On diagonal quantum channels

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Introduction. In quantum information theory, the notions of the channel and its capacity, giving a measure of ultimate information-processing performance of the channel, play a central role. For a comprehensive introduction to quantum channels, see [1]. Diagonal quantum channels have significant applications in communication and physics. There are some studies on different types of diagonal channels, for instance depolarizing channels [2, 3] and diagonal channels with constant Frobenius norm [4].

Definition 1. Quantum channel $\Phi : \mathbb{C}^{n\times n} \to \mathbb{C}^{n\times n}$ is called *diagonal*, if its representation with respect to an orthonormal basis β (constructed by the generalized Pauli matrices) is diagonal, i.e. $\Phi = \mathrm{diag}(1, a_1, a_2, ..., a_{n^2-1})$.

Theorem 1. ([5]) *For every diagonal quantum channel* Φ, *there is a collection of transition probabilities* $\{P_{kj}\}_{j=1}^{n}$, *i.e.* $P_{kj} \geq 0, \sum_{j=1}^{n} P_{kj} = 1$ *such that*

$$\Phi(|k\rangle\langle k|) = \sum_{j=1}^{n} P_{kj}|j\rangle\langle j| \qquad (k = 1, 2, ..., n).$$

Kraus representation for diagonal channel. Before we formulate the result of this section, we need to prove the following two lemmas [5].

Lemma 1. Let $\kappa = (x_1, x_2, ..., x_n)$ *where* x_i*'s are rows of* $n\times n$ *matrix* K, *then* $(K^* E_{ij} K)_{1\leq i,j\leq n} = (x_i^* x_j)_{1\leq i,j\leq n} = \kappa^*\kappa$.

Lemma 2. Let $\Phi : \mathbb{C}^{n\times n} \to \mathbb{C}^{n\times n}$ *be a quantum channel,* C_Φ *be its Choi matrix, and* $C_\Phi = R^* R$ *for some matrix* R. *If* κ_i*'s are rows of* R, *and* K_i*'s are associated matrices to* κ_i*'s in lemma* 1 $(1 \leq i \leq n^2)$ *then* $\{K_i\}_{i=1}^{n^2}$ *is a set of Kraus operators of* Φ.

Now we are in a position to assert the main result of this section:

Theorem 2. ([5]) *For hybrid depolarizing classical quantum channel*

$$\Phi = \mathrm{diag}(1, \underbrace{-p, ..., -p}_{N}, \underbrace{-p, ..., -p}_{N}, \underbrace{p, ..., p}_{n-1}),$$

Kraus operators can be determined in the following explicit form:

$$K_1 = \begin{pmatrix} \sqrt{a_0} & 0 & ... & 0 \\ 0 & \frac{b_0}{\sqrt{a_0}} & ... & 0 \\ ... & ... & ... & ... \\ 0 & 0 & ... & \frac{b_0}{\sqrt{a_0}} \end{pmatrix}, \quad K_2 = \begin{pmatrix} 0 & \sqrt{\frac{1-p}{n}} & ... & 0 \\ 0 & 0 & ... & 0 \\ ... & ... & ... & ... \\ 0 & 0 & ... & 0 \end{pmatrix},$$

$$..., K_n = \begin{pmatrix} 0 & 0 & ... & \sqrt{\frac{1-p}{n}} \\ 0 & 0 & ... & 0 \\ ... & ... & ... & ... \\ 0 & 0 & ... & 0 \end{pmatrix}, \quad K_{n+1} = \begin{pmatrix} 0 & 0 & ... & 0 \\ \sqrt{\frac{1-p}{n}} & 0 & ... & 0 \\ ... & ... & ... & ... \\ 0 & 0 & ... & 0 \end{pmatrix},$$

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$$K_{n+2} = \begin{pmatrix} 0 & 0 & ... & 0 \\ 0 & \sqrt{a_1} & ... & 0 \\ ... & ... & ... & ... \\ 0 & 0 & ... & \frac{b_1}{\sqrt{a_1}} \end{pmatrix}, ..., K_{2n} = \begin{pmatrix} 0 & 0 & ... & 0 \\ 0 & 0 & ... & \sqrt{\frac{1-p}{n}} \\ ... & ... & ... & ... \\ 0 & 0 & ... & 0 \end{pmatrix},$$

$$..., K_{n^2-1} = \begin{pmatrix} 0 & ... & 0 & 0 \\ 0 & ... & 0 & 0 \\ ... & ... & ... & ... \\ 0 & ... & \sqrt{\frac{1-p}{n}} & 0 \end{pmatrix}, \quad K_{n^2} = \begin{pmatrix} 0 & 0 & ... & 0 \\ 0 & 0 & ... & 0 \\ ... & ... & ... & ... \\ 0 & 0 & ... & \sqrt{a_{n-1}} \end{pmatrix},$$

where $a_m = (2p + \frac{1-p}{n})(1 + \frac{-p}{-pm+2p+\frac{1-p}{n}})$ *for* $m = 1, 2, ..., n-1$; $b_m = (2p + \frac{1-p}{n})(\frac{-p}{-pm+2p+\frac{1-p}{n}})$ *for* $m = 1, 2, ..., n-2$; $a_0 = p + \frac{1-p}{n}$, *and* $b_0 = -p$.

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