On informational completeness of covariant positive operator-valued measures

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Let G be a locally compact Abelian group with the Haar measure ν. Due to the Pontraygin duality there exists the unique Haar measure $\hat{\nu}$ on the group of characters $\hat{G}$ such that the composition of forward and inverse Fourier transforms of functions from $\mathcal{H} = L^2(G)$ gives the identity transformation. Following to [1-2] define a projective unitary representation of the group $\mathfrak{G} = \hat{G} \times G$ in $\mathcal{H}$ by the formula

$$(U_{\chi,g}f)(a) = \chi(a)f(a+g),\ \chi \in \hat{G},\ g \in G,\ f \in \mathcal{H}.$$

Then, the following statement holds true.

Theorem 1 [2]. Let f be a cyclic vector for the representation $(U_{\chi,g})$. Then,

$$\mathfrak{M}(B) = \int\limits_{B} |U_{\chi,g}f\rangle \langle U_{\chi,g}f|\, d\hat{\nu}(\chi)d\nu(g),\ B \subset \mathfrak{G},$$

is a positive operator-valued measure (POVM) on the space $\mathfrak{G}$.

Any POVM $\mathfrak{M}$ determines an affine map Φ from the set $\mathfrak{S}(\mathcal{H})$ consisting of all states (positive unit trace operators) in $\mathcal{H}$ to the set $\Pi(\mathfrak{G})$ of all probability distributions on $\mathfrak{G}$ defined by the formula

$$\Phi(\rho)[B] = Tr(\rho\mathfrak{M}(B))$$

for measurable $B \subset \mathfrak{G}$. The map Φ is known as a measurement channel. We call a POVM $\mathfrak{M}$ informationally complete if given a probability distribution $\pi \in \Phi(\mathfrak{S}(H))$ there exists the unique state ρ such that $\Phi(\rho) = \pi$.

We introduced a family of contractions

$$T_{\chi,g} = \int\limits_{\hat{G}\times G} \chi'(g)\overline{\chi(g')}d\hat{\nu}(\chi')d\nu(g'),\ \chi \in \hat{G},\ g \in G,$$

and proved that [3] there is a family of complex-valued functions $f(\chi, g),\ 0 < |f(\chi, g)| \leq 1$ such that

$$T_{\chi,g} = f(\chi, g)U_{\chi,g},\ \chi \in \hat{G},\ g \in G,$$

and has shown that the following theorem takes palce.

Theorem 2 [3]. The POVM $\mathfrak{M}$ is informationally complete. Moreover, the inverse formula for restoring a state ρ is given by

$$\rho = \int\limits_{\mathfrak{G}} d\hat{\nu}(\chi)d\nu(g)f(\chi, g)^{-1}U^*_{\chi,g} \int\limits_{\mathfrak{G}} d\hat{\nu}(\chi')d\nu(g')\chi(g')\overline{\chi'(g)}p_\rho(\chi', g'),$$

where

$$p_\rho(\chi, g) = \langle U_{\chi,g}f, \rho U_{\chi,g}f\rangle$$

is the density of probability distribution $\pi = \Phi(\rho)$.

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References

[1] G. Amosov. *On quantum tomography on locally compact groups.* // Phys. Lett. A. 2022. Vol. 431. No. 128002, 7 pp.

[2] G. Amosov. *On quantum channels generated by covariant positive operator-valued measures on a locally compact group.* // Quantum Inf. Process. 2022. Vol. 21. No. 312, 10 pp.

[3] G. Amosov. *On constructing informationally complete covariant positive operator-valued measures.* // https://arxiv.org/abs/2301.12492