The range of C_0-semigroups

H. Zwart [1]

Keywords: right invertible; closed range; strongly continuous semigroup.

MSC2020 codes: 47D06, 47A05

Introduction. Let $(T(t))_{t\geq 0}$ be a strongly continuous semigroup on the Hilbert space Z. It is well-know that if the operator $T(t)$ is surjective for one $t > 0$, then it is surjective for all $t \geq 0$, see [1]. In this paper we study the question if other properties of the range of $T(t)$ are independent of t. By means of a counter example we show that the range of a semigroup can be change from non-closed to closed and back again. Thus properties of the range will be time-dependent, in general.

An illustrative example

In this section we construct an example showing that the range of a strongly continuous semigroup can change from closed to non-closed, and back again.

Let $H_0^1(0,3)$ denote the Sobolev space consisting of all functions in $L^2(0,3)$ whose first derivative exists in $L^2(0,3)$ and which are zero at $\zeta = 3$. It is a Hilbert space with the norm

$$\|f\|_{H^1}^2 = \|f\|^2 + \|\dot{f}\|^2,$$

where the later norms denote the standard L^2-norms of f and its derivative. It is well-known that $H_0^1(0,3)$ is a Hilbert space with this norm.

As Hilbert space Z we take $Z = H_0^1(0,3) \oplus L^2(-1,1)$, and as semigroup we define

$$\begin{aligned} T(t)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} &= \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \quad \text{with} \\ y_1(\zeta) &= x_1(t+\zeta)\mathbf{1}_{[0,3]}(t+\zeta), \quad \zeta \in [0,3] \\ y_2(\zeta) &= x_1(t+\zeta)\mathbf{1}_{[-1,0]}(t+\zeta) + x_2(t+\zeta)\mathbf{1}_{[-1,1]}(t+\zeta), \quad \zeta \in [-1,1], \end{aligned}$$

where $\mathbf{1}_{[a,b]}$ denotes the indicator function on the interval $[a,b]$, and we have extended x_1 and x_2 by zero "outside their own interval".

Next we study the range at different time instances.

- $\mathbf{t = 1}$. At $t = 1$, the second component equals zero for $\zeta \in (0,1)$, whereas in the interval $(-1,0)$ it consists of a function in H^1 plus an L^2-function. Since this L^2 function can be constructed freely by using a proper choice of x_2, the range of the second component is closed. It is easy to see that the range of the first component is closed, and thus the range of $T(1)$ is closed.

- $\mathbf{t = 2.5}$. For $t = 2.5$ we see that $x_2(t+\zeta)\mathbf{1}_{[-1,1]}(t+\zeta)$ equals zero for all $\zeta \in [-1,1]$, and the second component of the semigroup consists out of shifted H^1 functions. Since H^1 is not closed in L^2, the range cannot be closed.

- $\mathbf{t > 3}$. For time instances larger than 3, the semigroup equals zero, and thus its range is closed.

[1] University of Twente, Department of Applied Mathematics, The Netherlands, Enschede. Eindhoven University of Technology, Department of Mechanical Engineering, The Netherlands Eindhoven. Email: h.j.zwart@utwente.nl

The above example shows that the range of a semigroup can change from closed to non-closed and back again. Using the above idea for the construction, it is not hard to see how examples can be constructed for which this change happens (infinitely) many times.

An open problem

In [2] it is shown that if the semigroup is left invertible, then its left inverse can be chosen to be a strongly continuous semigroup as well. Until now this result is only known for Hilbert spaces, and although the proof uses Hilbert space techniques, the problem is well-formulated in a general Banach space. Hence the research question is to investigate whether this results extends to Banach spaces.

Note that when $T(t)$ is surjective, its adjoint is left invertible, and so there is a direct connection with the range of the semigroup.

References

[1] G.Q. Xu, Y.F. Shang, *Characteristic of left invertible semigroups and admissibility of observation operators.* // Systems and Control Letters 2009. Vol. 58, P. 561–566 (2009).

[2] H. Zwart, *Left-invertible semigroups on Hilbert spaces.*// J. Evol. Equ. 2013. Vol. 13, P. 335–342.