Error of Chernoff approximations based on Chernoff function with a given coefficient at t^2

O. E. Galkin[1], S. Yu. Galkina[2]

Keywords: Chernoff product formula, approxition of C_0-semigroup, speed of convergence

MSC2010 codes: 47D03, 47D06, 35A35, 41A25

This talk is devoted to the error of Chernoff approximations [1,2,3] to strongly continuous one-parameter semigroups [4, 5] in the case when Chernoff function has a coefficient at t^2 which is known.

Let $(X, \|\cdot\|)$ be any Banach space and $\mathscr{L}(X)$ denotes the set of all bounded linear operators on X.

Definition 1 (see, for example, Engel, Nagel [5]). The family $\{G(t)\}_{t\geqslant 0}$ of bounded linear operators on the Banach space X is called the *strongly continuous (one-parameter) semigroup (and also the C_0–semigroup)*, if it is strongly continuous, $G(0) = I$ and for all $t, s \geqslant 0$ the equality $G(t+s) = G(t)G(s)$ is true.

Definition 2 (see, for example: Engel, Nagel [5]). *Generator of a strongly continuous semigroup* $\{G(t)\}_{t\geqslant 0}$ *on the Banach space* X is the operator $A\colon D(A) \to X$, defined by the equality $Ax = \lim\limits_{t\to+0}(G(t)x - x)/t$ for all x from the domain $D(A)$, where

$$D(A) = \{\ x \in X \mid \lim_{t\to+0}(G(t)x - x)/t \text{ exists }\}.$$

In 1968 Paul Chernoff proved the following theorem:

Theorem 1 (Chernoff [6]). Let X be a Banach space, $F(t)$ be a strongly continuous function from $[0, \infty)$ to a subset of the compressing operators from $\mathscr{L}(X)$, with $F(0) = I$. Suppose that the closure A of the strong derivative $F'(0)$ is the generator of the contracting C_0–semigroup $\{e^{tA}\}_{t\geqslant 0}$. Then $[F(t/n)]^n$ converges to e^{tA} in a strong operator topology.

Let us note that this theorem does not contain an estimate of the rate of convergence, that is, an estimate of the form

$$\|[F(t/n)]^n x - e^{tA}x\| \leq C(t, x, n) \to 0 \quad (n \to \infty).$$

In 2022 was published the theorem that provides such estimate under certain conditions:

Theorem 2 (Galkin, Remizov [3]). Suppose that:

1) $T > 0$, $M_1 \geq 1$, $w \geq 0$. $(A, D(A))$ is generator of C_0-semigroup $(e^{tA})_{t\geqslant 0}$ in a Banach space X, such that $\|e^{tA}\| \leqslant M_1 e^{wt}$ for $t \in [0, T]$.

2) There are a mapping $F\colon (0, T] \to \mathscr{L}(X)$ and constant $M_2 \geq 1$ such that we have $\|(F(t))^k\| \leqslant M_2 e^{kwt}$ for all $t \in (0, T]$ and all $k \in \mathbb{N} = \{1, 2, 3, \ldots\}$.

3) $m \in \mathbb{N} \cup \{0\}$, $p \in \mathbb{N}$, subspace $\mathcal{D} \subset D(A^{m+p})$ is $(e^{tA})_{t\geqslant 0}$-invariant.

4) There exist such functions $K_j\colon (0, T] \to [0, +\infty)$, $j = 0, 1, \ldots, m+p$ that for all $t \in (0, T]$ and all $f \in \mathcal{D}$ we have

$$\left\| F(t)f - \sum_{k=0}^{m} \frac{t^k A^k f}{k!} \right\| \leqslant t^{m+1} \sum_{j=0}^{m+p} K_j(t) \|A^j f\|.$$

[1]HSE University, Russian Federation, Nizhny Novgorod city. Email: olegegalkin@yandex.ru

[2]HSE University, Russian Federation, Nizhny Novgorod city. Email: svetlana.u.galkina@mail.ru

Then: for all $t > 0$, all integer $n \geq t/T$ and all $f \in \mathcal{D}$ we have

$$\|(F(t/n))^n f - e^{tA} f\| \leq \frac{M_1 M_2 t^{m+1} e^{wt}}{n^m} \sum_{j=0}^{m+p} C_j(t/n) \|A^j f\|,$$

$C_{m+1}(t) = K_{m+1}(t)e^{-wt} + M_1/(m+1)!$, $C_j(t) = K_j(t)e^{-wt}$ $(j \neq m+1)$.

Let us consider particular example. *Example 1.* Suppose that $\|e^{tA}\| \leq M_1 e^{wt}$, $\|F(t)\| \leq M_2 e^{wt}$, where $w \geq 0$,

$$\|F(t)x - x - tAx\| \leq K_2 t^2 \|A^2 x\|$$

for all $x \in D(A^2)$ and $t \in (0;1]$. Then $m = 1$, $K_0(t) = K_1(t) = 0$ for any $t \in (0;1]$. So theorem 2 states that for any fixed $t > 0$, all $x \in D(A^2)$ and all integer $n \geq t$ the following estimate is true, having the following asymptotic behaviour as $n \to \infty$:

$$\|(F(t/n))^n x - e^{tA} x\| \leq \frac{M_1 M_2 t^2 e^{wt}}{n} \left(K_2 e^{-wt/n} + \frac{M_1}{2} \right) \|A^2 x\| \leq$$

$$\leq M_1 M_2 (K_2 + M_1/2) \frac{t^2 e^{wt}}{n} \|A^2 x\|.$$

So the question arises: what is the lower estimate of the error $\|(F(t/n))^n x - e^{tA} x\|$?

In 2018, Ivan Remizov formulated the following conjecture:

Conjecture 1 (Remizov [7]). Let $(e^{tA})_{t \geq 0}$ be a C_0-semigroup in a Banach space X, and F is a Chernoff function for operator A (recall that this implies $F(0) = I$ and $F'(0) = A$ but says nothing about $F''(0)$) and number $T > 0$ is fixed. Suppose that vector x is from intersection of domains of operators $F'(t)$, $F''(t)$, $F'''(t)$, $F''''(t)$, $F'(t)F''(t)$, $(F'(t))^2 F''(t)$, $(F''(t))^2$ for each $t \in [0,T]$, and suppose that if $Z(t)$ is any of these operators then function $t \to Z(t)x$ is continuous for each $t \in [0,T]$. Then there exists such a number $C_x > 0$, that for each $t \in [0,T)$ and each $n \in \mathbb{N}$ the following inequality holds, where $B = F''(0)$:

$$\|(F(t/n))^n x - e^{tA} x - \frac{t^2}{2n} e^{tA} (B - A^2) x\| \leqslant \frac{C_x}{n^2}.$$

Unfortunately, this hypothesis can only be true if the operators A and B commute. We prove the following theorem:

Theorem 3. Suppose that:

1) C_0-semigroup $(e^{tA})_{t \geqslant 0}$ in a Banach space X has bounded generator $A \in \mathscr{L}(X)$.

2) $T > 0$ and there are a mapping $F\colon [0,T] \to \mathscr{L}(X)$ and constants $M \geq 1$, $w \geq 0$ such that $\|(F(t))^k\| \leqslant M e^{kwt}$ for all $t \in [0,T]$, $k \in \mathbb{N}$.

3) There exist such bounded operator $B \in \mathscr{L}(X)$ and constant $K \geq 0$ that for all $t \in [0,T]$ we have

$$\big\|F(t) - I - tA - \frac{t^2}{2} B\big\| \leqslant K t^3.$$

Then: there exists such a number $C > 0$, that for each $t \in [0,T]$ and each $n \in \mathbb{N}$ the following inequality holds:

$$\big\|(F(t/n))^n - e^{tA} - \frac{t^2}{2n} \int_0^1 e^{tsA} (B - A^2) e^{t(1-s)A} ds\big\| \leq \frac{C}{n^2}.$$

If A and B commute then:

$$\big\|(F(t/n))^n - e^{tA} - \frac{t^2}{2n} e^{tA} (B - A^2)\big\| \leq \frac{C}{n^2}.$$

Acknowledgements. The authors thank Ivan Remizov for setting the task and his interest in the work.

The authors are partially supported by Laboratory of Dynamical Systems and Applications NRU HSE, of the Ministry of science and higher education of the RF grant ag. no. 075-15-2022-1101.

References

[1] Ya.A. Butko. *The Method of Chernoff Approximation.*// In book *Semigroups of Operators – Theory and Applications*. Springer International Publishing, 2020.

[2] A.V. Vedenin, V.S. Voevodkin, V.D. Galkin, E.Yu. Karatetskaya, I.D. Remizov. *Speed of Convergence of Chernoff Approximations to Solutions of Evolution Equations.*// Math. Notes. 2020. Vol. 108. No. 3. P. 451–456.

[3] O.E. Galkin, I.D. Remizov. *Rate of Convergence of Chernoff Approximations of Operator C_0-Semigroups.* // Math. Notes. 2022. Vol. 111. No. 2. P. 305–307.

[4] E. Hille, R.S. Phillips. *Functional Analysis and Semi-Groups.* American Mathematical Society, 1975.

[5] K.-J. Engel, R. Nagel. *One-Parameter Semigroups for Linear Evolution Equations.* Springer, 2000.

[6] P.R. Chernoff. *Note on product formulas for operator semigroups.* // J. Functional Analysis. 1968. Vol. 2. No. 2. P. 238–242.

[7] I.D. Remizov. *On estimation of error in approximations provided by Chernoff's product formula.* // International Conference "ShilnikovWorkshop-2018" dedicated to the memory of outstanding Russian mathematician Leonid Pavlovich Shilnikov. Book of abstracts. Lobachevsky State University of Nizhny Novgorod. 2018. December 17–18. P. 38–41.