Statistical properties of synchronous soliton collisions

T. V. Tarasova [1], A. V. Slunyaev [2]

Keywords: soliton collisions, Korteweg-de Vries equation, statistical moments.

MSC2020 codes: 00A79, 35C08, 35Q53

The concept of soliton gas, soliton or integrable turbulence appeared shortly after the discovery of outstanding properties of solitons – isolated waves, maintained by a stable balance between the counteracting effects of nonlinearity and dispersion. They exhibit an exceptional stability which becomes apparent through elastic interactions with other solitons and quasilinear waves. The existence of solitons originates from the fact of integrability of a series of nonlinear equations, such as the Korteweg-de Vries equation or the Nonlinear Schrödinger equation, by the Inverse Scattering Transform. From this perspective, solitons correspond to waves of the discrete spectrum of the associated scattering problem, which do not disperse and represent the large-time asymptotic solution of the Cauchy problem for localized initial conditions. Solitons exist in various fields of physics: hydrodynamics, plasma and optics, and play a particular role in the dynamics of nonlinear waves. They are deeply intertwined with the problem of the emergence of rogue waves, represented by extreme deviations from the average wave amplitude.

For the description of the soliton gas dynamics kinetic equations were derived [1,2]. They characterize the transport of the soliton spectral density, but due to the violation of the wave linear superposition principle, do not provide information about the wave solution itself (which can be water surface displacement, intensity of electromagnetic fields, etc.). In particular, the questions about the probability distribution for wave amplitudes or about the values of the wave field statistical moments remain unanswered. Multisoliton solutions, which can be formally written in a closed form using the Inverse Scattering Transform or related methods for integrable equations, are very cumbersome, what makes their analytical and even numerical analysis difficult. The direct numerical simulation of evolution equations is commonly used to study the soliton gas evolution, which also becomes complicated in the case of a dense gas (i.e. when many solitons interact simultaneously) [3].

The focus of this study is made on the dynamics of soliton interactions governed by the classic Korteweg – de Vries (KdV) equation, which has the standard dimensionless form

$$u_t + 6uu_x + u_{xxx} = 0, \tag{1}$$

where the real functions $u(x,t)$ describes the wave field, the variable $x \in (-\infty, +\infty)$ serves as a space coordinate, $t \in (-\infty, +\infty)$ is the time. Its exact N-soliton solution $u_N(x,t)$ can be obtained via consecutive Darboux transformations, which allow a compact representation

$$u_N(x,t) = 2\frac{\partial^2}{\partial x^2} \ln W_N(\Psi_1, \Psi_2, ...\Psi_N). \tag{2}$$

Here $W(\cdot)$ denotes the Wronskian for N "seed" functions $\psi_{2s-1} = \cosh\theta_{2s-1}$, $\psi_{2s} = \sinh\theta_{2s}$ for integer $s \geq 1$, where the phases are $\theta_j = k_j(x - V_j t - x_j)$, $j = 1, 2, ..., N$. The parameters k_j specify the soliton amplitudes $A_j = 2k_j^2$ and velocities $V_j = 4k_j^2$, while the constants x_j are responsible for the respective positions of solitons at a given time. The solution (2) is always positive, $u_N(x,t) > 0$. The use of an ultra-high-precision procedure made it possible to compute

[1]National Research University Higher School of Economics, Russia, Nizhny Novgorod; Institute of Applied Physics RAS, Group for modeling of extreme wave phenomena in the ocean, Russia, Nizhny Novgorod. Email: tvtarasova.nn@gmail.com

[2]National Research University Higher School of Economics, International Laboratory of Dynamical Systems and Applications, Russia, Nizhny Novgorod; Institute of Applied Physics RAS, Group for modeling of extreme wave phenomena in the ocean, Russia, Nizhny Novgorod. Email: Slunyaev@ipfran.ru

the exact N-soliton solutions (2) when N is large [4] and to calculate their statistical moments $\mu_n(t) = \int_{-\infty}^{+\infty} u_N^n(x,t)dx$, $n \in \mathbb{N}$, with high accuracy.

In this work we present a general idea that dense ensembles of KdV-type solitons of the same sign can be considered as strongly-nonlinear / small-dispersion wave states, what allows to express the statistical moments in terms of the spectral parameters of the associated scattering problem. A particular case when dense soliton states can occur is synchronous multisoliton collisions (see Fig. 1), for which the reference locations of all the solitons at $t = 0$ coincide with the coordinate origin, $x_j = 0$, $j = 1, ..., N$. This property can be formalized through the following symmetry condition, $u_N(-x,-t) = u_N(x,t)$. The soliton amplitudes are set decaying exponentially, so that they form a geometric series with the ratio $d > 1$, $A_j = 1/d^{j-1}$, $j = 1, ..., N$.

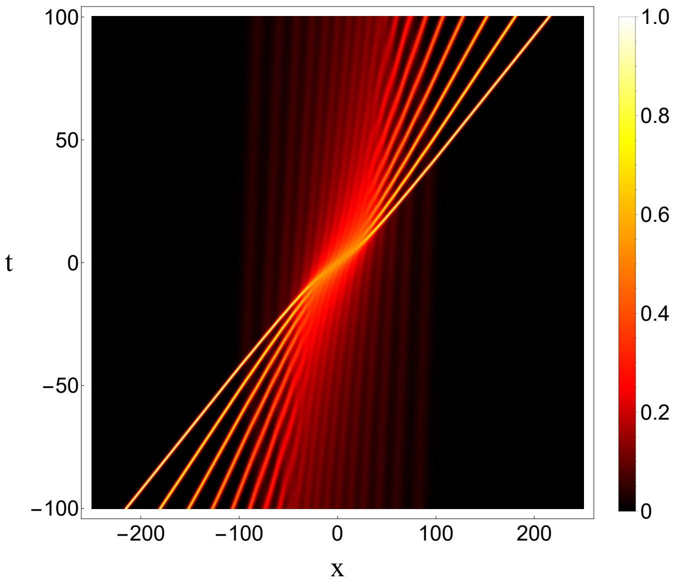

Figure 1: Interaction of $N = 20$ KdV solitons with $d = 1.2$

Time dependences of statistical moments are investigated for many-soliton solutions. It is shown that during the interaction of solitons of the same sign the wave field is effectively smoothed out. When d is sufficiently close to 1, and N is large, the statistical moments remain approximately constant within long time spans, when the solitons are located most densely. This quasi-stationary state is characterized by greatly reduced statistical moments and by the density of solitons close to some critical value. This state may be treated as the small-dispersion limit, what makes it possible to analytically estimate all high-order statistical moments. While the focus of the study is made on the Korteweg–de Vries equation and its modified version, a much broader applicability of the results to equations that support soliton-type solutions is discussed.

The work was supported by the Russian Science Foundation (Project No. 19-12-00253) and by the Foundation for the Advancement of Theoretical Physics and Mathematics “BASIS” (Project No. 22-1-2-42).

References

[1] V.E. Zakharov. *Kinetic equation for solitons.* // JETP 33, 538-541 (1971).

[2] G.A. El, A.M. Kamchatnov. *Kinetic equation for a dense soliton gas.* // Phys. Rev. Lett. 95, 204101 (2005).

[3] E.G. Didenkulova, A.V. Kokorina, A.V. Slunyaev. *Numerical simulation of soliton gas within the Korteweg-de Vries type equations.* // Computational Technologies 24(2), 52-66 (2019). doi: 10.25743/ICT.2019.24.2.005

[4] T.V. Tarasova, A.V. Slunyaev. *Properties of synchronous collisions of solitons in the Korteweg – de Vries equation.* // Communications in Nonlinear Science and Numerical Simulation 118, 107048 (2022). doi: 10.1016/j.cnsns.2022.107048

[5] A.V. Slunyaev, T.V. Tarasova. *Statistical properties of extreme soliton collisions.* // Chaos 32, 101102 (2022). doi: 10.1063/5.0120404