Some Chebyshev type Inequalities for Riemann-Liouville type integral operator

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Introduction. In this work, some weighted Chebyshev type inequalities are obtained by using a more general fractional integral operator, than the Riemann-Liouvile one.

Let $0 \leq a < b < \infty$, f and g be two integrable functions on $[a,b]$ and

$$T(f,g) := \int_a^b f(t)g(t)dt - \frac{1}{(b-a)}\left(\int_a^b f(x)\,dx\right)\left(\int_a^b g(x)dx\right). \tag{1}$$

The Chebyshev functional (1) has many applications in numerical quadrature, transform theory, probability, study of existence of solutions of differential equations and in statistical problems.

In the following we give some basic definitions.

Definition 1. For $1 \leq p < \infty$ we denote by $L_p := L_p(0,\infty)$ the set of all Lebesgue measurable functions f such that

$$\|f\|_p = \left(\int_0^\infty |f(x)|^p dx\right)^{\frac{1}{p}} < \infty.$$

Definition 2. The Riemann-Liouville fractional integral operators of order $\alpha \geq 0$ of function $f(x) \in L_1[a,b], -\infty < a < b < +\infty$ are defined by

$$J_{a+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt,\ x > a, \tag{2}$$

$$J_{b-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)dt,\ \ x < b. \tag{3}$$

The following definition was introduced in [3].

Definition 3. Let $\alpha > 0, \beta \geq 1, 1 \leq p < \infty$ and the integral operator $\mathbf{K}_{u,v}^{\alpha,\beta}$ of the form

$$\mathbf{K}_{u,v}^{\alpha,\beta} f(x) = \frac{v(x)}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1}\left[\ln\left(\frac{x}{t}\right)\right]^{\beta-1} f(t)\,u(t)dt, \tag{4}$$

defined from L_p to L_p space,with locally integrable non-negative weight functions u and v on $(0,\infty)$.

Remark 1. If $v(x) = u(x) = 1, \beta = 1,$ the operator $\mathbf{K}_{1,1}^{\alpha,1}$ coincides with the classical Riemann-Liouville fractional integral operator.

The following theorem was proved in [2].

Theorem 1. Let f and g be two synchronous functions on $(0,\infty)$. Then for all $t > 0, \alpha > 0,$

$$J^{\alpha}(fg)(t) \geq \frac{\Gamma(\alpha+1)}{t^{\alpha}} J^{\alpha} f(t) J^{\alpha} g(t). \tag{5}$$

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The inequality (5) is reversed if the functions are asynchronous on $(0, \infty)$.

The following theorem was proved in [1].

Theorem 2. Let $\{f_i\}_{1 \leq i \leq n}$ be n positive increasing functions on $(0, \infty)$ then for all $x > 0, \alpha > 0$,

$$J^\alpha \left(\prod_{i=1}^{i=n} f_i \right)(x) \geq (J^\alpha(1)(x))^{(1-n)} \prod_{i=1}^{i=n} J^\alpha f_i(x). \tag{6}$$

To simplify we denote by $\mathbf{K} := \mathbf{K}_{u,v}^{\alpha,\beta}$, and $k(x,t) := (x-t)^{\alpha-1} \ln^{\beta-1}\left(\frac{x}{t}\right) \neq 0$, thus the integral operator in the inequality (4) takes the following form

$$\mathbf{K}f(x) = \frac{v(x)}{\Gamma(\alpha)} \int_0^x k(x,t)\, f(t)\, u(t) dt, \qquad x > 0. \tag{7}$$

Theorem 3. Let f, g be two synchronous functions on $(0, \infty)$, u and v locally integrable non-negative weight functions.Then

$$\mathbf{K}(fg)(x) \geq (\mathbf{K}(1))^{-1}\, \mathbf{K}f(x)\, \mathbf{K}g(x), \tag{8}$$

where $\mathbf{K}(1)(x) = \frac{v(x)}{\Gamma(\alpha)} \int_0^x k(x,t)\, u(t)\, dt.$

The inequality (8) is reversed if the functions are asynchronous on $(0, \infty)$.

Remark 2. By applying Theorem 3, for $v(x) = u(x) = 1, \beta = 1,$ we obtain Theorem 1.

Theorem 4. Let $\{f_i\}_{1 \leq i \leq n}$ be n positive increasing functions on $[0, \infty[$ u and v locally integrable non-negative weight functions, then for all $x > 0$

$$\mathbf{K} \left(\prod_{i=1}^{i=n} f_i \right)(x) \geq (\mathbf{K}(1)(x))^{(1-n)} \prod_{i=1}^{i=n} \mathbf{K} f_i(x). \tag{9}$$

Remark 3. By applying Theorem 4, for $v(x) = u(x) = 1, \beta = 1,$, we obtain Theorem 2.

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