Open problems in operator semigroup theory (OPSO 2021 and 2022)

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DRAFT

Generation problems

Problem 1 (Inverse generator problem)**.** Let H be a Hilbert space over $\mathbb{C}$ and let $A : D(A) \subseteq H \to H$ be the infinitisimal generator of a bounded semigroup on H. Assume further that A^{-1} exists as a densely defined, closed operator.

Is A^{-1} the infinitesimal generator of a bounded semigroup?

Comments. This problem was originally posed (for Banach spaces) by R. de Laubenfels in [1]. However, it is not hard to show that for general Banach spaces the answer to the problem is negative, see e.g. [4]. The counter example can be chosen such that the strongly continuous semigroup generated by A is a contraction semigroup, whereas in Hilbert spaces the answer to the problem is positive for a generator of a contraction semigroup (almost trivial to show).

There is a strong relation between the inverse generator problem and the question whether the Cayley transform of A is power bounded, i.e. if $\sup_n \|A_d^n\| < \infty$, where $A_d = (I + A)(I - A)^{-1}$, see [3]. The latter question is related to numerical analysis, since this Cayley transform pops up when applying the Crank-Nicolson scheme to the differential equation $\dot{x}(t) = Ax(t)$.

When the answer to the inverse generator problem is positive, then the (strong) stability of the semigroup generated by A is equivalent to the (strong) stability of the semigroup generated by A^{-1}. Furthermore, it is equivalent to the strong stability of A_d, i.e., $\lim_{n\to\infty} A_d^n x = 0$ for all $x \in H$, [3].

For finite-dimensional Hilbert spaces H, it is clear that the problem has a positive answer. For these spaces the question is; if there exists a constant c independent of the dimension of H such that $\sup_{t\geq 0} \|e^{A^{-1}t}\| \leq c \sup_{t\geq 0} \|e^{At}\|$.

In 2017, a nice survey on the problem appeared [2]. In that paper, the interested reader can find more results and references on the inverse generator problem.

Communicated by Hans Zwart.

References:

[1] R. deLaubenfels. Inverses of generators. *Proceedings of the American Mathematical Society*, Vol. 104, 2, pp. 443–448, 1988.

[2] A. Gomilko. Inverses of semigroup generators: a survey and remarks. *Banach Center Publications*, Vol. 102, pp. 107–142, 2017.

[3] B.-Z. Guo and H. Zwart. On the relation between stability of continuous- and discrete-time evolution equations via the Cayley transform. *Integral Equations and Operator Theory*, Vol. 54, 3, pp. 349 – 383, 2006.

[4] H. Zwart Is A^{-1} an infinitesimal generator? *Perspectives in Operator Theory. Banach Center Publications*, Vol. 75, pp. 303–313, 2007.

Problem 2 (An analytic semigroup generation problem)**.** This problem, arising in the theory of Gaussian open quantum systems, consists in proving that a dissipative operator G which is quadratic in creation and annihilation operators (or, after a unitary transformation, a differential operator quadratic in partial derivatives ∂_j and multiplication by coordinates x_k) generates an analytic semigroup. The solution has applications in the proof of strong positivity, irreducibility and regularity properties of Gaussian open quantum systems.

Let a_j, a_k^* be the annihilation and creation operators on $\ell^2(\mathbb{N}^d; \mathbb{C})$ defined by closure from their action on the canonical orthonormal basis $(e(n_1, \ldots, n_d))_{n\in\mathbb{N}^d}$

$$\begin{aligned} a_j\, e(n_1, \ldots, n_d) &= \sqrt{n_j}\; e(n_1, \ldots, n_{j-1}, n_j - 1, \ldots, n_d), \\ a_k^*\, e(n_1, \ldots, n_d) &= \sqrt{n_k + 1}\; e(n_1, \ldots, n_{k-1}, n_k + 1, \ldots, n_d). \end{aligned}$$

Let H, L_ℓ be the closures of operators defined on the canonical basis by

$$\begin{aligned} H &= \sum_{j,k=1}^{d} \left(\Omega_{jk} a_j^* a_k + \frac{\kappa_{jk}}{2} a_j^* a_k^* + \frac{\overline{\kappa}_{jk}}{2} a_j a_k \right) + \sum_{j=1}^{d} \left(\frac{\zeta_j}{2} a_j^* + \frac{\bar{\zeta}_j}{2} a_j \right), \\ L_\ell &= \sum_{k=1}^{d} \left(\overline{v}_{\ell k} a_k + u_{\ell k} a_k^* \right) \qquad \ell = 1, \ldots, 2d \end{aligned}$$

where $\Omega := (\Omega_{jk})_{1 \le j,k \le d} = \Omega^*$ and $\kappa := (\kappa_{jk})_{1 \le j,k \le d} = \kappa^{\mathrm{T}} \in M_d(\mathbb{C})$, are $d \times d$ complex matrices with Ω Hermitian and κ symmetric, $V = (v_{\ell k})_{1 \le \ell \le 2d, 1 \le k \le d}$, $U = (u_{\ell k})_{1 \le \ell \le 2d, 1 \le k \le d}$ are $2d \times d$ complex matrices $\zeta = (\zeta_j)_{1 \le j \le d} \in \mathbb{C}^d$.

It is not difficult to show (see e.g. [1] Proposition 4.9) that the closure of the operator G defined on the canonical basis by

$$G = -\mathrm{i}H - \frac{1}{2} \sum_{\ell=1}^{2d} L_\ell^* L_\ell$$

generates a C_0 contraction semigroup $P = (P_t)_{t \ge 0}$ on $\ell^2(\mathbb{N}^d; \mathbb{C})$.

Suppose that the non-degeneracy condition (block-matrix form)

$$\mathbb{K} = \begin{bmatrix} V^{\mathrm{T}} \\ U^* \end{bmatrix} \begin{bmatrix} \overline{V} & U \end{bmatrix} = \begin{bmatrix} V^{\mathrm{T}} \overline{V} & V^{\mathrm{T}} U \\ U^* \overline{V} & U^* U \end{bmatrix} > 0$$

holds, then sufficient conditions for P to be an analytic semigroup are also available.

The problem it to find a *classification* of the set of parameters $\Omega, \kappa, U, V, \zeta$ for which P is analytic.

Alternatively, by the unitary correspondence of the above basis with multidimensional Hermite polynomials (multiplied by $\exp(-|x|^2/2)$ normalized), one can formulate the problem with differential operators

$$a_j = \frac{1}{\sqrt{2}} \left(x_j + \frac{\partial}{\partial x_j} \right), \qquad a_k^* = \frac{1}{\sqrt{2}} \left(x_k - \frac{\partial}{\partial x_k} \right)$$

In this case strict positivity of $\mathbb{K}$ implies

$$\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \mathbb{K} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} > 0$$

from which one finds the strong ellipticity condition for the self-adjoint part $G_0 = -(1/2) \sum_{\ell=1}^{2d} L_\ell^* L_\ell$ of G

$$\mathrm{Re} \sum_{j,k=1}^{d} \left(U^* U + V^{\mathrm{T}} \overline{V} - V^{\mathrm{T}} U - U^* \overline{V} \right)_{jk} \overline{z}_j z_k > \epsilon \|z\|^2$$

for $z = (z_j)_{1\le j\le d} \in \mathbb{C}^d$.

Thinking of spectra of G_0 and H one wonders if the semigroup P generated by P is analytic when $\mathbb{K} > 0$ and H is bounded from below or from above.

Communicated by Franco Fagnola.

References:

[1] F. Fagnola, Quantum Markov Semigroups and Quantum Markov Flows. *Proyecciones* **18** n.3 1–144 (1999).

Problem 3 (Lumer–Phillips for transition semigroups)**.** Let Ω be a Polish space. We call a semigroup $T = (T(t))_{t\ge 0}$ of contractions on $C_b(\Omega)$ a *transition semigroup* if

(i) for every $f \in C_b(E)$ we have $T(t)f \to T(s)f$ locally uniformly whenever $t \to s$ and

(ii) for every $t \ge 0$ and every uniformly bounded sequence $(f_n)_{n\in\mathbb{N}} \subset C_b(\Omega)$ that converges locally uniformly to f, we have that $T(t)f_n$ converges locally uniformly to f.

Is there a chracterization of the generators of such a semigroup akin to the Lumer–Phillips theorem?

Comments. The name 'transition semigroup' is inspired by applications in probability theory, where semigroups with the above properties frequently appear as transition semigroups of Markov processes (typically, these semigroups are additionally positive). We should point out that semigroups of this kind (at least at first glance) do not fit into the theory of semigroups on locally convex spaces (see, e.g. [5]) as this theory requires equicontinuity of the operators involved. But even the simplest example of the heat semigroup on $C_b(\mathbb{R}^d)$ shows that one cannot expect equicontinuity with respect to the topology τ_{co} of uniform convergence on compact subsets of $\mathbb{R}^d$.

Consequently, in the literature several approaches were developed where this equicontinuity condition was weakened such as the 'theory of of weakly continuous semigroups' by Cerrai [1] (where instead of $C_b(\Omega)$ one works on the space $BUC(\Omega)$) or the 'theory of bi-continuous semigroups' by Kühnemund [3]; both approaches allow for a Hille–Yosida type generation result. On the other hand, [4, Theorem 4.4] shows that conditions (i) and (ii) the above definition already entail equicontinuity: Not with respect to τ_{co} but with respect to the so-called *strict topology* β_0 (which agrees with τ_{co} on $\|\cdot\|_\infty$-bounded subsets of $C_b(\Omega)$). This allows us to use the results from [5] after all to characterize the generators of transition semigroups.

Thus, characterizations of generators of transition semigroups are available in the literature. However, to the best of my knowledge, none of these Hille–Yosida type theorems was ever used to establish that a certain operator generates a transition semigroup (even though many examples of such semigroups and also their generators are known). This is not as surprising as it might seem, for even in the setting of strongly continuous semigroups the Hille–Yosida theorem is difficult to apply. This is due to the fact that this result requires us (in the case of bounded semigroups) to prove uniform boundedness of the family $\{\lambda^n R(\lambda, A)^n : \lambda > 0, n \in \mathbb{N}\}$, which is difficult in concrete examples. In the case of non-strongly continuous semigroups one would have to establish equicontinuity of this family of operators – an even harder task.

The 'weapon of choice' to prove that a given operator generates a strongly continuous semigroup is rather the Lumer–Phillips theorem (see [2, Theorem II.3.15]) which, however, only characterizes the generators of *contraction* semigroups. The main advantage of the Lumer–Phillips theorem is that one does not have to consider powers of the resolvent. Indeed, given dissipativity, we only need to check the so-called *range condition*, i.e. we need to prove that $\lambda - A$ has dense range for some $\lambda > 0$.

It would be very interesting to have a Lumer–Phillips type result for transition semigroups. Part of the problem is to find out how such a result should look like. Here, usability is (in my opinion) more important than generality. If we can obtain a sufficient condition for generation (which might make use of additional properties that one has at hand in many possible applications such as positivity of the resolvent or the strong Feller property for the resolvent or ...) this would be already be very nice.

Communicated by Markus Kunze.

References:

[1] S. Cerrai, *A Hille-Yosida theorem for weakly continuous semigroups*, Semigroup Forum, 49 (1994), pp. 349 – 367.

[2] K. Engel and R. Nagel, *One-parameter semigroups for linear evolution equations*, vol. 194 of Graduate Texts in Mathematics, Springer-Verlag, New York, 2000.

[3] F. Kühnemund, *A Hille-Yosida theorem for bi-continuous semigroups*, Semigroups Forum, 67 (2003), pp. 205 – 225.

[4] M. Kunze, *Continuity and equicontinuity of semigroups on norming dualpairs*, Semigroup Forum, 79 (2009), pp. 540 – 560.

[5] K. Yosida, *Functional Analysis*, vol. 123 of Grundlehren der mathematischen Wissenschaften, Springer, 1980.

Long-term behaviour of semigroups

Problem 4 (Tauberian Theorem for Semigroups of Kernel Operators)**.** Let $E := L^p(\Omega, \mu)$ for $1 \le p \le \infty$ and a σ-finite measure space (Ω, μ) and let $(T_t)_{t\in[0,\infty)} \subseteq \mathcal{L}(E)$ be a strongly continuous semigroup on E such that T_t is a positive *kernel operators* for each $t > 0$, meaning that there exists a measurable function $k_t \colon \Omega \times \Omega \to \mathbb{R}_+$ such that

$$(T_t f)(y) = \int_\Omega k_t(y, x) f(x) \mathrm{d}\mu(x)$$

for almost every $y \in \Omega$.

Assume that $(T_t)_{t\in[0,\infty)}$ is *mean ergodic*, meaning that the limit

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T T_t f \mathrm{d}t$$

exists in E for all $f \in E$. Does it follow that (T_t) is *strongly convergent*, i.e. $\lim_{t\to\infty} T_t f$ exists for all $f \in E$?

Comments. It is well known that positive semigroups of kernel operators are strongly convergent provided that the semigroups possesses a fixed point $f \in E$ satisfying $f(x) > 0$ for almost every $x \in \Omega$. This has been proven in a very general setting in [1, Theorem 3.5]. The existence of such an "quasi-interior" fixed point is crucial and cannot be omitted: For instance, the Gaussian semigroup on $L^1(\mathbb{R})$ is not strongly convergent and fulfils all assumptions of [1, Theorem 3.5] except that it does not possess a quasi-interior fixed point. On the other hand, the Gaussian semigroup is not mean ergodic on $L^1(\mathbb{R})$. In fact, using [1, Theorem 3.5] it is not difficult to show the following Tauberian theorem:

Let $(T_t)_{t\in[0,\infty)}$ be a positive, bounded and mean ergodic C_0-semigroup on $L^1(\Omega)$ for any measure space Ω. If T_{t_0} is kernel operator for some $t_0 > 0$, then $(T_t)_{t\in[0,\infty)}$ is strongly convergent.

The theorem above and similar results for semigroups on spaces of measures can be found in [2, Theorems 2.1, 4.1, 5.4]. This gives rise to the conjecture that the existence of a quasi-interior fixed point in [1, Theorem 3.5] can always, i.e. for every $1 \le p \le \infty$ or – more generally – for every Banach lattice E, be omitted in case of a mean ergodic C_0-semigroup $(T_t)_{t\in(0,\infty)}$.

In support of this conjecture one may note that the Gaussian semigroup on $L^p(\mathbb{R})$ for $p \in (1, \infty)$ is mean ergodic and converges strongly to 0.

Communicated by Moritz Gerlach.

References:

[1] M. Gerlach and J. Glück. Convergence of Positive Operator Semigroups. *Trans. Amer. Math. Soc.*, 372(9):6603–6627, 2019.

[2] M. Gerlach, J. Glück, and M. Kunze. Stability of transition semigroups and applications to parabolic equations. preprint, 2020

Positivity

Problem 5 (Infinite speed of propagation)**.** Let $E := L^p(\Omega, \mu)$ for $1 \leq p < \infty$ and a σ-finite measure space (Ω, μ) and let $:= (T_t)_{t\in[0,\infty)} \subseteq \mathcal{L}(E)$ be a strongly continuous semigroup of positive kernel operators on E, meaning that for each $t > 0$ there exists a measurable function $k_t \colon \Omega \times \Omega \to \mathbb{R}_+$ such that

$$(T_t f)(y) = \int_\Omega k_t(y, x) f(x) \mathrm{d}\mu(x)$$

for almost every $y \in \Omega$.

Assume that (T_t) is *irreducible*, i.e. for all non-zero and positive $f, g \in E_+$ there exists $t \geq 0$ such that $T_t f \wedge g \neq 0$ (where $\wedge$ denotes the infimum in the Banach lattice E). Does it follow that (T_t) is *expanding*, i.e. $T_t f > 0$ almost everywhere for all $t > 0$ and all non-zero positive $f \in E_+$?

Comments. The notion of irreducibility is of great importance – for instance in the spectral theory of positive semigroups – and obviously, every expanding semigroup is irreducible. On the other hand, the rotation semigroup on the unit circle is an easy example of an irreducible semigroup that fails to be expanding.

However, under certain circumstances the two properties, irreducible and expanding, are equivalent. For instance, every holomorphic positive semigroup is known to be expanding if it is irreducible [2, Theorem C-III 3.2]. A surprisingly little known fact is that the same holds for all positive semigroups on atomic Banach lattices like ℓ^p for $1 \leq p < \infty$. This is due to a fact which is referred to as "Lévy's Theorem" in literature: for every stochastic transition matrix $(p_{i,j})$ one either has $p_{i,j}(t) = 0$ or $p_{i,j}(t) > 0$ for all $t > 0$. The proof given by K. L. Chung in the appendix of [3] for this statement can easily be transferred to the setting of semigroups on atomic spaces. Since operators on atomic spaces often serve as a prototype for general kernel operators, it is natural to ask whether the same implication is true for semigroups of kernel operators.

The notation *expanding* is not used in a uniform matter in the literature; the property is, for instance, also referred to as "strongly positive" or "positivity improving". We would like to advertise the more systematic naming convention of [1, Section 9.1].

Communicated by Moritz Gerlach.

References:

[1] Y. A. Abramovich and C. D. Aliprantis. *An invitation to operator theory*, volume 50 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2002.

[2] W. Arendt, A. Grabosch, G. Greiner, U. Groh, H. P. Lotz, U. Moustakas, R. Nagel, F. Neubrander, and U. Schlotterbeck. *One-parameter semigroups of positive operators*, volume 1184 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1986.

[3] K. L. Chung. On the boundary theory for Markov chains. *Acta Mathematica*, 110:19 – 77, 1963.

Problem 6 (Positive commutator problem)**.** Let E be a Banach lattice and let $C\colon E \to E$ be a positive quasinilpotent compact operator. Do there exist positive operators $A, B\colon E \to E$ such that $C = AB - BA$ with one of A and B compact?

Comments. Given an associative algebra $\mathcal{A}$, the natural question is to determine all commutators of $\mathcal{A}$. Shoda [11] proved that a matrix $C \in \mathbb{M}_n(F)$ is a commutator if and only if the trace of C is zero. Wintner [12] proved that the identity in a unital Banach algebra is not a commutator. By passing to the Calkin algebra, Wintner's result immediately implies that a bounded operator on a Banach space which is of the form $\lambda I + K$ for some nonzero scalar λ and a compact operator K is not a commutator. Henceforth, researchers tried to characterize which operators on a given Banach space are commutators. The complete characterization of commutators in the Banach algebra $\mathcal{B}(\mathcal{H})$ of all bounded operators on an infinite-dimensional Hilbert space $\mathcal{H}$ is due to Brown and Pearcy [4]. They proved that a bounded operator C on $\mathcal{H}$ is a commutator if and only if it is not of the form $\lambda I + K$ for some nonzero scalar λ and some operator K from the unique maximal ideal in $\mathcal{B}(\mathcal{H})$. Apostol ([1, 2]) proved that a bounded operator on either ℓ^p $(1 < p < \infty)$ or c_0 is a commutator if and only if it is not of the form $\lambda I + K$ where $\lambda \neq 0$ and

K is compact. In the case of the Banach space ℓ^1 the same characterization was obtained by Dosev in [5]. In the case of the Banach space ℓ^∞ Dosev and Johnson [6] proved that a bounded operator is a commutator if and only if it is not of the form $\lambda I + K$ where $\lambda \neq 0$ and K is strictly singular.

The study of positive commutators of positive operators on a given Banach lattice was initiated in [3]. The assumption on positivity of A, B and $C := AB - BA$ may lead to some restrictions on the commutator. Namely, the authors proved that the positive commutator of positive compact operators is quasinilpotent. They also posed a question whether the same is true under the assumption that one of operators is compact. This question was affirmatively and independently solved by R. Drnovšek [7] and N. Gao [9]. Inspired by a result of Schneeberger [10] asserting that a compact operator acting on a separable L^p space ($1 \leq p < \infty$) is a commutator, in [8] authors prove that a positive compact operator acting on a separable L^p space ($1 \leq p < \infty$) is a commutator of positive operators. In [8] authors provide a technical condition under which the answer to the proposed problem is affirmative for positive operators on ℓ^p space ($1 \leq p \leq \infty$) satisfying this technical condition.

Communicated by Marko Kandić.

References:

[1] C. Apostol. Commutators on ℓ_p spaces. *Rev. Roumaine Math. Appl.* 17, 1513–1534, 1972.

[2] C. Apostol. Commutators on c_0-spaces and on ℓ_∞-spaces. *Rev. Roumaine Math. Appl.* 18, 1025–1032, 1973.

[3] J. Bračič, R. Drnovšek, Y. B. Farforovskaya, E. L. Rabkin and J. Zemánek. On positive commutators. *Positivity*, 14(3), 431–439, 2010.

[4] A. Brown and C. Pearcy. Structure of commutators of operators. *Ann. of Math.* 82, 112–127, 1965.

[5] D. T. Dosev. Commutators on ℓ_1. *J. Funct. Anal.* 256, 3490–3509, 2009.

[6] D. Dosev and W. B. Johnson. Commutators on l^∞., *Bull. Lond. Math. Soc.* 42(1), 155–169, 2010.

[7] R. Drnovšek. Once more on positive commutators. *Studia Math.*, 211(3), 241–245, 2012.

[8] R. Drnovšek and M. Kandić. Positive operators as commutators of positive operators. *Studia Math.* 245 (2), 185–200, 2019.

[9] N. Gao. On commuting and semi-commuting positive operators. *Proc. Amer. Math. Soc*, 142(8), 2733–2745, 2014.

[10] C. Schneeberger. Commutators on a separable L^p-space. *Proc. Amer. Math. Soc.* 28, 464–472, 1971.

[11] K. Shoda. Einige Satze Uber Matrizen. *Japanese J. Math.* 13, 361–365, 1937.

[12] A. Wintner, The unboudnedness of quantum-mechanical matrices. *Phys. Rev.* 71, 738–739, 1947.

Eventual Positivity

Problem 7 (Stability of eventually positive semigroups)**.** Let M be a von Neumann algebra equipped with a normal, semi-finite, and faithful trace τ and $p \in [1, \infty)$. Let $(e^{tA})_{t\in[0,\infty)}$ be an individually eventually positive C_0-semigroup on $L^p(M, \tau)$.

Is $s(A) = \omega_0(A)$?

Comments. A C_0-semigroup $(e^{tA})_{t\in[0,\infty)}$ on an ordered Banach space E is said to be *individually* eventually positive if for each $0 \leq f \in E$, there exists $t_0 \geq 0$ such that $e^{tA}f \geq 0$ for all $t \geq t_0$. If the time t_0 can be chosen independently of the initial value f, then we call the semigroup *uniformly eventually positive*.

For the case $p = 1, 2$, the answer is positive as shown in [2, Theorem 6.2.6] (see also [1, Theorem 7.8]); note that the proofs given in the aforementioned references are for the commutative L^p-spaces but they can be adapted to the non-commutative setting. If the semigroup is uniformly eventually positive and E is the classical L^p-space, then a positive answer is given by Vogt [3]. However, for the non-commutative L^p-spaces with $p \neq 1, 2$, the answer is not known even for positive semigroups.

This problem was posed by Ralph Chill.

Communicated by Sahiba Arora.

References:

[1] Daniel Daners, Jochen Glück, James B. Kennedy, Eventually positive semigroups of linear operators, *J. Math. Anal. Appl.*, 433, 1561–1593, 2016.

[2] Jochen Glück, Invariant sets and long time behaviour of operator semigroups, *Ph.D. Dissertation*, Universität Ulm, 2016.

[3] Hendrik Vogt, Stability of uniformly eventually positive C_0-semigroups on L_p-spaces, *Preprint*, arXiv: 2110.02310v2, 2021.

Problem 8 (Characterisation of eventual invariance by means of form methods)**.** Let $(e^{tA})_{t\in[0,\infty)}$ be a C_0-semigroup on a Hilbert space H, where the operator $-A$ is associated to a form $a : V \times V \to H$ with form domain $V \subseteq H$. Let $C \subseteq H$ be closed and convex. Give a characterisation, in terms of a and V, of one or both of the following properties:

(a) The semigroup leaves C *individually eventually invariant*, i.e., for each $c \in C$ exists a time $t_0 \in [0,\infty)$ such that $e^{tA}c \in C$ for all $t \geq t_0$.

(b) The semigroup leaves C *uniformly eventually invariant*, i.e., there exists a time $t_0 \in [0,\infty)$ such that $e^{tA}c \in C$ for all for each $c \in C$ and all $t \geq t_0$.

An answer would be particularly interesting in the case where H is an L^2-space and C is the usual positive cone in H – in this case, the properties mentioned in (a) and (b) become the properties *individual eventual positivity* and *uniform eventual positivity* that are mentioned in several further problems here.

Comments. Invariance rather than *eventual invariance* can indeed be characterised by means of a and V. This is a very useful result due to Ouhabaz [**?**, ...]. The criterion becomes particularly simple if H is an L^2-space and C is the usual positive cone. If, in addition, a is symmetric, this result is much older and goes back to Beurling and Deny [**?**, ...].

A characterisation of (individual or uniform) eventual positivity by means of form methods might have the potential to considerably extent the applicability of the theory of eventual positivity.

Communicated by Jochen Glück.

References:

[1] Author 1 and Author 2. On very interesting problems. *Journal of examples and counterexamples*, 145(5), 144–156, 1998.

[2] Author. On even more interesting problems. *Rejecta Mathematica*, 144(4), 133–157, 2001.

L^∞-bounds

Problem 9 (L^∞-boundedness problem)**.** Let (Ω, μ) be a σ-finite measure space. Let V be a dense subspace of $L^2(\Omega)$ (complex-valued functions) and suppose that $\mathfrak{a} : V \times V \to \mathbb{C}$ is a closed sectorial sesquilinar form, i.e. linear in the first and antilinear in the second argument. Here, sectoriality means that, for some $\theta \in [0, \frac{\pi}{2})$,

$$\forall u \in V : \mathfrak{a}(u,u) \in \Sigma_\theta := \{z \in \mathbb{C} \setminus \{0\} : |\arg z| \le \theta\} \cup \{0\},$$

and closedness of $\mathfrak{a}$ means that V is complete for the norm

$$\|u\|_{\mathfrak{a}} := \Big(\operatorname{Re} \mathfrak{a}(u,u) + \|u\|^2_{L^2(\Omega)}\Big)^{1/2}.$$

Let the linear operator A in $L^2(\Omega)$ be associated with a $\mathfrak{a}$ in the sense that, for $u, h \in L^2(\Omega)$,

$$u \in D(A) \text{ and } Au = h \quad \iff \quad u \in V \text{ and } \forall v \in V : \mathfrak{a}(u,v) = \langle u, h\rangle,$$

where $\langle u, h\rangle := \int_\Omega f\overline{h}\, d\mu$ denotes the usual scalar product in $L^2(\Omega)$. Then $-A$ is negative generator of a bounded analytic semigroup $(T(\cdot))$ in $L^2(\Omega)$, which is contractive on the sector $\Sigma_{\frac{\pi}{2}-\theta}$.

The semigroup $(T(t))_{t\ge 0}$ is called *L^∞-bounded*, if there exists $M > 0$ such that

$$\|T(t)f\|_\infty \le M\|f\|_\infty, \qquad \text{for all } f \in L^2(\Omega) \cap L^\infty(\Omega). \tag{1}$$

Can L^∞-boundedness of $(T(t))_{t\ge 0}$ be characterized in terms of the sesquilinear form $\mathfrak{a}$?

Comments. The problem came up in discussions with Sönke Blunck (at the end of the 1990s). The case $M = 1$ in (1), i.e., *L^∞-contractivity* of $(T(t))_{t\ge 0}$, is characterized in terms of the sesquilinear form $\mathfrak{a}$ by the well-know Beurling-Deny criterion (see, e.g., [3, Theorem 2.7] or [5, Section 2.2]), namely by the condition

$$\forall u \in V : \operatorname{sgn} u(|u| - 1)_+ \in V \text{ and } \operatorname{Re} \mathfrak{a}(u, \operatorname{sgn} u(|u| - 1)_+) \ge 0. \tag{2}$$

Here $v_+ := -((-v) \wedge 0)$, where $\wedge$ denotes the pointwise minimum, and $\operatorname{sgn} u := \frac{u}{|u|}1_{\{u\ne 0\}}$ denotes the sign of the function u.

A characterization of *L^∞-boundedness* of $(T(t))_{t\ge 0}$ could certainly be very useful (for the restrictions that L^∞-contractivity imposes on the coefficients of second order elliptic operators on domains we refer to [5, Section 4.3]).

It is clear that $(T(t))_{t\ge 0}$ is L^∞-bounded if there exists a function $g \in L^\infty(\Omega)$ with $g > 0$ μ-a.e. and $1/g \in L^\infty(\Omega)$ satisfying

$$\|gT(t)f\|_\infty \le \|gf\|_\infty \qquad \text{for all } f \in L^\infty(\Omega) \cap L^2(\Omega), \tag{3}$$

since the norm $f \mapsto \|gf\|_\infty$ is equivalent to $\|\cdot\|_\infty$. A modification of condition (2) characterizes (3), namely

$$\forall u \in V : \operatorname{sgn} u(|u| - g)_+ \in V \text{ and } \operatorname{Re} \mathfrak{a}(u, \operatorname{sgn} u(|u| - g)_+) \geq 0. \tag{4}$$

The proof can be done similar to the proof of equivalence of (2) and L^∞-contractivity. One can also resort to invariance results in $L^2(\Omega)$ and consider the closed convex set

$$K_g := \{f \in L^2(\Omega) : |f| \leq g \ \mu\text{-a.e.}\}.$$

Then it is not hard to check that the projection P_g of $L^2(\Omega)$ onto K is given by $P_g f := \operatorname{sgn} f(|f| \wedge g)$ ($P_g(f)$ is the best approximation of f in K). Then observe $u - P_g(u) = \operatorname{sgn} u(|u| - g)_+$ and apply [4, Theorem 2.1] (we also refer to [5, Section 2.1]).

In view of the comments to Problem 9 the following seems natural to ask.

Problem 10 (L^∞-contraction for a weight problem)**.** In the setting of Problem 9 assume that the semigroup $(T(t))_{t\geq 0}$ is L^∞-bounded, i.e., satisfies (1) for some $M \geq 1$. Does there exist a function $g \in L^\infty(\Omega)$ satisfying $g > 0$ μ-a.e., $1/g \in L^\infty(\Omega)$, and (3)?

Comments. In general this might be more than one can hope for.

Hence we are lead to the following.

Problem 11 (Characterization of L^∞-contraction for a weight)**.** In the setting of Problem 9, can one characterize those L^∞-bounded semigroups $(T(t))_{t\geq 0}$, for which a function $g \in L^\infty(\Omega)$ satisfying $g > 0$ μ-a.e., $1/g \in L^\infty(\Omega)$, and (3) exists?

Comments. The question is whether $(T(t))_{t\geq 0}$ can be made to be a contractive semigroup on the $\|\cdot\|_\infty$-closure of $L^2(\Omega) \cap L^\infty(\Omega)$ in $L^\infty(\Omega)$ for an equivalent norm of the special form $f \mapsto \|gf\|_\infty$. There is, of course and well-known from semigroup theory, a norm $|||\cdot|||$ on $L^2(\Omega) \cap L^\infty(\Omega)$, equivalent to $\|\cdot\|_\infty$, such that

$$|||T(t)f||| \leq |||f||| \qquad \text{for all } f \in L^\infty(\Omega) \cap L^2(\Omega).$$

One can take $|||f||| := \sup_{t\geq 0} \|T(t)f\|_\infty$, which satisfies

$$\|f\|_\infty \leq |||f||| \leq M\|f\|_\infty$$

by assumption (1). So the problem might be seen as an L^∞-counterpart to the question if every bounded C_0-semigroup on a Hilbert space can be made contractive for an equivalent scalar product. The answer to this question is known to be negative and there is a nice characterization of those bounded analytic C_0-semigroups that are contractive for an equivalent scalar product in terms of bounded imaginary powers (and bounded H^∞-calculus) for the negative generator (see [2]).

Still another question seems natural.

Problem 12 (Weight construction for L^∞-contraction)**.** In the setting of Problem 9, assume that a function $g \in L^\infty(\Omega)$ satisfying $g > 0$ μ-a.e., $1/g \in L^\infty(\Omega)$, and (3) exists. How can we find or construct such a function g?

Comments. In the general situation of Problems 9, 10, 11, and 12 it might well be that positivity of the semigroup $(T(t))_{t\geq 0}$ can help, i.e. the assumption that $f \geq 0$ a.e. on Ω implies $T(t)f \geq 0$ a.e. on Ω for all $t > 0$. Recall that positivity of the semigroup can be characterized in terms of the sesquilinear form $\mathfrak{a}$ (see [3], [5]).

Assume for the following that the semigroup $(T(t))_{t\geq 0}$ is positive and that the measure space (Ω, μ) is *finite*. Then $L^\infty(\Omega) \subseteq L^2(\Omega)$ and hence $T(t)f \in L^2(\Omega)$ is defined for any $f \in L^\infty(\Omega)$. In this situation, L^∞-contractivity of $(T(t))_{t\geq 0}$ is characterized by $T(t)1_\Omega \leq 1_\Omega$ μ-a.e. for all $t > 0$ where 1_Ω denotes the characteristic function of Ω.

Now let $g \in L^\infty(\Omega)$ such that $g > 0$ μ-a.e. on and $1/g \in L^\infty(\Omega)$. Then, consequently, (3) is characterized by $T(t)g^{-1} \leq g^{-1}$ μ-a.e. for all $t > 0$ (since this is equivalent to L^∞-contractivity of the positive semigroup $(gT(t)g^{-1})_{t\geq 0}$).

In particular, one has (3) for $g \in L^\infty(\Omega)$ with $g > 0$ μ-a.e. if $h := 1/g \in L^\infty(\Omega)$ is an *eigenfunction* for an eigenvalue $\lambda \geq 0$ of A: Recalling that $-A$ is the generator of $(T(t))_{t\geq 0}$ we obtain $T(t)h = e^{-\lambda t}h \leq h$ μ-a.e. for all $t > 0$.

Specializing further, take $\Omega \subset \mathbb{R}^d$ a bounded domain (with μ the Lebesgue measure) and the usual Dirichlet form $\mathfrak{a}(u, v) := \int_\Omega \nabla u \cdot \overline{\nabla v}\, dx$ with form domain $V := V_N := H^1(\Omega)$ (Neumann boundary conditions) or $V := V_D := H_0^1(\Omega)$ (Dirichlet boundary conditions), and denote the associated operators by A_N and A_D, respectively (the negative Laplacian on Ω with Neumann/Dirichlet boundary conditions).

Both semigroups are well-known to be positive and L^∞-contractive, i.e. they satisfy (3) for $g = 1_\Omega$. Now 1_Ω is an eigenfunction of A_N for the eigenvalue 0. However, for A_D we do not even have $1_\Omega \in V_D$. We do have an eigenfunction $h \in L^\infty(\Omega)$, $h > 0$ μ-a.e. on Ω, for the first eigenvalue $\lambda_0 > 0$ of A_D, but $1/h \notin L^\infty(\Omega)$ due to $h \in V_D = H_0^1(\Omega)$. Hence, considering (positive) eigenfunctions is not sufficient, in general.

It might be more adequate to consider *positive subeigenfunctions*, we refer to [1, II-C Section 3] for the notion of positive subeigenvectors and their role in the characterization of positivity of semigroups on Banach lattices.

Communicated by Peer Kunstmann.

References:

[1] W. Arendt et al. *One-parameter Semigroups of Positive Operators.* Edited by R. Nagel. Springer Lecture Notes Math. 1184, Springer, Berlin, Heidelberg, 1986.

[2] Christian Le Merdy. The similarity problem for bounded analytic semigroups on Hilbert space. *Semigroup Forum* 56 (1998) 205–224.

[3] El-Maati Ouhabaz. L^∞-contractivity of semigroups generated by sectorial forms. *J. London Math. Soc.* (2) 46 (1992) 529–542.

[4] El-Maati Ouhabaz. Invariance of Closed Convex Sets and Domination Criteria for Semigroups. *PotentialAnalysis* 5: 611–625, 1996.

[5] El-Maati Ouhabaz. *Analysis of Heat Equations on Domains.* Princeton University Press, Princeton, 2005.

Further problems (still need to be ordered)

Problem 13 (Charles J. K. Batty)**.** Prove or disprove that non-analytic growth bound $\zeta(T)$ of a C_0-semigroup T coincides with critical growth bound $\omega_{crit}(T)$

Comments. One has

$$\zeta(T) \leq \omega_{ess}(T) \quad \text{and} \quad \zeta(T) = \omega_{crit}(T)$$

in each of the following cases:

- T is a C_0-semigroup on Hilbert space,
- T is a C_0-semigroup on a Banach space, T has L^p-resolvent, $p \in (1, \infty)$,
- T is eventually differentiable.

Comments. (Charles Batty, April 2021)

This problem has not been solved so far. Note that there is some vague similarity to Open Problem 4.

Relevance. This problem is interesting because $\zeta(T)$ and $\omega_{crit}(T)$ are both candidates to be variants of the exponential growth bound $\omega_0(T)$, modulo analytic functions and spectral bounds of the generator modulo horizontal strips. Some variants of standard results have been obtained using $\zeta(T)$ instead of $\omega_(T)$ have been established. See, for example, Section 5 of my article in Perspectives in operator theory, 39–53, Banach Center Publ., 75, Polish Acad. Sci. Inst. Math., Warsaw, 2007.

References:

C. J. K. Batty, M. D. Blake, S. Srivastava: A non-analytic growth bound of Laplace transforms and semigroup of operators. Int. Eq. Op. Th. 45 (2003), no.2, 125-154.

Source: R. Nagel's list of problems collected in 2003 in the workshop in Bari.

Problem 14 (Charles J. K. Batty and Klaus-Jochen Engel)**.** Prove or disprove that semigroup is immediatly norm continuous if and only if
$\|R(is, A)\| \to 0$ as $|s| \to \infty$.

Comment (Charles Batty). Open Problem 3 was answered negatively, by Ralph Chill and Yuri Tomilov (J. Funct. Anal. 256 (2009), no. 2, 352-384), and independently by Tamás Mátrai (Israel J. Math. 168 (2008), 1-28).
References:

• O. El-Mennaoui, K.J. Engel: Towards a characterization of eventually norm continuous semigroup on Banach spaces, Quaest. Math. 19 (1996), 183-190. • O. Blasco, J. Martines: Norm continuity and related notions for semigroup on Banach spaces, Archiv Math. 66 (1996), 470-478. • V. Goerrsmeyer, L. Weis: Norm continuity of C_O-semigroups, Studia Math. 134 (1999), 169-178.

Source: R. Nagel's list of problems collected in 2003 in the workshop in Bari.

Problem 15 (Jerome A. Goldstein)**.** Which C_0-semigrops are "asymptotically analytic"?

Comment. First, we explain the notion of "asyptotically analytic semigroups". Let B be a positive selfadjoint operator in Hilbert space X, and let $a > 0$. Let u be a solution of the graph equation

$$u'' + 2au' + Bu = 0.$$

This problem is governed by a C_0-semigroup on energy space besed on X. Eckstein-Goldstein-Leggas [EJDE, Proc, Conf. 03, 1999] proved that

$$u(t) = v(t) + w(t),$$

where u satisfies the heat equation

$$2av' + Bv = 0$$

and $\|w(t)\| = o(\|v(t)\|)$ as t tends to infinity. This leads to the notion of "asymptotically analytic".

Let $(T(t))_{t\geq 0}$ be a C_0-semigroup on a Banach space X, let $S := (s(t))_{t\geq 0}$ be an analytic C_0-semigroup on a Banach space Y and let be P a (somehow natural) bounded linear operator from X to Y. We call S "asymptotically analytic" if there exist S, P as above such that from all $f \in X$ there is $g \in Y$ so that

$$T(t)f = S(t)g + w(t), \quad t \geq 0,$$

where $\|w(t)\| = o(\|S(t)g\|)$ as t tends to infinity.

Thus asymptotically analytic semigroups have the asymptotics of analytic semigroups, except for errors that are relatively small asymptotically. As a first step it would be of interest to have some results in the case of $Y = X$ and $P = I$. More generally, which C_0-groups (such as those governing second order equations) and which nonanalytic semigroups governing FDE's are asyptotically anatilic?

Comments. (Charles Batty, April 2021) Note that Open Problem 2 and Open Problem 4 have some vague similarity.

Source: R. Nagel's list of problems collected in 2003 at the workshop in Bari.

Problem 16 (Birgit Jacob, Hans Zwart)**.** Let $(T(t))_{t\geq 0}$ be a contraction semigroup with generation A on a Hilbert space H. Consider the following properties.

(i) There exist $m \geq 0$ such that

$$\|(\lambda - A)x\| \geq m|Re\lambda|\|x\| \quad \text{for all } Re\lambda < 0 \text{ and all } x \in H.$$

(ii) There exist $m_1 > 0$ such that

$$\|T(t)x\| \geq m_1\|x\| \quad \text{for all } t \geq 0, x \in H$$

Does (i) imply (ii)?

Comment. Condition (ii) always implies (i). This is not true if $(T(t))_{t\geq 0}$ is only bounded, but it is true if $\lambda \in \rho(A)$ with $Re\lambda < 0$.

Comment (Charles Batty, April 2021) I do not know of an answer to this problem. In a paper by Xu and Shang (Systems Control Lett. 58 (2009), no. 8, 561-566, a related result on Banach spaces is stated in Theorem 2.4. Their proof is seriously flawed, but a proof is given in a paper by Geyer and myself (J. Operator Theory 78 (2017), no. 2, 473-500); see Theorem 5.4 and Example 5.6.
Source: R. Nagel's list of problems collected in 2003 at the workshop in Bari.

Problem 17 (Yuri Latushkin)**.** Consider the following properties of the generation A of a C_0-semigroups $(T(t))_{t\geq 0}$.

1. $\operatorname{rg} A$ closed.
2. $\operatorname{rg}(1 - T(t))$ closed for one $t > 0$.

Does (ii) imply (i)?

Comment. This is a version of a "special inclusion theorem". The converse implication does not hold.
Source: R. Nagel's list of problems collected in 2003 at the workshop in Bari.

Problem 18 (Alessandra Lunardi)**.** Let A and B be generators of C_0 semigroups. Under which condition does

$$C := \overline{A^2 + B^2}$$

generate an analytic semigroup?

Source: R. Nagel's list of problems collected in 2003 at the workshop in Bari.

Problem 19 (Alessandra Lunardi)**.** Let $(T(t))_{t\geq 0}$ be a not necessary strongly continious semigroup on a Banach space X and consider the following condition.

(i) $(0, \infty) \ni t \mapsto T(t) \in L(X)$ is analytic.

(ii) $t \mapsto T(t)$ is analytic on a sector containing $\mathbb{R}_+$

(iii) $\|T(t)\| \leq Me^{t\omega}$, $\frac{d}{dt}T(t) \in L(X)$ and $\|\frac{d}{dt}T(t)\| \leq \frac{M}{t}e^{t\omega}$ for some constants ω, M.

Under which assumption added to (i), (ii) or (iii) does there exist a sectorial operator A generating $(T(t))_{t\geq 0}$?

Source: R. Nagel's list of problems collected in 2003 at the workshop in Bari.

Problem 20 (Alessandra Lunardi)**.** Study the "backward uniqueness property", i.e., characterize injective C_0-semigroups. Apply the result to the backward uniqueness property for non-autonomous Cauchy problems $u'(t) = A(t)u(t)$, $A(t)$ sectorial, by looking at the corresponding evolution semigroup.

Comments. Why this problem is important

Source: R. Nagel's list of problems collected in 2003 at the workshop in Bari.

Problem 21 (Alessandra Lunardi)**.** Consider "non-C_0-semigroups", e.g., bi-continious semigroup and describe appropriate regularization properties.

Comments. Compare the Ornstein-Uhlebeck semigroup in $C_b(\mathbb{R}^n)$.

Source: R. Nagel's list of problems collected in 2003 at the workshop in Bari.

Problem 22 (R. Nagel)**.** Let A and B the generators of two communiting C_0-semigroups on a Banach space and let G be the generator of corresponding product semigroup. Find (the most general) conditions implying

$$D(G) = D(A) \cap D(B).$$

Comments. This yields abstract "maximal regularity" results.

Source: R. Nagel's list of problems collected in 2003 at the workshop in Bari.

Problem 23 (R. Nagel)**.** Let $(T(t))_{t\geq 0}$ be a C_0-semigroup whith growth bound

$$\omega_0 := \inf\{\omega \in \mathbb{R} : \|T(t)\| \leq M^\omega \cdot e^{t\omega} \text{ for } t \geq 0\}$$

Find condition such that ω_0 is minimum, i.e.,

$$\|T(t)\| \leq M_0 \cdot e^{t\omega_0} \text{ for } t \geq 0$$

Comments. This corresponds to a characterization of boundedness for semigroups.

Source: R. Nagel's list of problems collected in 2003 at the workshop in Bari.

Problem 24 (H. Zwart)**.** (i) Does every bounded C_0-semigroup on a Hilbert space have a bounded rational calculus?

(ii) When is a C_0-semigroups on a Hilbert space similar to a contraction semigroup?

Comments. The answer to the first question is no. A bounded semigroup on a Hilbert space has a bounded rational calculus if and only if it has a bounded H_∞-calculus, see thesis of Markus Haase.

Comments.(Charles Batty)

In Problem (i) and the Comment above, the "boundedness" of the rational calculus is being interpreted as meaning boundedness with respect to the H^∞-norm. There are one or two issues as to what is the domain of those functions and whether or not the generator is injective, but basically the answer is correct. An alternative to Haase's thesis is his functional calculus book, Sections 5.3.4 and 5.3.5.

Instead of considering the H^∞-norm, one may consider Banach algebras in different norms that are embedded in $H^\infty(\mathbb{C}_+)$, where $\mathbb{C}_+$ is the open right half-plane. One example is the (Hille)-Phillips calculus, where the norm comes from measures on $[0, \infty)$. Alexander Gomilko, Yuri Tomilov and I have recently shown that if $-A$ is the generator of a bounded C_0-semigroup on a Hilbert space, then there is a bounded $\mathcal{B}$-calculus for A. Here $\mathcal{B}$ is a Banach algebra of "analytic Besov" functions on the right half-plane. This algebra is considerably bigger than the Phillips algebra, and the $\mathcal{B}$-norm is considerably smaller than the Phillips norm but it is bigger than the H^∞-norm.

On Banach spaces, the $\mathcal{B}$-calculus exists if and only if A satisfies the condition introduced by Gomilko, and independently by Shi and Feng, in 1999 and 2000. In particular, it exists if A is sectorial of angle less than $\pi/2$, so $-A$ generates a bounded holomorphic C_0-semigroup. For those operators, there are two further calculi, $\mathcal{D}$-calculus and $\mathcal{H}$-calculus, which extend the calculus to larger classes of functions than $\mathcal{B}$, with smaller norms.

Relevance. This problem was/is related to the inverse generator problem and questions concerning the powers of the co-generator of bounded semigroups. For both problems, the answers have been negative in general, and some positive partial answers have been obtained, but the answer for bounded semigroups on Hilbert spaces is unknown. The extended calculi provide systematic ways to approach such problems, instead of using ad hoc methods each time. For the functions $((z-1)(z+1)^{-1})^n$ the Hille-Phillips norm grows like $n^{1/2}$, the $\mathcal{B}$-norm grows like $\log n$, and the $\mathcal{D}$-norms are uniformly bounded.

Source: R. Nagel's list of problems collected in 2003 at the workshop in Bari.

Problem 25 (R. Nagel)**.** $A+B$ problem. Let A and B be generators of C_0-semigroups on a Banach space X.

(i) Define the sum $C = A + B$ such that C becomes a (maximal) closed operator on X and $Cx = \tilde{A}x + \tilde{B}x$ for all $x \in X$ and some extrapolated operators A and B.

(ii) Find assumptions on A and B such that C remains a generator on X, thereby unifying known perturbation results.

(iii) A test case is the following. Let $\mathcal{X} = C_0(\mathbb{R}, X)$ or $\mathcal{X} = L^p(\mathbb{R}, X)$, and take $Af = f'$ and $Bf(s) = C(s)f(s)$ for appropriate $f \in \mathcal{X}$ and closed operator $C(s)$ on X.

Source: R. Nagel's list of problems collected in 2003 at the workshop in Bari.

Problem 26 (R. Nagel)**. Non-autonomous abstract Cauchy problems.** For unbounded linear operators $A(t)$ on a Banach space X and for a starting time t_0, characterize the well-posedness of the non-autonomous Cauchy problem

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) \quad \text{for } t \geq t_0 \\ x(t_0) &= x_0 \end{aligned}$$

by a Hille-Yosida type condition for an operator G generating an evolution semigroup on $\mathcal{X} = C_0(\mathbb{R}, X)$ or $\mathcal{X} = L^p(\mathbb{R}, X)$.

Source: R. Nagel's list of problems collected in 2003 at the workshop in Bari.

Problem 27 (Communicated by Rainer Nagel)**.** Characterizing Koopman groups on Hilbert spaces:

Use the Perron Frobenius spectral theory of positive C_0-groups to characterize unitary C_0-groups on a Hilbert space H that are unitarily isomorphic to a Koopman group on an L^2-space.

References:

- Alexandre I. Danilenko, and Mariusz Lemańczyk: Spectral multiplicities for ergodic flows. Discrete Contin. Dyn. Syst. 33, No. 9, 4271-4289 (2013). • D. V. Anosov: pectral multiplicity in ergodic theory, Proc. Steklov Inst. Math. 290, Suppl. 1, S1-S44 (2015); translation from Sovrem. Probl. Mat. 2003, No. 3, 3–84 (2003).

Source: Manuscript of R. Nagel provided during OPSO 2021 conference.

Problem 28 (D. Seifert)**.** Let $(T(t))_{t\geq 0}$ be a bounded C_0-semigroup on a (complex) Banach space X, and let A denote its generator. Suppose that $\sigma(A) \cap i\mathbb{R} = \emptyset$ and that there exists $\alpha > 0$ such that $\|(is - A)^{-1}\| = O(|s|^\alpha)$ as $|s| \to \infty$. Find $\beta \in [0, 1]$ depending on the geometric properties of the space X (for instance its Fourier type, its type or its cotype) such that

$$\|T(t)A^{-1}\| = O\left(\frac{\log(t)^{\beta/\alpha}}{t^{1/\alpha}}\right), \qquad t \to \infty.$$

Comments. It was shown by Batty and Duyckaerts (2008) that one may always take $\beta = 1$; see also Chill and Seifert (2016). Batty and Duyckaerts moreover showed that negative values of β are in general not permissible. It is reasonable, therefore, to restrict attention to values of β lying in $[0, 1]$. Borichev and Tomilov (2010) showed that one may take $\beta = 0$, yielding the best possible upper bound, if X is a Hilbert space. This result may be viewed as a special case of more recent results appearing in papers by Batty, Chill and Tomilov (2016) and Rozendaal, Seifert and Stahn (2019). Borichev and Tomilov also showed, by considering the left-shift semigroup on a certain subspace of $\mathrm{BUC}(\mathbb{R}_+)$ with an appropriate norm, that one may have

$$\limsup_{t\to\infty} \frac{t^{1/\alpha}}{\log(t)^{1/\alpha}} \|T(t)A^{-1}\| > 0.$$

Hence one cannot in general hope to do better than $\beta = 1$ unless one imposes additional assumptions on X; see also Debruyne and Seifert (2019).

Relevance. This problem is important from a theoretical point of view, as its solution would elegantly complement our current understanding of polynomial stability of C_0-semigroups. Furthermore, there are likely to be interesting applications to concrete evolution equations on L^p-spaces and other (non-Hilbertian) Banach spaces with non-trivial geometric properties.

References:

• C.J.K. Batty, R. Chill, and Y. Tomilov. Fine scales of decay of operator semigroups. *J. Eur. Math. Soc. (JEMS)*, 18(4):853–929, 2016.
• C.J.K. Batty and T. Duyckaerts. Non-uniform stability for bounded semi-groups on Banach spaces. *J. Evol. Equ.*, 8(4):765–780, 2008.
• A. Borichev and Y. Tomilov. Optimal polynomial decay of functions and operator semigroups. *Math. Ann.*, 347(2):455–478, 2010.
• R. Chill and D. Seifert. Quantified versions of Ingham's theorem. *Bull. Lond. Math. Soc.*, 48(3):519–532, 2016.
• G. Debruyne and D. Seifert. An abstract approach to optimal decay of functions and operator semigroups. *Israel J. Math.*, 233(1):439–451, 2019.
• G. Debruyne and D. Seifert. Optimality of the quantified Ingham-Karamata theorem for operator semigroups with general resolvent growth. *Arch. Math. (Basel)*, 113(6):617–627, 2019.
• J. Rozendaal, D. Seifert, and R. Stahn. Optimal rates of decay for operator semigroups on Hilbert spaces. *Adv. Math.*, 346:359–388, 2019.

Problem 29 (A.E. Teretenkov)**.** Let $\mathcal{B}$ be a Banach space. Let $\mathcal{P}$ be a projection on the *finite-dimensional* Banach subspace of $\mathcal{B}$. Let $\mathcal{L}^0 + \lambda\mathcal{L}$ be a generator of C_0-semigroup $\mathcal{U}_t^\lambda$ on $\mathcal{B}$ for all $\lambda \in [0, \lambda_{\text{sup}})$. Let $\mathcal{U}_t^0$ leave both $\mathcal{P}\mathcal{B}$ and its complement $(I - \mathcal{P})\mathcal{B}$ invariant, let $\mathcal{P}\mathcal{L}\mathcal{P} = 0$. Denote $\mathcal{L}_t \equiv (\mathcal{U}_t^0)^{-1}\mathcal{L}\mathcal{U}_t^0$. Let the integrals

$$\int_{-\infty}^{t} dt_1 \ldots \int_{-\infty}^{t_{k-1}} dt_k \mathcal{P}\mathcal{L}_{t_1} \ldots \mathcal{L}_{t_k}\mathcal{P}, \qquad k = 1, \ldots, n+1$$

finite for all $t \geqslant 0$. If is it possible to find such a λ-dependent operator $r^{n,\lambda}$ on $\mathcal{P}\mathcal{B}$, which is polynomial in λ, and λ-dependent semigroup $u_t^{n,\lambda}$ on $\mathcal{P}\mathcal{B}$, whose generator is polynomial in λ, such that

$$\mathcal{P}(\mathcal{U}^0_{\frac{t}{\lambda^2}})^{-1}\mathcal{U}^\lambda_{\frac{t}{\lambda^2}}\mathcal{P} = u_t^{n,\lambda} r^{n,\lambda} + O(\lambda^{2n+2}), \qquad \lambda \to +0$$

for all $t > 0$? Let us emphasize that we do not assume here asymptotic behavior to be uniform in t, it is just assumed for each fixed $t > 0$. If there are counterexamples, what further restrictions should be assumed to obtain such asymptotic estimate?

Relevance. This problem is important for derivation of perturbative corrections to Markovian quantum master equations. It seems to be a possible direction of generalization of the classical results by E.B. Davies to higher orders of perturbation theory in λ and seems to be hold in a simple example discussed in arXiv:2008.02820. It also seems to be necessary for strict perturbative derivation of master equations recently obtained by A.S. Trushechkin.

References:

- E.B. Davies. *Commun. Math. Phys.* 39, no. 2 (1974): 91–110.
- A.E. Teretenkov. arXiv:2008.02820
- A.S. Trushechkin. *Proc. Steklov Inst. Math.* 313 (2021): to appear.