \dots, n$, $D^{\gamma_i} z \in C([t_0, t_1]; \mathcal{Z})$, $i = 1, 2, \dots, q$, condition (2) are satisfied, inclusion $(t, D^{\alpha-m-\varrho} z(t), D^{\alpha-m-\varrho+1} z(t), \dots, D^{\alpha-1} z(t), D^{\gamma_1} z(t), D^{\gamma_2} z(t), \dots, D^{\gamma_q} z(t)) \in Z$ for $t \in [t_0, t_1]$ and equality (1) for $t \in (t_0, t_1]$ hold.

Definition 1. A tuple of operators $(A_1, A_2, \dots, A_{m-1}, B_1, B_2, \dots, B_n, C_1, C_2, \dots, C_r)$, which are linear and closed in a Banach space $\mathcal{Z}$, belongs to the class $\mathcal{A}_\alpha^{n,r}(\theta_0, a_0)$ for some $\theta_0 \in (\pi/2, \pi)$, $a_0 \geq 0$, if

(i) $\mathcal{D}$ is dense in $\mathcal{Z}$;

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(ii) for all $\lambda \in S_{\theta_0,a_0} := \{\mu \in \mathbb{C} : |\arg(\mu - a_0)| < \theta_0\}$, $p = 0, 1, \dots, m-1$ we have

$$R_\lambda \cdot \left(I - \sum_{j=p+1}^{m-1} \lambda^{j-m} A_j\right) \in \mathcal{L}(\mathcal{Z});$$

(iii) for any $\theta \in (\pi/2, \theta_0)$, $a > a_0$, there exists such a $K(\theta, a)$, that for all $\lambda \in S_{\theta,a}$, $p = 0, 1, \dots, m-1$ we have

$$\left\| R_\lambda \cdot \left(I - \sum_{j=p+1}^{m-1} \lambda^{j-m} A_j\right) \right\|_{\mathcal{L}(\mathcal{Z})} \le \frac{K(\theta, a)}{|\lambda - a||\lambda|^{\alpha-1}}.$$

Definition 2. Let $p \in \{0, 1, \dots, m-1\}$; a strongly continuous family of operators $\{S_p(t) \in \mathcal{L}(\mathcal{Z}) : t > 0\}$ is called *p-resolving* for equation (1), if next conditions are satisfied:

(i) for $t > 0$ $S_p(t)[D_{A_j}] \subset D_{A_j}$, $S_p(t)A_j x = A_j S_p(t)x$ for all $x \in D_{A_j}$, $j = 1, 2, \dots, m-1$; $S_p(t)[D_{B_l}] \subset D_{B_l}$, $S_p(t)B_l x = B_l S_p(t)x$ for all $x \in D_{B_l}$; $S_p(t)[D_{C_s}] \subset D_{C_s}$, $S_p(t)C_s x = C_s S_p(t)x$ for all $x \in D_{C_s}$;

(ii) for every $z_p \in \mathcal{D}$ $S_p(t)z_p$ is a solution of linear ($B \equiv 0$) problem (1), (2) with $z_l = 0$ for every $l \in \{0, 1, \dots, m-1\} \setminus \{p\}$.

A p-resolving family of operators for $p \in \{0, 1, \dots, m-1\}$ is called *analytic*, if it has the analytic extension to a sector $\Sigma_{\psi_0} := \{t \in \mathbb{C} : |\arg t| < \psi_0, t \ne 0\}$ for some $\psi_0 \in (0, \pi/2]$. An analytic p-resolving family of operators $\{S_p(t) \in \mathcal{L}(\mathcal{Z}) : t > 0\}$ has a type (ψ_0, a_0) for some $\psi_0 \in (0, \pi/2]$, $a_0 \in \mathbb{R}$, if for all $\psi \in (0, \psi_0)$, $a > a_0$ there exists such a $C(\psi, a)$, that for all $t \in \Sigma_\psi$ the inequality $\|S_p(t)\|_{\mathcal{L}(\mathcal{Z})} \le C(\psi, a)|t|^{\alpha-m+p}e^{a\mathrm{Re}\, t}$ is satisfied.

Theorem 1. Let $m - 1 < \alpha \le m \in \mathbb{N}$, $\alpha_1 < \alpha_2 < \dots < \alpha_n < \alpha$, $m_l - 1 < \alpha_l \le m_l \in \mathbb{N}$, $\alpha_l - m_l \ne \alpha - m$, $m^* := m^*(\alpha, \alpha_1, \alpha_2, \dots, \alpha_n)$, $\beta_1 > \beta_2 > \dots > \beta_r \ge 0$, A_j, $j = 1, 2, \dots, m-1$, B_l, $l = 1, 2, \dots, n$, C_s, $s = 1, 2, \dots, r$, are linear and closed operators, $\mathcal{D}$ dense $\mathcal{Z}$. Then there are p-resolving families of operators $\{S_p(t) \in \mathcal{L}(\mathcal{Z}) : t > 0\}$ of the type (θ_0, a_0) for equation (1) for all $p = m^*, m^*+1, \dots, m-1$, if and only if $(A_1, A_2, \dots, A_{m-1}, B_1, B_2, \dots, B_n, C_1, C_2, \dots, C_r) \in \mathcal{A}_\alpha^{n,r}(\theta_0, a_0)$. Moreover,

$$S_p(t) = Z_p(t) := \frac{1}{2\pi i} \int_\Gamma \lambda^{m-1-p} R_\lambda \left(I - \sum_{j=p+1}^{m-1} \lambda^{j-m} A_j\right) e^{\lambda t} d\lambda, \quad p = m^*, m^*+1, \dots, m-1,$$

where $\Gamma := \Gamma^+ \cup \Gamma^- \cup \Gamma^0$, $\Gamma^0 := \{\lambda \in \mathbb{C} : \lambda = a + r_0 e^{i\varphi}, \varphi \in (-\theta, \theta)\}$, $\Gamma^\pm := \{\lambda \in \mathbb{C} : \lambda = a + r_0 e^{\pm i\theta}, r \in [r_0, \infty)\}$, $\theta \in (\pi/2, \theta_0)$, $a > a_0$, $r_0 > 0$.

Theorem 2. Let $m - 1 < \alpha \le m \in \mathbb{N}$, $0 < \alpha_1 < \alpha_2 < \dots < \alpha_n < \alpha$, $m_l - 1 < \alpha_l \le m_l \in \mathbb{N}$, $\alpha_l - m_l \ne \alpha - m$, $\gamma_1 < \gamma_2 < \dots < \gamma_q < \alpha - 1$, $n_i - 1 < \gamma_i \le n_i \in \mathbb{Z}$, $\gamma_i - n_i \ne \alpha - m$, $i = 1, 2, \dots, q$, $(A_1, A_2, \dots, A_{m-1}, B_1, B_2, \dots, B_n, C_1, C_2, \dots, C_r) \in \mathcal{A}_\alpha^{n,r}(\theta_0, a_0)$, $z_k \in \mathcal{D}$, $k = \mu_0^*, \mu_0^* + 1, \dots, m-1$, Z be open in $\mathbb{R} \times \mathcal{Z}^{m+\varrho+q}$, $(t_0, 0, 0, \dots, 0, z_{\mu_0^*}, z_{\mu_0^*+1}, \dots, z_{m-1}, 0, 0, \dots, 0) \in Z$, the mapping $B \in C(Z; \mathcal{D})$ be locally Lipschitz continuous with respect to the phase variables. Then there exists $t_1 > t_0$, such that problem (1), (2) has a unique solution on $(t_0, t_1]$.

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