Clark measures and composition operators in several variables

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Introduction. Let B_n denote the open unit ball of $\mathbb{C}^n$, $n \geq 1$, and let ∂B_n denote the unit sphere. We also use symbols $\mathbb{D}$ and $\mathbb{T}$ for the unit disk B_1 and the unit circle ∂B_1, respectively.

Given $k \in \mathbb{N}$ and $n_j \in \mathbb{N}$, $j = 1, 2, \ldots, k$, let

$$\mathcal{D} = \mathcal{D}[n_1, n_2, \ldots, n_k] = B_{n_1} \times B_{n_2} \cdots \times B_{n_k} \subset \mathbb{C}^{n_1+n_2+\cdots+n_k}.$$

Model examples of $\mathcal{D}$ are B_n and the polydisk $\mathbb{D}^n$. Let $C(z,\zeta) = C_{\mathcal{D}}(z,\zeta)$ denote the Cauchy kernel for $\mathcal{D}$. Let $\partial\mathcal{D}$ denote the distinguished boundary $\partial B_{n_1} \times \partial B_{n_2} \cdots \times \partial B_{n_k}$ of $\mathcal{D}$. Then

$$C_{\mathcal{D}}(z,\zeta) = \prod_{j=1}^{k} \frac{1}{(1 - \langle z_j, \zeta_j \rangle)^{n_j}}, \quad z = (z_1, z_2, \ldots, z_k) \in \mathcal{D},\ \zeta = (\zeta_1, \zeta_2, \ldots, \zeta_k) \in \partial\mathcal{D},$$

where $z_j = (z_{j,1}, z_{j,2}, \ldots, z_{j,n_j}) \in B_{n_j}$ and $\zeta_j = (\zeta_{j,1}, \zeta_{j,2}, \ldots, \zeta_{j,n_j}) \in \partial B_{n_j}$. The corresponding Poisson type kernel is given by the formula

$$P(z,\zeta) = \frac{C(z,\zeta)C(\zeta,z)}{C(z,z)}, \quad z \in \mathcal{D},\ \zeta \in \partial\mathcal{D}.$$

Clark measures. Let $M(\partial\mathcal{D})$ denote the space of complex Borel measures on $\partial\mathcal{D}$. Given an $\alpha \in \mathbb{T}$ and a holomorphic function $\varphi : \mathcal{D} \to \mathbb{D}$, the quotient

$$\frac{1 - |\varphi(z)|^2}{|\alpha - \varphi(z)|^2} = \operatorname{Re}\left(\frac{\alpha + \varphi(z)}{\alpha - \varphi(z)}\right), \quad z \in \mathcal{D},$$

is positive and pluriharmonic. Therefore, there exists a unique positive measure $\sigma_\alpha = \sigma_\alpha[\varphi] \in M(\partial\mathcal{D})$ such that

$$P[\sigma_\alpha](z) = \operatorname{Re}\left(\frac{\alpha + \varphi(z)}{\alpha - \varphi(z)}\right), \quad z \in \mathcal{D}.$$

After the seminal paper of Clark [3], various properties and applications of the measures σ_α on the unit circle $\mathbb{T}$ have been obtained; see [1] for further details and references in several variables.

Let Σ denote the normalized Lebesgue measure on $\partial\mathcal{D}$. Specific properties of Clark measures are illustrated by the following theorem on disintegration of Lebesgue measure.

Theorem 1. Let $\varphi : \mathcal{D} \to \mathbb{D}$ be a holomorphic function and let $\sigma_\alpha = \sigma_\alpha[\varphi]$, $\alpha \in \mathbb{T}$. Then

$$\int_{\mathbb{T}} \int_{\partial\mathcal{D}} f \, d\sigma_\alpha \, dm_1(\alpha) = \int_{\partial\mathcal{D}} f \, d\Sigma$$

for all $f \in C(\partial\mathcal{D})$.

Essential norms of composition operators. Let $\mathcal{H}ol(\mathcal{D})$ denote the space of holomorphic functions in $\mathcal{D}$. For $0 < p < \infty$, the classical Hardy space $H^p = H^p(\mathcal{D})$ consists of those $f \in \mathcal{H}ol(\mathcal{D})$ for which

$$\|f\|_{H^p}^p = \sup_{0<r<1} \int_{\partial\mathcal{D}} |f(r\zeta)|^p \, d\Sigma(\zeta) < \infty.$$

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Each holomorphic function $\varphi : \mathcal{D} \to \mathbb{D}$ generates the composition operator $C_\varphi : \mathcal{H}ol(\mathbb{D}) \to \mathcal{H}ol(\mathcal{D})$ by the following formula:

$$(C_\varphi f)(z) = f(\varphi(z)), \quad z \in \mathcal{D}.$$

It is well known that C_φ maps $H^2(\mathbb{D})$ into $H^2(\mathcal{D})$. So, a natural problem is to characterize the compact operators $C_\varphi : H^2(\mathbb{D}) \to H^2(\mathcal{D})$. A more general problem is to compute or estimate the essential norm of the composition operator under consideration. For the unit disk $\mathbb{D}$, a solution to this problem in terms of the Nevanlinna counting function was given in the seminal paper of Shapiro [4]. A solution in terms of the family $\sigma_\alpha[\varphi]$, $\alpha \in \mathbb{T}$, was later obtained by Cima and Matheson [2]. Extending the theorem of Cima and Matheson to several variables, we prove the following result:

Theorem 2. Let $\varphi : \mathcal{D} \to \mathbb{D}$ be a holomorphic function. Then the essential norm of the composition operator $C_\varphi : H^2(\mathbb{D}) \to H^2(\mathcal{D})$ is equal to the following quantity:

$$\sqrt{\sup\{\|\sigma_\alpha^s\| : \alpha \in \mathbb{T}\}},$$

where σ_α^s denotes the singular part of the Clark measure $\sigma_\alpha = \sigma_\alpha[\varphi]$.

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