Frequency-domain conditions for the exponential stability of compound cocycles generated by delay equations and effective dimension estimates of global attractors

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Illustrative example: Mackey-Glass equation

Consider the Mackey-Glass equation

$$\dot{x}(t) = -\tau_0\gamma_0 x(t) + \tau_0\beta_0 f(x(t-1)), \tag{1}$$

where $\tau_0, \beta_0, \gamma_0 > 0$ are parameters and for an even integer k the nonlinearity is given by

$$f(y) = \frac{y}{1+y^k} \tag{2}$$

It is well-known that the model exhibits chaotic behavior for a range of parameters.

Problem: How to estimate the dimension of the resulting attractor?

Operators and delay equations: main space

For some $\tau > 0$ consider the main Hilbert space

$$\mathbb{H} = L_2([-\tau, 0]; \mu; \mathbb{R}^n), \tag{3}$$

where $\mu = \mu_L + \delta_0$ is the sum of the Lebesgue measure on $[-\tau, 0]$ and the δ-measure concentrated at 0.
For $\phi(\cdot) \in \mathbb{H}$ we consider

$$R_0^{(1)}\phi := \phi(0) \in \mathbb{R}^n \text{ and } R_1^{(1)}\phi := \phi\big|_{(-\tau,0)} \in L_2(-\tau, 0; \mathbb{R}^n). \tag{4}$$

We define an (unbounded) operator A in $\mathbb{H} = L_2([-\tau, 0]; \mu; \mathbb{R}^n)$ by

$$R_0^{(1)}(A\phi) = \widetilde{A}\phi \text{ and } R_1^{(1)}(A\phi) = \frac{d}{d\theta}\phi, \tag{5}$$

where $\widetilde{A} \colon C([-\tau, 0]; \mathbb{R}^n) \to \mathbb{R}^n$ is a bounded linear operator.
For scalar ($n = 1$) equations we often have $\widetilde{A}\phi = \alpha\phi(0) + \beta\phi(-\tau)$.

Operators and delay equations: additive symmetrization of A

Recall that A is given by

$$R_0^{(1)}(A\phi) = \widetilde{A}\phi \text{ and } R_1^{(1)}(A\phi) = \frac{d}{d\theta}\phi, \tag{6}$$

For the adjoint A^* of A in $\mathbb{H} = L_2([-\tau, 0]; \mu; \mathbb{R}^n)$ we have

$$R_1^{(1)}(A^*\psi) = -\frac{d}{d\theta}\psi \tag{7}$$

due to integration by parts. Thus, $R_1^{(1)}(A + A^*)\phi = 0$, that is the additive symmetrization $A + A^*$ has kernel with finite codimension $\leq n$.

As a consequence, the Liouville trace formula (at least in the standard inner product) cannot be utilized to obtain effective dimension estimates.

Operators and delay equations: nonautonomous systems

Let us consider a semiflow $(\mathcal{P}, \pi)$ on a complete metric space $\mathcal{P}$. Let $\mathbb{U} := \mathbb{R}^{r_1}$ and $\mathbb{M} := \mathbb{R}^{r_2}$ be endowed with some (not necessarily Euclidean) inner products. We consider the class of nonautonomous delay equations in $\mathbb{R}^n$ over $(\mathcal{P}, \pi)$ given by

$$\dot{x}(t) = \widetilde{A}x_t + \widetilde{B}F'(\pi^t(\mathfrak{p}))Cx_t, \tag{8}$$

where $\widetilde{A} \colon C([-\tau, 0]; \mathbb{R}^n) \to \mathbb{R}^n$, $C \colon C([-\tau, 0]; \mathbb{R}^n) \to \mathbb{M}$ are bounded linear operators; $\widetilde{B} \colon \mathbb{U} \to \mathbb{R}^n$ is a linear operator and $F' \colon \mathcal{P} \to \mathcal{L}(\mathbb{M}; \mathbb{U})$ is a continuous mapping such that for some $\Lambda > 0$ we have

$$\|F'(\mathfrak{p})\|_{\mathcal{L}(\mathbb{M};\mathbb{U})} \leq \Lambda \text{ for all } \mathfrak{p} \in \mathcal{P}. \tag{9}$$

Operators and delay equations: nonautonomous systems (continuation)

We study the class of delay equations in $\mathbb{R}^n$ over $(\mathcal{P}, \pi)$ given by

$$\dot{x}(t) = \widetilde{A}x_t + \widetilde{B}F'(\pi^t(\mathfrak{p}))Cx_t, \tag{10}$$

System (10) can be treated as an abstract evolution equation in $\mathbb{H} = L_2([-\tau, 0]; \mu; \mathbb{R}^n)$ given by

$$\dot{\xi}(t) = A\xi(t) + BF'(\pi^t(\mathfrak{p}))C\xi(t), \tag{11}$$

where A is the operator associated with $\widetilde{A}$; $B \colon \mathbb{U} \to \mathbb{H}$ is the *boundary operator* such that $R_0^{(1)}B\eta = \widetilde{B}\eta$ and $R_1^{(1)}B\eta = 0$ for $\eta \in \mathbb{U}$ and $C\phi := CR_1^{(1)}\phi$ for $\phi \in \mathbb{H}$.
It can be shown that (11) generates a cocycle Ξ in $\mathbb{H}$ over $(\mathcal{P}, \pi)$. Let Ξ_m be its extension to the m-fold exterior power $\mathbb{H}^{\wedge m}$ of $\mathbb{H}$.

Problem: Provide conditions for the uniform exponential stability of Ξ_m.

Our method: consider Ξ (resp. Ξ_m) as a perturbation of the C_0-semigroup generated by A (resp. its multiplicative extension).

Operators and delay equations: eventually compact C_0-semigroup $G(t)$ generated by A

Recall $\widetilde{A}\colon C([-\tau,0];\mathbb{R}^n) \to \mathbb{R}^n$ is a bounded linear operator and A is given by

$$R_0^{(1)}(A\phi) = \widetilde{A}\phi \text{ and } R_1^{(1)}(A\phi) = \frac{d}{d\theta}\phi, \tag{12}$$

The operator A is defined on the domain $\mathcal{D}(A)$ given by the embedding of $\phi \in W^{1,2}(-\tau,0;\mathbb{R}^n)$ into $\psi \in \mathbb{H}$ such that $R_0^{(1)}\psi = \phi(0)$ and $R_1^{(1)}\psi = \phi$.

It can be shown that A generates an eventually compact C_0-semigroup $G = G(t)$, where $t \geq 0$.

Operators and delay equations: compound operators

We define $G^{\otimes m}(t)$ as the m-fold multiplicative tensor product of $G(t)$. It can be shown that $G^{\otimes m} = G^{\otimes m}(t)$, where $t \geq 0$, is an eventually compact C_0-semigroup in the m-fold tensor product $\mathbb{H}^{\otimes m}$ of $\mathbb{H}$. Analogously, $G^{\wedge m}(t)$ can be defined as the restriction of $G^{\otimes m}(t)$ to the m-fold exterior power $\mathbb{H}^{\wedge m}$ of $\mathbb{H}$.

Let $A^{[\otimes m]}$ be the generator of $G^{\otimes m}$ called the *m-fold additive compound* of A. Its restriction $A^{[\wedge m]}$ to $\mathbb{H}^{\wedge m}$ is the generator of $G^{[\wedge m]}$ and it is called the *m-fold antisymmetric additive compound* of A.

Spectra of $A^{[\otimes m]}$ and $A^{[\wedge m]}$

Theorem

We have $\operatorname{spec}(A^{[\wedge m]}) \subseteq \operatorname{spec}(A^{[\otimes m]})$ *and*

$$\operatorname{spec}(A^{[\otimes m]}) = \left\{ \sum_{j=1}^{m} \lambda_j \mid \lambda_j \in \operatorname{spec}(A) \text{ for any } j \in \{1, \dots, m\} \right\}. \tag{13}$$

Moreover, any $\lambda_0 \in \operatorname{spec}(A^{[\otimes m]})$ *is an isolated spectral point and there exist finitely many, say* N*, distinct* m*-tuples* $\left(\lambda_1^k, \dots, \lambda_m^k\right) \in \mathbb{C}^m$ *for* $1 \leq k \leq N$ *such that*

$$\lambda_0 = \sum_{j=1}^{m} \lambda_j^k \text{ and } \lambda_j^k \in \operatorname{spec}(A). \tag{14}$$

Spectra of $A^{[\otimes m]}$ and $A^{[\wedge m]}$ (continuation)

Theorem (continuation)

In addition, each λ_j^k is an isolated spectral point of A and for the corresponding spectral subspaces $\mathbb{L}_{A^{\otimes m}}(\lambda_0)$ and $\mathbb{L}_A(\lambda_j^k)$ we have

$$\mathbb{L}_{A^{[\otimes m]}}(\lambda_0) = \bigoplus_{k=1}^{N} \bigotimes_{j=1}^{m} \mathbb{L}_A(\lambda_j^k). \tag{15}$$

Moreover, $\lambda_0 \in \operatorname{spec}(A^{[\wedge m]})$ if and only if $\Pi_m^{\wedge}\mathbb{L}_{A^{[\otimes m]}}(\lambda_0) \neq \{0\}$. In this case the spectral subspace of $A^{[\wedge m]}$ w.r.t. λ_0 is given by

$$\mathbb{L}_{A^{[\wedge m]}}(\lambda_0) = \Pi_m^{\wedge}\mathbb{L}_{A^{[\otimes m]}}(\lambda_0) = \mathbb{L}_{A^{[\otimes m]}}(\lambda_0) \cap \mathbb{H}^{\wedge m}. \tag{16}$$

Operators and delay equations: description of $\mathbb{H}^{\otimes m}$

Recall $\mu = \mu_L + \delta_0$.

Theorem

For the space $\mathbb{H} = L_2([-\tau, 0]; \mu; \mathbb{R}^n)$, *the mapping*

$$\phi_1 \otimes \ldots \otimes \phi_m \mapsto (\phi_1 \otimes \ldots \otimes \phi_m)(\theta_1, \ldots, \theta_m) := \phi_1(\theta_1) \otimes \ldots \otimes \phi_m(\theta_m) \tag{17}$$

induces a natural isometric isomorphism between $\mathbb{H}^{\otimes m}$ *and*

$$\mathcal{L}_m^{\otimes} := L_2([-\tau, 0]^m; \mu^{\otimes m}; (\mathbb{R}^n)^{\otimes m}). \tag{18}$$

In particular, its restriction to $\mathbb{H}^{\wedge m}$ *gives an isometric isomorphism onto the subspace* $\mathcal{L}_m^{\wedge}$ *of antisymmetric functions*[a].

[a] Such functions satisfy for each permutation $\sigma \in \mathbb{S}_m$

$$\Phi(\theta_{\sigma(1)}, \ldots, \theta_{\sigma(m)}) = (-1)^{\sigma} T_{\sigma^{-1}} \Phi(\theta_1, \ldots, \theta_m). \tag{19}$$

$\mu^{\otimes m}$-almost everywhere on $[-\tau, 0]^m$; T_σ is the transposition operator in $(\mathbb{R}^n)^{\otimes m}$.

Operators and delay equations: k-faces of $[-\tau, 0]^m$ w.r.t. $\mu^{\otimes m}$

Now let us choose $1 \leq k \leq m$ integers $1 \leq j_1 < \ldots < j_k \leq m$ and define the set $\mathcal{B}_{j_1 \ldots j_k}$ (a *k-face* of $[-\tau, 0]^m$ w.r.t. $\mu^{\otimes m}$) as

$$\mathcal{B}_{j_1 \ldots j_k} = \{0\}^{j_1 - 1} \times (-\tau, 0) \times \{0\}^{j_2 - 1} \times (-\tau, 0) \ldots . \tag{20}$$

We also put $\mathcal{B}_0 := \{0\}^m$ denoting the set corresponding to the unique 0-face w.r.t. $\mu^{\otimes m}$ and consider it as $\mathcal{B}_{j_1 \ldots j_k}$ for $k = 0$.
From the definition of $\mu = \mu_L + \delta_0$ we have that $\mu^{\otimes m}$ can be decomposed into the orthogonal sum given by

$$\mu^{\otimes m} = \sum_{k=0}^{m} \sum_{j_1 \ldots j_k} \mu_L^k(\mathcal{B}_{j_1 \ldots j_k}), \tag{21}$$

where $\mu_L^k(\mathcal{B}_{j_1 \ldots j_k})$ denotes the k-dimensional Lebesgue measure on $\mathcal{B}_{j_1 \ldots j_k}$ and $\mu_L^0(\mathcal{B}_0)$ denotes the δ-measure concentrated at $\mathcal{B}_0 = \{0\}^m$.

Operators and delay equations: restriction operators

We define the *restriction operator* $R^{(m)}_{j_1 \dots j_k}$ as

$$\mathcal{L}^{\otimes}_{m} \ni \Phi \mapsto R^{(m)}_{j_1 \dots j_k}\Phi := \Phi\big|_{\mathcal{B}_{j_1 \dots j_k}} \in L_2((-\tau,0)^k;(\mathbb{R}^n)^{\otimes m}) \tag{22}$$

Let $\partial_{j_1 \dots j_k}\mathcal{L}^{\otimes}_{m}$ denote the subspace of $\mathcal{L}^{\otimes}_{m}$ where all the restriction operators except possibly $R^{(m)}_{j_1 \dots j_k}$ vanish. We call $\partial_{j_1 \dots j_k}\mathcal{L}^{\otimes}_{m}$ the *boundary subspace on* $\mathcal{B}_{j_1 \dots j_k}$. Clearly, the space $\mathcal{L}^{\otimes}_{m}$ decomposes into the orthogonal inner sum as

$$\mathcal{L}^{\otimes}_{m} = \bigoplus_{k=0}^{m} \bigoplus_{j_1 \dots j_k} \partial_{j_1 \dots j_k}\mathcal{L}^{\otimes}_{m}, \tag{23}$$

where each boundary subspace $\partial_{j_1 \dots j_k}\mathcal{L}^{\otimes}_{m}$ is naturally isomorphic to the space $L_2((-\tau,0)^k;(\mathbb{R}^n)^{\otimes m})$ via the restriction operator $R^{(m)}_{j_1 \dots j_k}$.

Operators and delay equations: example $m = 2$, $n = 1$

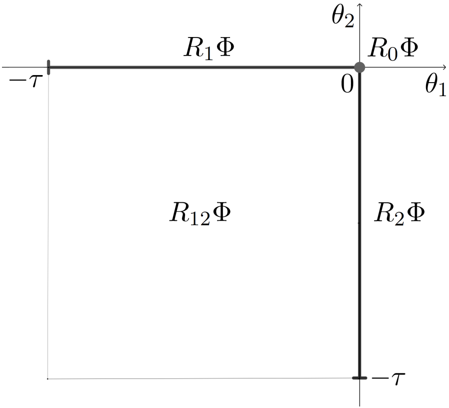

Figure: A representation of an element Φ from $L_2([-\tau, 0]^2; \mu^{\otimes 2}; \mathbb{R})$ via its four restrictions $R_0\Phi$, $R_1\Phi$, $R_2\Phi$ and $R_{12}\Phi$.

Operators and delay equations: action of $A^{[\otimes m]}$

Let $\mathcal{W}^2_D((-\tau,0)^k;(\mathbb{R}^n)^m)$ be the space of $\Phi \in L_2((-\tau,0)^k;(\mathbb{R}^n)^m)$ with L_2-summable diagonal derivative $\sum_{l=1}^{k}\frac{\partial}{\partial\theta_l}\Phi$.

Theorem

For the m-fold additive compound $A^{[\otimes m]}$ of A and any $\Phi \in \mathcal{D}(A^{[\otimes m]}$ we have $R_{j_1\ldots j_k}\Phi \in \mathcal{W}^2_D((-\tau,0)^k;(\mathbb{R}^n)^m)$ and[a]

$$R_{j_1\ldots j_k}\left(A^{[\otimes m]}\Phi\right) = \sum_{l=1}^{k}\frac{\partial}{\partial\theta_l}R_{j_1\ldots j_k}\Phi + \sum_{j\notin\{j_1,\ldots,j_k\}}\widetilde{A}^{(k)}_{j,J(j)}R_{jj_1\ldots j_k}\Phi, \quad (24)$$

for any $0 \le k \le m$, $1 \le j_1 < j_2 < \ldots < j_k \le m$.

[a] Here $R_{j_1\ldots j_k}\Phi$ is considered as a function of $\theta_1,\ldots,\theta_k$ and $\widetilde{A}^{(k)}_{j,J(j)}$ is an operator associated with $\widetilde{A}$.

Compound delay equations: structural Cauchy formula

For $T > 0$ let $\Phi_\nu(\cdot)$ be a mild solution on $[0, T]$ to

$$\dot{\Phi}(t) = (A^{[\otimes m]} + \nu I)\Phi(t) + \eta(t), \tag{25}$$

where $\eta(\cdot) \in L_2(0, T; \mathcal{L}^{\otimes m})$. Put $\rho_\nu(t) := e^{\nu t}$.

Theorem (Structural Cauchy formula)

For every $1 \leq k \leq m$ and $1 \leq j_1 < \ldots < j_k \leq m$ there exist functions $X = X_{j_1 \ldots j_k} \in L_2(\mathcal{C}^k_T; (\mathbb{R}^n)^{\otimes m})$ and $Y = Y_{j_1 \ldots j_k} \in L_2(0, T; L_2((-\tau, 0)^k; (\mathbb{R}^n)^{\otimes m}))$ such that $R_{j_1 \ldots j_k}\Phi_\nu$ is given by the sum of the ρ_ν-adornment of X and ρ_ν-twisting of Y

$$R_{j_1 \ldots j_k}\Phi(t) = \Phi_{X,\rho_\nu}(t) + \Psi_{Y,\rho_\nu}(t) \text{ for all } t \in [0, T]. \tag{26}$$

Structural Cauchy formula: adorned functions

For $T > 0$ define the set

$$\mathcal{C}_T^m = \bigcup_{t \in [0,T]} \left([-\tau, 0]^m + \underline{t} \right), \tag{27}$$

where $\underline{t} = (t, \dots, t) \in \mathbb{R}^m$.
For simplicity, let $\rho(t) = \rho_\nu(t) = e^{\nu t}$ and fix a Hilbert space $\mathbb{F}$. Then for each $X \in L_2(\mathcal{C}_T^m; \mathbb{F})$ we define a function $\Phi(t)$ for $t \in [0, T]$ as

$$\Phi(t) = \Phi_{X,\rho}(t) := \rho(t) X(t + \cdot_1, \dots, t + \cdot_m) \in L_2((-\tau, 0)^m; \mathbb{F}). \tag{28}$$

In this case we say that Φ is a ρ*-adornment* of X or that Φ is ρ*-adorned*) over $\mathcal{C}_T^m$. It is clear that Φ determines X uniquely.

Structural Cauchy formula: spaces of adorned functions

We define the space $\mathcal{Y}^2_\rho(0,T;L_2(-\tau,0;\mathbb{F}))$ of all ρ-adorned over $\mathcal{C}^m_T$ functions $\Phi(\cdot)$ and endow it with the norm given by

$$\|\Phi(\cdot)\|_{\mathcal{Y}^2_\rho(0,T;L_2(-\tau,0;\mathbb{F}))} :=$$
$$= \left(\int_{(-\tau,0)^m} \left| X(\overline{\theta}) \right|^2_{\mathbb{F}} d\overline{\theta} + \sum_{j=1}^{m} \int_{\mathcal{B}_{\hat{j}}} d\widehat{\theta}_j(\overline{\theta}) \int_0^T \left| \rho(t) X(\overline{\theta} + \underline{t}) \right|^2_{\mathbb{F}} dt \right)^{1/2}, \tag{29}$$

where $d\widehat{\theta}_j$ is the $(m-1)$-dimensional Lebesgue measure on the $(m-1)$-face $\mathcal{B}_{\hat{j}} = \mathcal{B}_{1\ldots\hat{j}\ldots m}$.
In the case $T = \infty$ we additionally require that the norm in (29) is finite.

Structural Cauchy formula: twisted functions

Now let $T_m(t)$, where $t \geq 0$, be the diagonal translation semigroup in $L_2((-\tau,0)^m;\mathbb{F})$, i.e.

$$(T_m(t)\Phi)(\overline{\theta}) = \begin{cases} \Phi(\overline{\theta}+\underline{t}), \text{ if } \overline{\theta}+\underline{t} \in (-\tau,0)^m, \\ 0, \text{ otherwise }. \end{cases} \tag{30}$$

Here $\overline{\theta} = (\theta_1,\ldots,\theta_m) \in [-\tau,0]^m$ and $\underline{t} = (t,\ldots,t) \in \mathbb{R}^m$.
For a given $T > 0$ let $\Psi(\cdot)$ be a function on $[0,T]$ taking values in $L_2((-\tau,0)^m;\mathbb{F})$ such that

$$\Psi(t) = \Psi_{Y,\rho}(t) := \rho(t)\int_0^t T_m(t-s)Y(s)ds \text{ for all } t \in [0,T] \tag{31}$$

for some $Y(\cdot) \in L_2(0,T;L_2((-\tau,0)^m;\mathbb{F}))$. We say that Ψ is a ρ-*twisting* of Y or simply that Ψ is ρ-*twisted*. It can be shown that Ψ determines Y uniquely.

Structural Cauchy formula: spaces of twisted functions

We consider the space $\mathcal{T}^2_\rho(0, T; L_2((-\tau, 0)^m; \mathbb{F}))$ of ρ-twisted functions and endow it with the norm

$$\|\Psi(\cdot)\|_{\mathcal{T}^2_\rho(0,T;L_2((-\tau,0)^m;\mathbb{F}))} := \left(\int_0^T \|\rho(t)Y(t)\|^2_{L_2((-\tau,0)^m;\mathbb{F})} dt \right)^{1/2}. \tag{32}$$

For $T = \infty$ we require the value in (32) to be finite.

Structural Cauchy formula: uniqueness

It turns out that the spaces $\mathcal{Y}^2_\rho(0,T;L_2(-\tau,0;\mathbb{F}))$ and $\mathcal{T}^2_\rho(0,T;L_2((-\tau,0)^m;\mathbb{F}))$ are linearly independent, i.e.

$$\begin{aligned} \Phi_{X,\rho}(t) + \Psi_{Y,\rho}(t) = 0 \text{ for all } t \in [0,T] \\ \text{if and only if} \\ \Phi_{X,\rho}(t) = \Psi_{Y,\rho}(t) = 0 \text{ for all } t \in [0,T]. \end{aligned} \tag{33}$$

Compound delay equations: structural Cauchy formula (continuation)

For $T > 0$ let $\Phi_\nu(\cdot)$ be a mild solution on $[0, T]$ to

$$\dot{\Phi}(t) = (A^{[\otimes m]} + \nu)\Phi(t) + \eta(t), \tag{34}$$

where $\eta(\cdot) \in L_2(0, T; \mathcal{L}^{\otimes m})$. Put $\rho_\nu(t) := e^{\nu t}$.

Theorem (Structural Cauchy formula, continuation)

...such that $R_{j_1 \ldots j_k}\Phi_\nu$ is given by the sum of the ρ_ν-adornment of X and ρ_ν-twisting of Y

$$R_{j_1 \ldots j_k}\Phi(t) = \Phi_{X,\rho_\nu}(t) + \Psi_{Y,\rho_\nu}(t) \text{ for all } t \in [0, T]. \tag{35}$$

Moreover, the norms of Φ_{X,ρ_ν} in $\mathcal{Y}^2_\rho(0, T; L_2(-\tau, 0; (\mathbb{R}^n)^{\otimes m}))$ and Ψ_{Y,ρ_ν} in $\mathcal{T}^2_\rho(0, T; L_2((-\tau, 0)^m; (\mathbb{R}^n)^{\otimes m})))$ can be estimated in terms of $|\Phi_\nu(0)|_{\mathcal{L}^{\otimes m}}$, $\|\Phi_\nu(\cdot)\|_{L_2(0,T;\mathcal{L}^{\otimes m})}$ and $\|\eta(\cdot)\|_{L_2(0,T;\mathcal{L}^{\otimes m})}$ with some uniform in T constant.

Measurement operators

For given Hilbert spaces $\mathbb{F}$ and $\mathbb{M}_\gamma$, let $\gamma(\theta) \in \mathcal{L}(\mathbb{F};\mathbb{M}_\gamma)$ be an operator-valued function of bounded variation on $\theta \in [-\tau, 0]$.
For given $1 \leq J \leq k$ we consider the operator C^γ_J from $C([-\tau,0]^{k+1};\mathbb{F})$ to $C([-\tau,0]^k;\mathbb{M}_\gamma)$ given by

$$C^\gamma_J \Phi(\overline{\theta}_{\hat{J}}) = \int_{-\tau}^{0} d\gamma(\theta_J)\Phi(\theta_1, \dots, \theta_{k+1}), \tag{36}$$

where $\overline{\theta}_{\hat{J}} := (\theta_1, \dots, \hat{\theta}_J, \dots, \theta_{k+1})$.

For example, for $k = 1$ and $d\gamma = \delta_{-\tau}$ we have $(C^\gamma_1 \Phi)(\theta) = \Phi(-\tau, \theta)$ and $(C^\gamma_2 \Phi)(\theta) = \Phi(\theta, -\tau)$.

Pointwise measurement operators

We want to interpret the operator $\mathcal{I}_{C^{\gamma}_{J}}$ acting on $\Phi(\cdot)$ from $L_2(0,T;L_2((-\tau,0)^{k+1};\mathbb{F}))$ by pointwise measurement of C^{γ}_{J}, i.e.

$$(\mathcal{I}_{C^{\gamma}_{J}}\Phi)(t) = C^{\gamma}_{J}\Phi(t) \tag{37}$$

It turns out that it is possible to interpret $\mathcal{I}_{C^{\gamma}_{J}}$ as a bounded operator if we restrict ourselves with

$$\Phi(t) = \Phi_{X,\rho}(t) + \Psi_{Y,\rho}(t), \tag{38}$$

where $\Phi_{X,\rho} \in \mathcal{Y}^{2}_{\rho}(0,T;L_2(-\tau,0;\mathbb{F}))$ and $\Psi_{Y,\rho} \in \mathcal{T}^{2}_{\rho}(0,T;L_2(-\tau,0;\mathbb{F}))$. We call call such functions as in (38) *ρ-agalmanated*.

Nonautonomous systems in abstract form

Recall the class of nonautonomous delay equations in $\mathbb{R}^n$ over a semiflow $(\mathcal{P}, \pi)$ given by

$$\dot{x}(t) = \widetilde{A}x_t + \widetilde{B}F'(\pi^t(\mathfrak{p}))Cx_t, \tag{39}$$

and that system (39) can be treated as an abstract evolution equation in $\mathbb{H} = L_2([-\tau, 0]; \mu; \mathbb{R}^n)$ given by

$$\dot{\xi}(t) = A\xi(t) + BF'(\pi^t(\mathfrak{p}))C\xi(t). \tag{40}$$

where $B\colon \mathbb{U} \to \mathbb{H}$ is the boundary operator such that $R_0^{(1)}B\eta = \widetilde{B}\eta$ and $R_1^{(1)}B\eta = 0$ for $\eta \in \mathbb{U}$ and $C\phi := CR_1^{(1)}\phi$ for $\phi \in \mathbb{H}$.
Recall that (40) generates a cocycle Ξ in $\mathbb{H}$.

Compound delay equations: infinitesimal description of Ξ_m

Theorem

For any m *solutions* $\xi_1(t), \ldots, \xi_m(t)$ *of* (40) *with* $\xi_1(0)$, …, $\xi_m(0) \in \mathcal{D}(A)$, *the function* $\Phi(t) = \xi_1(t) \otimes \ldots \otimes \xi_m(t)$ *for* $t \geq 0$ *is a* C^1*-differentiable* $\mathcal{L}^{\otimes}_m$*-valued mapping such that* $\Phi(t) \in \mathcal{D}(A^{[\otimes m]})$, $\Phi(t) \in C([-\tau, 0]^m; (\mathbb{R}^n)^{\otimes m})$ *continuously depend on* $t \geq 0$ *and*[a]

$$\dot{\Phi}(t) = A^{[\otimes m]}\Phi(t) + \sum_{j_1 \ldots j_k} \sum_{j \notin \{j_1, \ldots, j_k\}} B_j^{j_1 \ldots j_k} F'_j(\pi^t(\mathfrak{p})) C^{(k)}_{j, J(j)} R_{j j_1 \ldots j_k} \Phi(t), \tag{41}$$

where the sum taken over all $1 \leq j_1 < \ldots < j_k \leq m$ *with* $0 \leq k \leq m-1$.

[a] Here $J(j) = J(j; j_1 \ldots j_k)$ denotes an integer J such that j is the J-th element of the set $\{j, j_1, \ldots, j_k\}$ arranged by increasing

Compound delay equations: definition of $C^{(k)}_{j,J}$

For each operator $C \colon C([-\tau,0];\mathbb{R}^n) \to \mathbb{M} = \mathbb{R}^{r_2}$ there exists a $(r_2 \times n)$-matrix of bounded variation $c(\theta)$ such that

$$C\phi = \int_{-\tau}^{0} dc(\theta)\phi(\theta) \text{ for all } \phi \in C([-\tau,0];\mathbb{R}^n). \tag{42}$$

Then for $j \in \{1,\dots,m\}$, we put $\gamma_j(\theta)$ to be the linear operator from $\mathbb{F} := (\mathbb{R}^n)^{\otimes m}$ to $\mathbb{M}_j := (\mathbb{R}^n)^{\otimes j-1} \otimes \mathbb{M} \otimes (\mathbb{R}^n)^{m-j}$ such that

$$x_1 \otimes \ldots \otimes x_j \otimes \ldots \otimes x_m \mapsto x_1 \otimes \ldots \otimes c(\theta)x_j \otimes \ldots x_m. \tag{43}$$

Then $\gamma_j(\theta) \in \mathcal{L}(\mathbb{F};\mathbb{M}_j)$ and we put $C^{(k)}_{j,J} := C^{\gamma}_J$ with $\gamma = \gamma_j$, and $\mathbb{M}_\gamma = \mathbb{M}_j$.

Compound delay equations: definition of $F'_j(\mathfrak{p})$

We define $F'_j(\mathfrak{p})$ as an operator form $\mathbb{M}_j = (\mathbb{R}^n)^{\otimes j-1} \otimes \mathbb{M} \otimes (\mathbb{R}^n)^{m-j}$ to $\mathbb{U}_j = (\mathbb{R}^n)^{\otimes j-1} \otimes \mathbb{U} \otimes (\mathbb{R}^n)^{m-j}$ by

$$x_1 \otimes \ldots \otimes x_j \otimes \ldots x_m \to x_1 \otimes \ldots \otimes F'(\mathfrak{p}) x_j \otimes \ldots x_m. \tag{44}$$

We use the same notation to denote the operator between spaces of functions taking values in $\mathbb{M}_j$ and $\mathbb{U}_j$ respectively where $F'_j(\mathfrak{p})$ is applied pointwisely.

Compound delay equations: definition of $B_j^{j_1 \dots j_k}$

Recall $\mathbb{U}_j = (\mathbb{R}^n)^{\otimes j-1} \otimes \mathbb{U} \otimes (\mathbb{R}^n)^{m-j}$.

For $0 \leq k \leq m-1$ we define a linear bounded operator $B_j^{(j_1 \dots j_k)}$ which takes an element $\Phi_{\mathbb{U}}$ from $L_2((-\tau,0)^k;\mathbb{U}_j)$ to an element from $\partial_{j_1 \dots j_k}\mathcal{L}_m^{\otimes}$ defined for $(\theta_1,\dots,\theta_m) \in \mathcal{B}_{j_1 \dots j_k}$ as

$$\left(B_j^{j_1 \dots j_k}\Phi_{\mathbb{U}}\right)(\theta_1,\dots,\theta_m) := (\mathrm{Id}_{\mathbb{R}_{1,j}} \otimes \widetilde{B} \otimes \mathrm{Id}_{\mathbb{R}_{2,j}})\Phi_{\mathbb{U}}(\theta_{j_1},\dots,\theta_{j_k}), \quad (45)$$

where $\mathbb{R}_{1,j} := (\mathbb{R}^n)^{\otimes(j-1)}$ and $\mathbb{R}_{2,j} := (\mathbb{R}^n)^{\otimes(m-j)}$.

Compound delay equations: associated control system in $\mathcal{L}_m^{\otimes}$

Let us consider the *control space* given by the outer orthogonal sum

$$\mathbb{U}_m^{\otimes} := \bigoplus_{j_1 \ldots j_k} \bigoplus_{j \notin \{j_1, \ldots, j_k\}} L_2((-\tau, 0)^k; \mathbb{U}_j), \tag{46}$$

where the indices $j_1 \ldots j_k$ and j are such that $1 \leq j_1 < \ldots < j_k \leq m$ with $0 \leq k \leq m-1$ and $j \in \{1, \ldots, m\}$. We define a *control operator* $B_m^{\otimes} \in \mathcal{L}(\mathbb{U}_m^{\otimes}; \mathcal{L}_m^{\otimes})$ as (see (45))

$$B_m^{\otimes} \eta := \sum_{j_1 \ldots j_k} \sum_{j \notin \{j_1, \ldots, j_k\}} B_j^{j_1 \ldots j_k} \eta^j_{j_1 \ldots j_k} \text{ for } \eta = (\eta^j_{j_1 \ldots j_k}) \in \mathbb{U}_m^{\otimes}. \tag{47}$$

We associate to the pair $(A^{[\otimes m]}, B_m^{\otimes})$ a control system as

$$\dot{\Phi}(t) = A^{[\otimes m]} \Phi(t) + B_m^{\otimes} \eta(t), \tag{48}$$

where $\eta(\cdot) \in L_2(0, T; \mathbb{U}_m^{\otimes})$.

Compound delay equations: subspace $\mathcal{L}_m^{\wedge}$: definition

Recall that for $\mathbb{H} = L_2([-\tau, 0]; \mu; \mathbb{R}^n)$ the m-fold exterior product $\mathbb{H}^{\wedge m}$ is naturally isomorphic to the subspace $\mathcal{L}_m^{\wedge}$ of antisymmetric functions in $\mathcal{L}_m^{\otimes} = L_2([-\tau, 0]^m; \mu^{\otimes m}; (\mathbb{R}^n)^{\otimes m})$.

Recall that such functions satisfy for each permutation $\sigma \in \mathbb{S}_m$ the identity

$$\Phi(\theta_{\sigma(1)}, \ldots, \theta_{\sigma(m)}) = (-1)^{\sigma} T_{\sigma^{-1}} \Phi(\theta_1, \ldots, \theta_m). \tag{49}$$

$\mu^{\otimes m}$-almost everywhere on $[-\tau, 0]^m$. Here T_σ is the transposition operator in $(\mathbb{R}^n)^{\otimes m}$ w.r.t. σ, i.e.

$$T_\sigma(x_1 \otimes \ldots \otimes x_m) := x_{\sigma(1)} \otimes \ldots \otimes x_{\sigma(m)}. \tag{50}$$

Subspace $\mathcal{L}_m^{\wedge}$: antisymmetric relations

For each permutation $\sigma \in \mathbb{S}_m$ we have the identity

$$\Phi(\theta_{\sigma(1)}, \dots, \theta_{\sigma(m)}) = (-1)^{\sigma} T_{\sigma^{-1}} \Phi(\theta_1, \dots, \theta_m). \tag{51}$$

$\mu^{\otimes m}$-almost everywhere on $[-\tau, 0]^m$.
This relations induce antisymmetric relations on restrictions to k-faces. Namely, for any $0 \leq k \leq m$, any $1 \leq j_1 < \ldots < j_m \leq m$ and $\sigma \in \mathbb{S}_m$ we have

$$(R_{j_1 \dots j_k} \Phi)(\theta_{j_1}, \dots, \theta_{j_k}) = (-1)^{\sigma} T_{\sigma} (R_{\sigma^{-1}(j_1) \dots \sigma^{-1}(j_k)} \Phi)(\theta_{j_{\overline{\sigma}(1)}}, \dots, \theta_{j_{\overline{\sigma}(k)}}),$$
$$\text{for almost all } (\theta_{j_1}, \dots, \theta_{j_k}) \in (-\tau, 0)^k, \tag{52}$$

where $\overline{\sigma} \in \mathbb{S}_k$ is such that $\sigma^{-1}(j_{\overline{\sigma}(1)}) < \ldots < \sigma^{-1}(j_{\overline{\sigma}(k)})$.

Subspace $\mathcal{L}^{\wedge}_{m}$: decomposition

Note that the antisymmetric relations (52) link each $\partial_{j_1 \ldots j_k}\mathcal{L}^{\otimes}_{m}$ with other boundary subspaces on k-faces. Thus, it is convenient to define for a given $k \in \{0, \ldots, m\}$ the subspace

$$\partial_k \mathcal{L}^{\wedge}_{m} := \left\{ \Phi \in \bigoplus_{j_1 \ldots j_k} \partial_{j_1 \ldots j_k}\mathcal{L}^{\otimes}_{m} \mid \Phi \text{ satisfies } (52) \right\}, \tag{53}$$

where the sum is taken over all $1 \leq j_1 < \ldots < j_k \leq m$. We say that k is *improper* if $\partial_k \mathcal{L}^{\wedge}_{m}$ is zero. Otherwise we say that k is *proper*. For example, when $n = 1$, gives that any $k \leq m - 2$ is improper and only $k = m - 1$ and $k = m$ are proper.
Clearly, $\mathcal{L}^{\wedge}_{m}$ decomposes into the orthogonal sum of all $\partial_k \mathcal{L}^{\wedge}_{m}$ as

$$\mathcal{L}^{\wedge}_{m} = \bigoplus_{k=0}^{m} \partial_k \mathcal{L}^{\wedge}_{m}. \tag{54}$$

Definition of $\mathbb{U}^{\wedge}_{m}$

Consider $\eta = (\eta^{j}_{j_1 \ldots j_k}) \in \mathbb{U}^{\otimes}_{m}$ satisfying for all $k \in \{0, \ldots, m-1\}$, $1 \leq j_1 < \ldots < j_k \leq m$, $j \notin \{j_1, \ldots, j_k\}$ and any $\sigma \in \mathbb{S}_m$ the relations

$$\eta^{j}_{j_1 \ldots j_k}(\theta_{j_1}, \ldots, \theta_{j_k}) = (-1)^{\sigma} T_{\sigma^{-1}} \eta^{\sigma(j)}_{\sigma(j_{\overline{\sigma}(1)}) \ldots \sigma(j_{\overline{\sigma}(k)})}(\theta_{j_{\overline{\sigma}(1)}}, \ldots, \theta_{j_{\overline{\sigma}(k)}}),$$
$$\text{for almost all } (\theta_{j_1}, \ldots, \theta_{j_k}) \in (-\tau, 0)^k, \tag{55}$$

where $\overline{\sigma} \in \mathbb{S}_k$ is such that $\sigma(j_{\overline{\sigma}(1)}) < \ldots < \sigma(j_{\overline{\sigma}(k)})$.
Now we define $\mathbb{U}^{\wedge}_{m}$ as

$$\mathbb{U}^{\wedge}_{m} := \{\eta = (\eta^{j}_{j_1 \ldots j_k}) \in \mathbb{U}^{\otimes}_{m} \mid \eta \text{ satisfies (55) and } \eta^{j}_{j_1 \ldots j_k} = 0 \text{ for improper } k\}. \tag{56}$$

Compound delay equations: associated control system in $\mathcal{L}_m^{\wedge}$

Recall the system just considered in the antisymmetric case

$$\dot{\Phi}(t) = A^{[\wedge m]}\Phi(t) + \sum_{j_1 \ldots j_k} \sum_{j \notin \{j_1,\ldots,j_k\}} B_j^{j_1 \ldots j_k} F_j'(\pi^t(\mathfrak{p})) C_{j,J(j)}^{(k)} R_{j j_1 \ldots j_k} \Phi(t), \quad (57)$$

We associate to (57) the linear system in $\mathcal{L}_m^{\wedge}$ as

$$\dot{\Phi}(t) = A^{[\wedge m]}\Phi(t) + B_m^{\wedge}\eta(t), \quad (58)$$

where $\eta(\cdot) \in L_2(0,T;\mathbb{U}_m^{\wedge})$ and $B_m^{\wedge}$ is defined on $\mathbb{U}_m^{\wedge}$ by the restriction of $B_m^{\otimes}$ from $\mathbb{U}_m^{\otimes}$ to $\mathbb{U}_m^{\wedge}$.

Compound delay equations: Lipschitz quadratic constraint

Recall

$$\dot{\Phi}(t) = A^{[\wedge m]}\Phi(t) + \sum_{j_1 \ldots j_k} \sum_{j \notin \{j_1,\ldots,j_k\}} B_j^{j_1\ldots j_k} F_j'(\pi^t(\mathfrak{p})) C_{j,J(j)}^{(k)} R_{jj_1\ldots j_k}\Phi(t), \tag{59}$$

and

$$\dot{\Phi}(t) = A^{[\wedge m]}\Phi(t) + B_m^{\wedge}\eta(t). \tag{60}$$

Since $\|F'(\mathfrak{p})\|_{\mathcal{L}(\mathbb{U};\mathbb{M})} \leq \Lambda$, for $\eta_{j_1\ldots j_k}^{j}(t) = F_j'(\pi^t(\mathfrak{p}))C_{j,J(j)}^{(k)} R_{jj_1\ldots j_k}\Phi(t)$ we have the quadratic constraint $\mathcal{F}(\Phi(t),\eta(t)) \geq 0$ satisfied, where

$$\mathcal{F}(\Phi,\eta) = \\ = \sum_{j_1 \ldots j_k} \sum_{j \notin \{j_1,\ldots,j_k\}} (\Lambda^2 \|C_{j,J(j)}^{(k)} R_{jj_1\ldots j_k}\Phi\|^2_{L_2((-\tau,0)^k;\mathbb{M}_j)} - \\ - \|\eta_{j_1\ldots j_k}^{j}\|^2_{L_2((-\tau,0)^k;\mathbb{U}_j)}), \tag{61}$$

Extension of C^{γ}_{J} to $\mathbb{E}_{k+1}(\mathbb{F})$

We need to consider C^{γ}_{J} in a wider context. For this we define the space $\mathbb{E}_{m}(\mathbb{F})$ of all functions $\Phi \in L_{2}((-\tau,0)^{m};\mathbb{F})$ such that for any $j \in \{1,\ldots,m\}$ there exists $\Phi^{b}_{j} \in C([-\tau,0];L_{2}((-\tau,0)^{m-1};\mathbb{F})$ such that we have the identity in $L_{2}((-\tau,0)^{m-1};\mathbb{F})$ as[1]

$$\Phi\big|_{\mathcal{B}_{\hat{j}}+\theta e_{j}} = \Phi^{b}_{j}(\theta) \text{ for almost all } \theta \in [-\tau,0]. \tag{62}$$

Let us endow $\mathbb{E}_{m}(\mathbb{F})$ with the norm

$$\|\Phi\|_{\mathbb{E}_{m}(\mathbb{F})} := \sup_{1\leq j\leq m}\sup_{\theta\in[-\tau,0]}\|\Phi^{b}_{j}(\theta)\|_{L_{2}((-\tau,0)^{m-1};\mathbb{F})} \tag{63}$$

which makes $\mathbb{E}_{m}(\mathbb{F})$ a Banach space.
We have that C^{γ}_{J} can be extended to a bounded operator from $\mathbb{E}_{k+1}(\mathbb{F})$ to $L_{2}((-\tau,0)^{k};\mathbb{M}_{\gamma})$.

[1] Recall that we naturally identify $\mathcal{B}_{\hat{j}}+\theta e_{j}$ with $[-\tau,0]^{m-1}$ by omitting the j-th coordinate.

Intermediate Banach spaces $\mathbb{E}^{\otimes}_{m}$ and $\mathbb{E}^{\wedge}_{m}$

We define the Banach space $\mathbb{E}^{\otimes}_{m}$ through the outer direct sum as

$$\mathbb{E}^{\otimes}_{m} := \bigoplus_{k=0}^{m} \bigoplus_{j_1 \dots j_k} \mathbb{E}_k((\mathbb{R}^n)^{\otimes m}) \tag{64}$$

and endow it with any of standard norms. We embed the space $\mathbb{E}^{\otimes}_{m}$ into $\mathcal{L}^{\otimes}_{m}$ by naturally sending each element from the $j_1 \dots j_k$-th summand in (64) to $\partial_{j_1 \dots j_k} \mathcal{L}^{\otimes}_{m}$. We have that

$$\mathcal{D}(A^{[\otimes m]}) \subset \mathbb{E}^{\otimes}_{m} \subset \mathcal{L}^{\otimes}_{m}, \tag{65}$$

where all the embeddings are dense and continuous.
Let $\mathbb{E}^{\wedge}_{m}$ be the intersection of $\mathbb{E}^{\otimes}_{m}$ with $\mathcal{L}^{\wedge}_{m}$. Analogously, we have

$$\mathcal{D}(A^{[\wedge m]}) \subset \mathbb{E}^{\wedge}_{m} \subset \mathcal{L}^{\wedge}_{m}, \tag{66}$$

where all the embeddings are dense and continuous.

Measurement space $\mathbb{M}^{\otimes}_{m}$ and the operator $C^{\otimes}_{m}$

Consider the *measurement space* $\mathbb{M}^{\otimes}_{m}$ given by the outer orthogonal sum

$$\mathbb{M}^{\otimes}_{m} := \bigoplus_{j_1 \ldots j_k} \bigoplus_{j \notin \in \{j_1, \ldots, j_k\}} L_2((-\tau,0)^k; \mathbb{M}_j), \tag{67}$$

where the sum is taken over all $k \in \{0, \ldots, m-1\}$,
$1 \leq j_1 < \ldots < j_k \leq m$ and $j \in \{1, \ldots, m\}$.

Define $C^{\otimes}_{m} \in \mathcal{L}(\mathbb{E}^{\otimes}_{m}; \mathbb{M}^{\otimes}_{m})$ by

$$C^{\otimes}_{m}\Phi := \sum_{j_1 \ldots j_k} \sum_{j \notin \{j_1, \ldots, j_k\}} C^{(k)}_{j, J(j)} R_{j j_1 \ldots j_k} \Phi, \tag{68}$$

where the sum is taken in $\mathbb{M}^{\otimes}_{m}$.

Measurement space $\mathbb{M}^{\wedge}_{m}$ and the operator $C^{\wedge}_{m}$

Let us consider $M = (M^{j}_{j_1 \ldots j_k}) \in \mathbb{M}^{\otimes}_{m}$ which satisfy for all $k \in \{0, \ldots, m-1\}$, $1 \leq j_1 < \ldots < j_k \leq m$, $j \notin \{j_1, \ldots, j_k\}$ and any $\sigma \in \mathbb{S}_m$ the relations

$$M^{j}_{j_1 \ldots j_k}(\theta_{j_1}, \ldots, \theta_{j_k}) = (-1)^{\sigma} T_{\sigma^{-1}} M^{\sigma(j)}_{\sigma(j_{\overline{\sigma}(1)}) \ldots \sigma(j_{\overline{\sigma}(k)})}(\theta_{j_{\overline{\sigma}(1)}}, \ldots, \theta_{j_{\overline{\sigma}(k)}}),$$
$$\text{for almost all } (\theta_{j_1}, \ldots, \theta_{j_k}) \in (-\tau, 0)^k, \tag{69}$$

where $\overline{\sigma} \in \mathbb{S}_k$ is such that $\sigma(j_{\overline{\sigma}(1)}) < \ldots < \sigma(j_{\overline{\sigma}(k)})$.
We define $\mathbb{M}^{\wedge}_{m}$ as

$$\begin{aligned} \mathbb{M}^{\wedge}_{m} := \{ M = (M^{j}_{j_1 \ldots j_k}) \in \mathbb{M}^{\otimes}_{m} \mid M \text{ satisfies (69) and} \\ M^{j}_{j_1 \ldots j_k} = 0 \text{ for improper } k \}. \end{aligned} \tag{70}$$

Let $C^{\wedge}_{m}$ be the restriction of $C^{\otimes}_{m}$ to $\mathbb{E}^{\wedge}_{m}$. We have $C^{\wedge}_{m} \in \mathcal{L}(\mathbb{E}^{\wedge}_{m}; \mathbb{M}^{\wedge}_{m})$.

Lipschitz quadratic constraints via $C_m^{\wedge}$

One can rewrite the quadratic form

$$\mathcal{F}(\Phi,\eta) = \\ = \sum_{j_1\ldots j_k} \sum_{j \notin \{j_1,\ldots,j_k\}} (\Lambda^2 \| C^{(k)}_{j,J(j)} R_{jj_1\ldots j_k} \Phi \|^2_{L_2((-\tau,0)^k;\mathbb{M}_j)} - \\ - \| \eta^j_{j_1\ldots j_k} \|^2_{L_2((-\tau,0)^k;\mathbb{U}_j)}), \tag{71}$$

in a compact way using $C_m^{\wedge}$ as

$$\mathcal{F}(\Phi,\eta) = \Lambda^2 \| C_m^{\wedge} \Phi \|^2_{\mathbb{M}_m^{\wedge}} - \| \eta \|^2_{\mathbb{U}_m^{\wedge}}. \tag{72}$$

General quadratic constraints

One can generalize quadratic constraints as follows. Let $\mathcal{G}(M,\eta)$ be a bounded quadratic form of $M \in \mathbb{M}^{\wedge}_{m}$ and $\eta \in \mathbb{U}^{\wedge}_{m}$. Then we put

$$\mathcal{F}(\Phi,\eta) := \mathcal{G}(C^{\wedge}_{m}\Phi,\eta) \text{ for } \Phi \in \mathbb{E}^{\wedge}_{m} \text{ and } \eta \in \mathbb{U}^{\wedge}_{m}. \tag{73}$$

We say that $\mathcal{F}$ is a *quadratic constraint* if $\mathcal{F}(\Phi,\eta) \geq 0$ is satisfied for all $\Phi \in \mathbb{E}^{\wedge}_{m}$, any $\mathfrak{p} \in \mathcal{P}$ and $\eta \in \mathbb{U}^{\wedge}_{m}$ such that
$\eta^{j}_{j_1 \ldots j_k} = F'_j(\mathfrak{p})C^{(k)}_{j,J(j)}R_{jj_1\ldots j_k}\Phi$ for all $k \in \{0,\ldots,m-1\}$,
$1 < j_1 < \ldots j_k < m$ and $j \notin \{j_1,\ldots,j_k\}$.

Resolvent estimates in $\mathbb{E}^{\otimes}_{m}$ and $\mathbb{E}^{\wedge}_{M}$

Theorem

For regular (=nonspectral) points $p \in \mathbb{C}$ *of* $A^{[\otimes m]}$ *we have*

$$\|(A^{[\otimes m]} - pI)^{-1}\|_{\mathcal{L}(\mathcal{L}^{\otimes}_{m};\mathbb{E}^{\otimes}_{m})} \leq C_1(p) \cdot \|(A^{[\otimes m]} - pI)^{-1}\|_{\mathcal{L}(\mathcal{L}^{\otimes}_{m})} + C_2(p), \tag{74}$$

where the constants $C_1(p)$ *and* $C_2(p)$ *depend on* $\max\{1, e^{-\tau \operatorname{Re} p}\}$ *in a monotonically increasing way. Moreover, analogous statement holds for* $A^{[\wedge m]}$.

Frequency inequalities associated with $\mathcal{F}$

We associate with each $\mathcal{F}$ the frequency inequality on the line $\operatorname{Re} p = -\nu_0$ (with $\nu_0 \in \mathbb{R}$) avoiding the spectrum of $A^{[\wedge m]}$ as follows.

(FI) For some $\delta > 0$ and any p with $\operatorname{Re} p = -\nu_0$ we have

$$\mathcal{F}^{\mathbb{C}}(-(A^{[\wedge m]} - pI)^{-1} B^{\wedge}_{m} \eta, \eta) \leq -\delta \, |\eta|^2_{(\mathbb{U}^{\wedge}_{m})^{\mathbb{C}}} \text{ for any } \eta \in \left(\mathbb{U}^{\wedge}_{m}\right)^{\mathbb{C}}. \tag{75}$$

Here $\mathcal{F}^{\mathbb{C}}$ is the Hermitian extension of $\mathcal{F}$.

Existence of Lyapunov functionals for Ξ_m

Theorem

Suppose for some $\nu_0 \in \mathbb{R}$ the spectrum of $A^{[\wedge m]}$ avoids the line $-\nu_0 + i\mathbb{R}$ and there are exactly j eigenvalues with $\operatorname{Re}\lambda \geq -\nu_0$. Let the frequency inequality w.r.t. $\mathcal{F}$ defining a quadratic constraint be satisfied. Then there exists a bounded self-adjoint operator $P \in \mathcal{L}(\mathcal{L}^{\wedge}_m)$ such that for its quadratic form $V(\Phi) := (\Phi, P\Phi)_{\mathcal{L}^{\wedge}_m}$ and some $\delta_V > 0$ for the cocycle Ξ_m in $\mathcal{L}^{\wedge}_m$ corresponding to (40) we have

$$e^{2\nu_0 t} V(\Xi^t_m(\mathfrak{p}, \Phi)) - V(\Phi) \leq -\delta_V \int_0^t e^{2\nu_0 s} \left|\Xi^s_m(\mathfrak{p}, \Phi)\right|^2_{\mathcal{L}^{\wedge}_m} ds. \qquad (76)$$

for any $t \geq 0$, $\mathfrak{p} \in \mathcal{P}$ and $\Phi \in \mathcal{L}^{\wedge}_m$.
Moreover, $V(\cdot)$ is positive on the stable subspace $\mathcal{L}^s_m(\nu_0)$ and negative on the unstable subpsace $\mathcal{L}^u_m(\nu_0)$ of $A^{[\wedge m]} + \nu_0 I$.

Exponential stability of Ξ_m and gaps in the Sacker-Sell spectrum

In the case $j = 0$ and $\nu_0 > 0$, from (76) we have the uniform exponential stability of the cocycle Ξ_m with the exponent ν_0, i.e. for some $M(\nu_0) > 0$ we have

$$|\Xi^t_m(\mathfrak{p}, \Phi)|_{\mathcal{L}^\wedge_m} \leq M(\nu_0) e^{-\nu_0 t} |\Phi|_{\mathcal{L}^\wedge_m} \text{ for all } t \geq 0, \mathfrak{p} \in \mathcal{P}, \Phi \in \mathcal{L}^\wedge_m. \quad (77)$$

In the case $(\mathcal{P}, \pi)$ is a flow, from (76) we obtain that $-\nu_0$ is a gap of rank j in the Sacker-Sell spectrum of Ξ_m, i.e. the cocycle $e^{\nu_0 t}\Xi^t_m$ admits uniform exponential dichotomy with the unstable bundle of rank j. To construct the corresponding bundles, one may use our work [4]. Here it is important that the cocycle Ξ_m is uniformly eventually compact.

Numerical computation of frequency inequalities: self-adjoint nonlinearities

Suppose $\mathbb{M} = \mathbb{U}$ and that $F'(\mathfrak{p})$ is a self-adjoint operator satisfying $0 \leq (F'(\mathfrak{p})M, M) \leq \Lambda^2(M, M)$ for each $\mathfrak{p} \in \mathcal{P}$ and $M \in \mathbb{M}$. Then for the quadratic form $\mathcal{G}(M, \eta)$ of $M \in \mathbb{M}^{\wedge}_{m}$ and $\eta \in \mathbb{U}^{\wedge}_{m}$ given by

$$\mathcal{G}(M, \eta) := \Lambda(M, \eta)_{\mathbb{U}^{\wedge}_{m}} - (\eta, \eta)_{\mathbb{U}^{\wedge}_{m}}, \tag{78}$$

the associated quadratic form $\mathcal{F}(\Phi, \eta) := \mathcal{G}(C^{\wedge}_{m}\Phi, \eta)$ of $\Phi \in \mathbb{E}^{\wedge}_{m}$ and $\eta \in \mathbb{U}^{\wedge}_{m}$ defines a quadratic constraint.
Then the frequency inequality associated with $\mathcal{F}$ is equivalent to

$$\inf_{\omega \in \mathbb{R}} \inf_{\substack{\eta \in (\mathbb{U}^{\wedge}_{m})^{\mathbb{C}}, \\ \eta \neq 0}} \frac{(S_W(-\nu_0 + i\omega)\eta, \eta)_{(\mathbb{U}^{\wedge}_{m})^{\mathbb{C}}}}{|\eta|^2_{(\mathbb{U}^{\wedge}_{m})^{\mathbb{C}}}} + \Lambda^{-1} > 0, \tag{79}$$

where $S_W(p) := \frac{1}{2}(W(p) + W^*(p))$ is the additive symmetrization of $W(p) = -C^{\wedge}_{m}(A^{[\wedge m]} - pI)^{-1}B^{\wedge}_{m}$.

Numerical computation of frequency inequalities: approximation

Recall the frequency inequality associated with $\mathcal{F}$ is equivalent to

$$\inf_{\omega \in \mathbb{R}} \inf_{\substack{\eta \in (\mathbb{U}_m^{\wedge})^{\mathbb{C}}, \\ \eta \neq 0}} \frac{(S_W(-\nu_0 + i\omega)\eta, \eta)_{(\mathbb{U}_m^{\wedge})^{\mathbb{C}}}}{|\eta|^2_{(\mathbb{U}_m^{\wedge})^{\mathbb{C}}}} + \Lambda^{-1} > 0, \tag{80}$$

where $S_W(p) := \frac{1}{2}(W(p) + W^*(p))$ is the additive symmetrization of $W(p) = -C_m^{\wedge}(A^{[\wedge m]} - pI)^{-1}B_m^{\wedge}$.
Take an orthogonal basis $e_1, e_2, \dots$ in $(\mathbb{U}_m^{\wedge})^{\mathbb{C}} = (\mathbb{M}_m^{\wedge})^{\mathbb{C}}$. Let P_N be the orthogonal projector onto $\operatorname{Span}\{e_1, \dots, e_N\}$. Let us put

$$\alpha_N(\omega) := \inf_{\substack{\eta \in (\mathbb{U}_m^{\wedge})^{\mathbb{C}}, \\ \eta \neq 0}} \frac{(P_N S_W(-\nu_0 + i\omega)P_N\eta, P_N\eta)_{(\mathbb{U}_m^{\wedge})^{\mathbb{C}}}}{|P_N\eta|^2_{(\mathbb{U}_m^{\wedge})^{\mathbb{C}}}} \tag{81}$$

Numerical computation of frequency inequalities: pointwise convergence

Recall

$$\alpha_N(\omega) := \inf_{\substack{\eta \in (\mathbb{U}^{\wedge}_m)^{\mathbb{C}}, \\ \eta \neq 0}} \frac{(P_N S_W(-\nu_0 + i\omega) P_N \eta, P_N \eta)_{(\mathbb{U}^{\wedge}_m)^{\mathbb{C}}}}{|P_N \eta|^2_{(\mathbb{U}^{\wedge}_m)^{\mathbb{C}}}} \tag{82}$$

It can be shown that for each $\omega \in \mathbb{R}$ we have $\alpha_N(\omega) \to \alpha(\omega)$ as $N \to \infty$, where

$$\alpha(\omega) = \inf_{\substack{\eta \in (\mathbb{U}^{\wedge}_m)^{\mathbb{C}}, \\ \eta \neq 0}} \frac{(S_W(-\nu_0 + i\omega)\eta, \eta)_{(\mathbb{U}^{\wedge}_m)^{\mathbb{C}}}}{|\eta|^2_{(\mathbb{U}^{\wedge}_m)^{\mathbb{C}}}}. \tag{83}$$

Numerical computation of frequency inequalities: problems

For each $\omega \in \mathbb{R}$ we have $\alpha_N(\omega) \to \alpha(\omega)$ as $N \to \infty$, but

1. The convergence depends on ω: the wider interval of ω we want, the larger N we should take.
2. Computing $\alpha_N(\omega)$ requires solving the resolvent equation, that is a first-order PDE in the qube $[-\tau, 0]^m$ with boundary conditions containing both partial derivatives and delays, for each basis vector upto Nth.
3. For large N we deal with highly oscillating functions in the basis that cause high computational errors.
4. Unlike in the case $m = 1$, $\alpha(\omega)$ do not vanish as $\omega \to \infty$. But, in concrete examples, it seems to display an asymptotically as $\omega \to \infty$ periodic (or almost periodic) pattern.

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Thanks for your attention!

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