Analogues of Carleman's tangent approximation theorem

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Carleman's famous tangent approximation theorem derived in 1927 states that for every function $f \in C(\mathbb{R})$ and every error function ε, i.e. any positive function $\varepsilon \in C(\mathbb{R})$, there exists an entire function $g : \mathbb{C} \to \mathbb{C}$ such that

$$|f(t) - g(t)| < \varepsilon(t)$$

for all $t \in \mathbb{R}$ (see, for example, [1], Chap. 4, Sect. 3). Carleman's theorem has been further developed and refined in many papers (see bibliography in [1] and [2]). Carleman himself had already generalized his result by replacing $\mathbb{R}$ by more general curves and systems of curves in the complex plane. Many authors have studied, in connection with Carleman's theorem, approximation in combination with interpolation, as well as tangent approximation of smooth functions together with their derivatives. In addition, approximation with a certain rate of decrease of the error function was considered. Questions related to tangent and uniform approximation under restrictions on the growth of the approximating function were also studied. We also note the multidimensional analog of Carleman's theorem obtained by S. Sheinberg (see references in [1]). Carleman's theorem and its generalizations play an important role in the study of boundary properties of analytic functions and in the study of the distribution of their values (see [1], Chap. 4, Sect. 5).

The class of entire functions $g : \mathbb{C} \to \mathbb{C}$ coincides with the set of solutions of the differential equation

$$\left(\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2}\right) g = 0, \quad (x_1, x_2) \in \mathbb{R}^2.$$

In this regard, it is of interest to obtain analogues of Carleman's theorem in which the approximation is made by solutions of other linear partial differential equations in $\mathbb{R}^n$, $n \geq 2$, with constant coefficients. For the solutions of most of these equations, many important and useful properties of the class of entire functions are not fulfilled (for example, they as a rule do not form an algebra), which prevents them from obtaining analogues of Carleman's theorem by known methods. The simplest example is the class of eigenfunctions of the Laplace operator in $\mathbb{R}^2$, that is, the set of solutions of the equation

$$\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\right) g + \lambda g = 0, \quad (x_1, x_2) \in \mathbb{R}^2,$$

for $\lambda \neq 0$.

Here we study the approximation of continuous functions on rays in $\mathbb{R}^n$ by solutions of a multidimensional convolution equation of the form

$$g * T = 0, \tag{1}$$

where T is a given radial distribution with compact support in $\mathbb{R}^n$, $n \geq 2$. The theory of equations (1) originates in the work of the famous Romanian mathematician D. Pompeiu who considered the case when T is the indicator of a ball in $\mathbb{R}^n$ (see, e.g., [3], [4]). Equation (1)

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as well as its various analogues and generalizations have been intensively studied over the past fifty years by F. John, J. Delsarte, J.D. Smith, L. Zalcman, C.A. Berenstein, and others (see the overviews in [3], [4] and monographs [5]–[7] which provide extensive bibliographies). We note that with an appropriate choice of T they characterize such important classes of functions as functions with zero spherical (or ball) means, functions with the property of mean values from the theory of harmonic functions, and also solutions of elliptic differential equations of the form

$$p(\Delta)g = 0,$$

where Δ is the Laplace operator in $\mathbb{R}^n$, and p is an arbitrary algebraic polynomial other than the identical constant.

Everywhere in what follows, $\mathbb{R}^n$ is a Euclidean space of dimension $n \geq 2$. Denote by $\mathcal{D}'(\mathbb{R}^n)$ (respectively, $\mathcal{E}'(\mathbb{R}^n)$) the space of distributions (respectively, distributions with compact supports) in $\mathbb{R}^n$, $\mathcal{D}(\mathbb{R}^n)$ is the space of finite infinitely differentiable functions in $\mathbb{R}^n$, $\mathcal{E}(\mathbb{R}^n) = C^\infty(\mathbb{R}^n)$.

Let $T \in \mathcal{E}'(\mathbb{R}^n)$, $T \neq 0$. For every $f \in \mathcal{D}'(\mathbb{R}^n)$, the convolution $f * T$ is defined by the equality

$$\langle f * T, \varphi \rangle = \langle f_y, \langle T_x, \varphi(x + y) \rangle \rangle\,, \quad \varphi \in \mathcal{D}(\mathbb{R}^n),$$

as a distribution in $\mathcal{D}'(\mathbb{R}^n)$ (the index at the bottom of the distributions f and T means the action on the specified variable). A distribution of the class

$$\mathcal{D}'_T(\mathbb{R}^n) = \{f \in \mathcal{D}'(\mathbb{R}^n) : f * T = 0\}$$

is called mean periodic with respect to T.

Let $SO(n)$ be the rotation group of $\mathbb{R}^n$. A distribution $T \in \mathcal{E}'(\mathbb{R}^n)$ is called radial if it is invariant under the group $SO(n)$, i.e.

$$\langle T, \varphi(\tau x) \rangle = \langle T, \varphi(x) \rangle \quad \text{for all} \quad \varphi \in \mathcal{E}(\mathbb{R}^n), \quad \tau \in SO(n).$$

Denote by $\mathcal{E}'_\natural(\mathbb{R}^n)$ the set of all radial distributions $T \in \mathcal{E}'(\mathbb{R}^n)$. The simplest example of distribution in the class $\mathcal{E}'_\natural(\mathbb{R}^n)$ is the Dirac delta function δ_0 with support at zero, i.e.

$$\langle \delta_0, \varphi \rangle = \varphi(0), \quad \varphi \in \mathcal{E}(\mathbb{R}^n).$$

Let $\mathbb{S}^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$, $l \in \mathbb{S}^{n-1}$, and assume that $a \in \mathbb{R}^n$. As usual, the ray in $\mathbb{R}^n$ with vertex a in direction l is the set

$$L_{a,l} = \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n : x_j = a_j + tl_j,\, t \geq 0,\, j = 1, \ldots, n\}.$$

Theorem 1. Let $T \in \mathcal{E}'_\natural(\mathbb{R}^n)$ and

$$T \neq c\delta_0, \quad c \in \mathbb{C}\backslash\{0\}. \tag{2}$$

Suppose also that $a \in \mathbb{R}^n$, $l \in \mathbb{S}^{n-1}$, and $g \in C(L_{a,l})$. Then for every positive function $\varepsilon \in C(L_{a,l})$ there exists a function $f \in (\mathcal{E} \cap \mathcal{D}'_T)(\mathbb{R}^n)$ satisfying the conditions

(i) for every $x \in L_{a,l}$

$$|g(x) - f(x)| < \varepsilon(x); \tag{3}$$

(ii) there exists a function $w \in C^\infty(\mathbb{R}^2)$ such that

$$f(x) = w\big((x, l), \sqrt{|x|^2 - (x, l)^2}\,\big) \tag{4}$$

for all $x \in \mathbb{R}^n$.

By the arbitrariness of $\varepsilon \in C(L_{a,l})$, inequality (3) guarantees the tangent approximation of g on $L_{a,l}$ by smooth solutions to (1). Note that (4) means that the approximating function f is radial in any hyperplane orthogonal to the ray $L_{a,l}$.

Observe that (2) is necessary in Theorem 1. Indeed, if $T = c\delta_0$ for some $c \in \mathbb{C}\backslash\{0\}$ then the zero function is the only solution to (1); therefore, the claim of Theorem 1 fails.

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