Динамические системы для математического моделирования: многомерный хаос в численных экспериментах

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Classification of chaotic attractors

Chaos generated by nonlinear dynamical systems was discovered in the middle of 20 century. Chaotic dynamics if fundamental property of nonlinear systems, which was observed almost all areas of science. During this time a lot investigations of different chaos properties were carried out, scenarios of formation chaos were described, a lot different dynamical models with chaos were suggested, and different approaches where dynamical chaos was used were developed.

https://www.behance.net/gallery/7618879/Strange-Attractors

Система Ресслера

Model

$$\dot{x} = -y - z,$$
$$\dot{y} = x + py$$
$$\dot{z} = q + (x - r)z$$

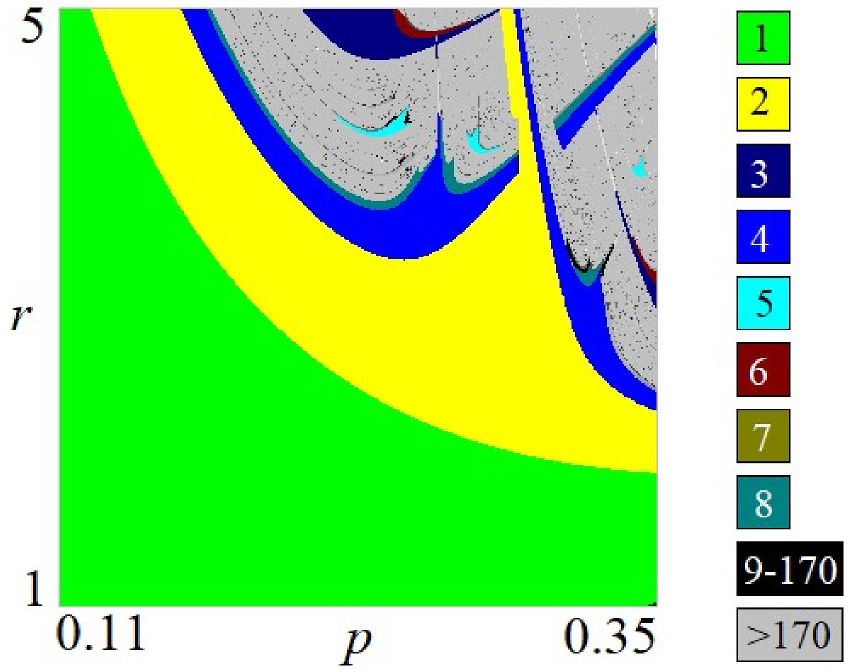


Chart of dynamical regimes, q=0.2

Система Ресслера

Analysis of equilibrium points stability

$$x_0 = \frac{r \mp \sqrt{r^2 - 4pq}}{2}, \quad y_0 = \frac{-r \pm \sqrt{r^2 - 4pq}}{2p}, \quad z_0 = \frac{r \mp \sqrt{r^2 - 4pq}}{2p}$$

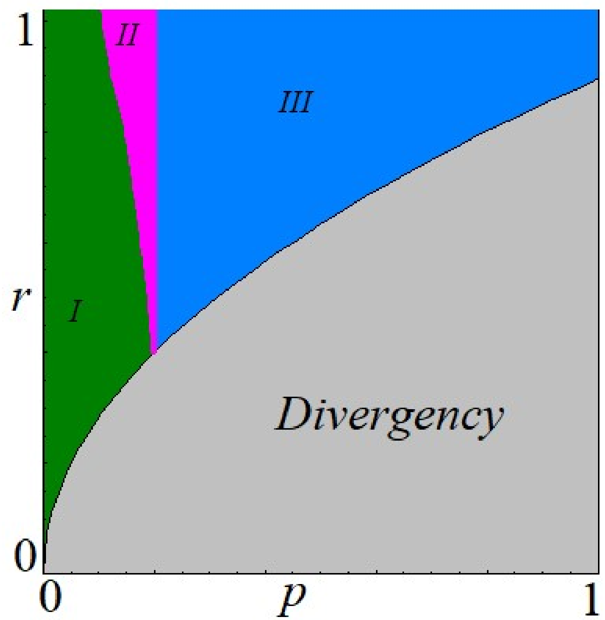
Saddle-node bifurcation of equilibrium points

$$r^2 = 4pq$$

I – stable focus, saddle-focus (1,2)

II – saddle-focus (1,2), saddle-focus (2,1)

III – unstable focus, saddle-focus (2,1)



Parameter plane of equilibrium points stability

Генератор Анищенко-Астахова

Model

$$\dot{x} = mx + y - xz,$$
$$\dot{y} = -x$$
$$\dot{z} = -gz + l(x)x^2,$$
$$l(x) = \begin{cases} 0, x \le 0, \\ 1, x > 0 \end{cases}$$

$$\ddot{y} - (m - z)\dot{y} + y = 0$$

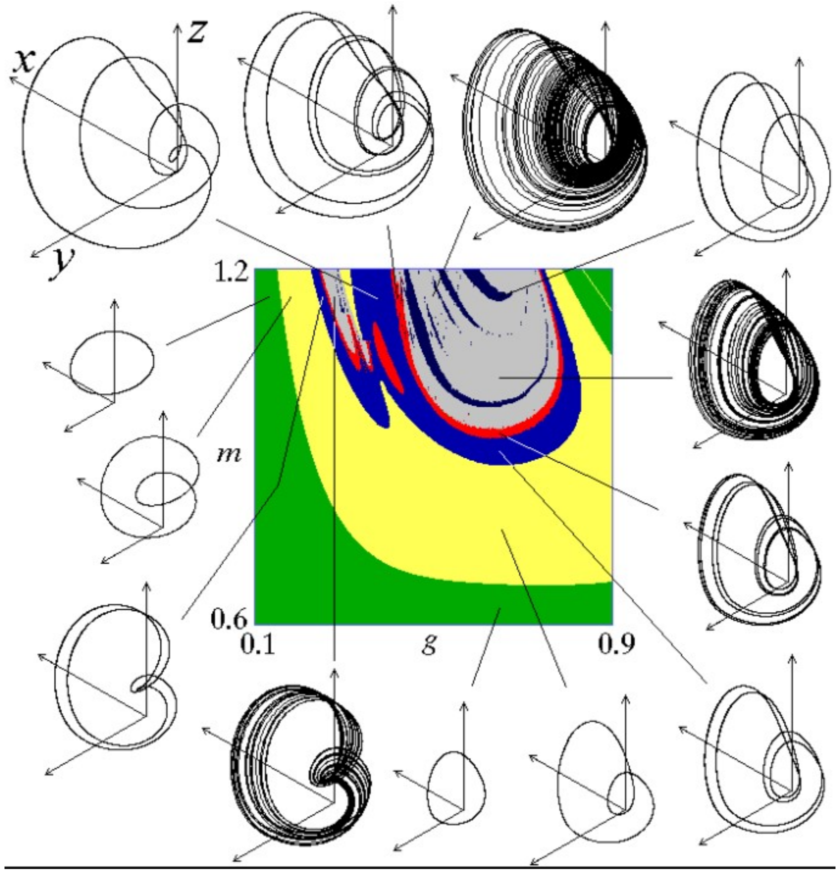
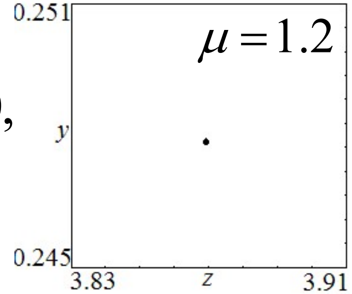


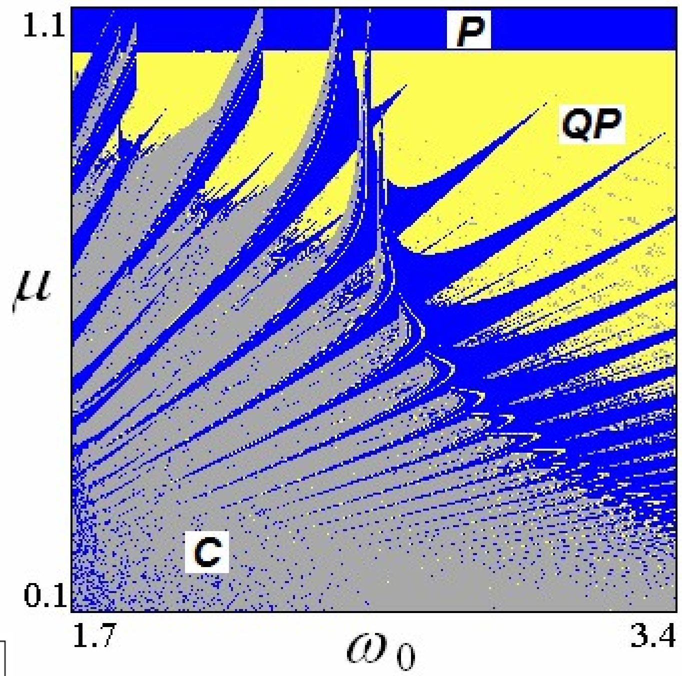
Chart of dynamical regimes

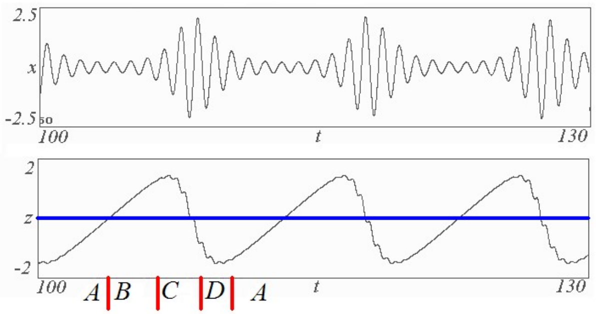
Afraimovich-Shilnikov's scenario, quasiperiodic generator

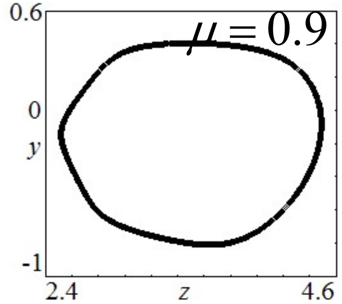
$$\ddot{x} - (\lambda + z + x^2 - \tfrac{1}{2}x^4)\dot{x} + \omega_0^2 x = 0,$$

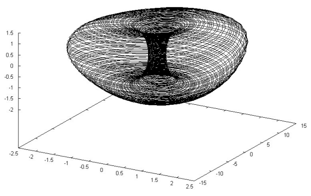
$$\dot{z} = \mu - x^2,$$

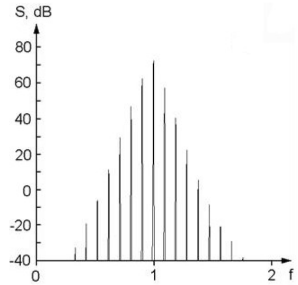


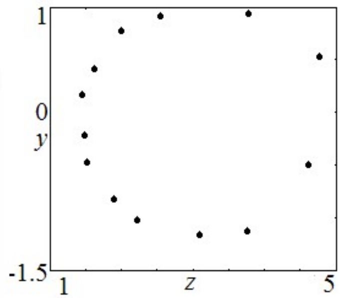




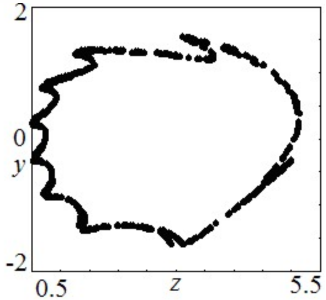


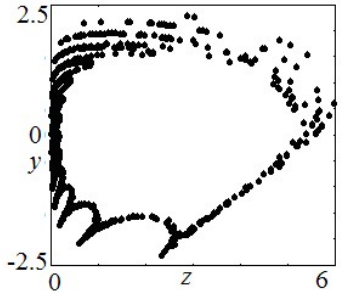






$\lambda = 0, \omega_0 = 2.7$

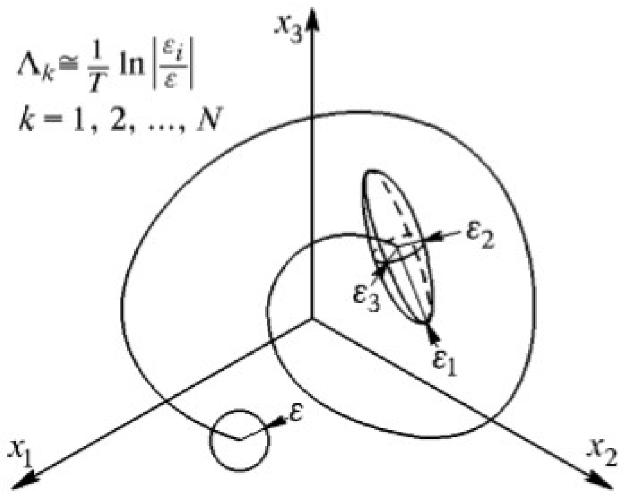




Lyapunov exponents

The simplest way to detect chaotic dynamics is to calculate the largest Lyapunov exponents. In dependence on the spectrum of Lyapunov exponents can be classified different types of oscillations:

- periodic oscillations	(0, -, -, …);
-quasiperiodic oscillations	(0, 0, …, 0, -, -, …)
-chaotic oscillations	(+, 0, -, -, …)
-chaos with additional zero LE	(+, 0, …, 0, -, -, …).
-hyperchaotic oscillations	(+, +, …, +, 0, -, -, …).



At the present in literature there is a lot examples of models with hyperchaos, but the question about scenarios of occurring hypechaos and chaos with additional zero LE is open.

Scenarios of birth of hyperchaos

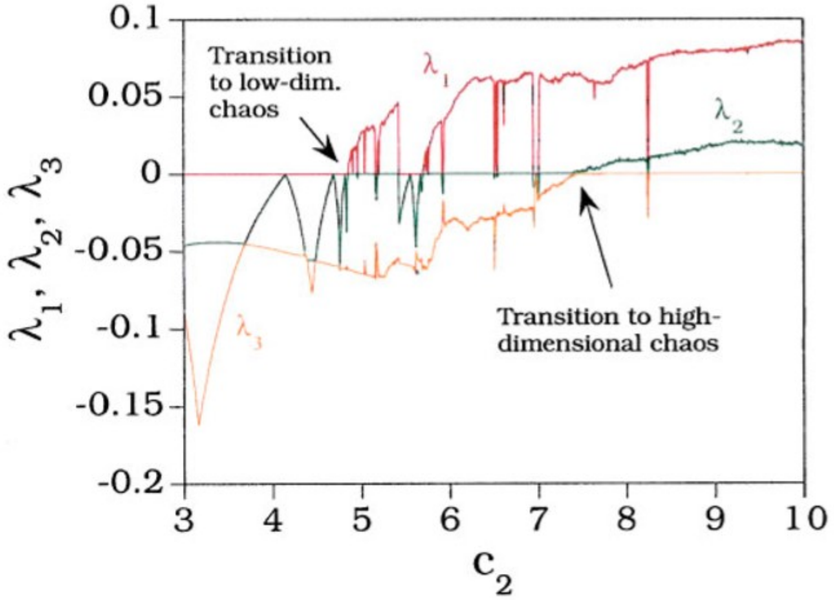


FIG. 3. (Color) First three Lyapunov exponents of the coupled Rössler system.

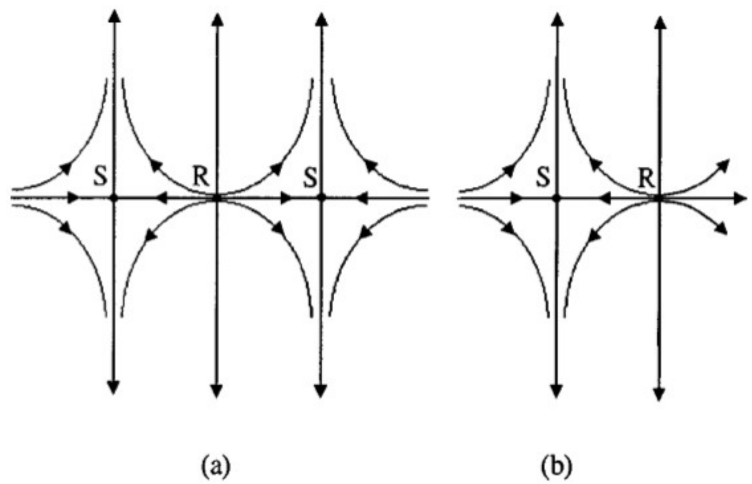
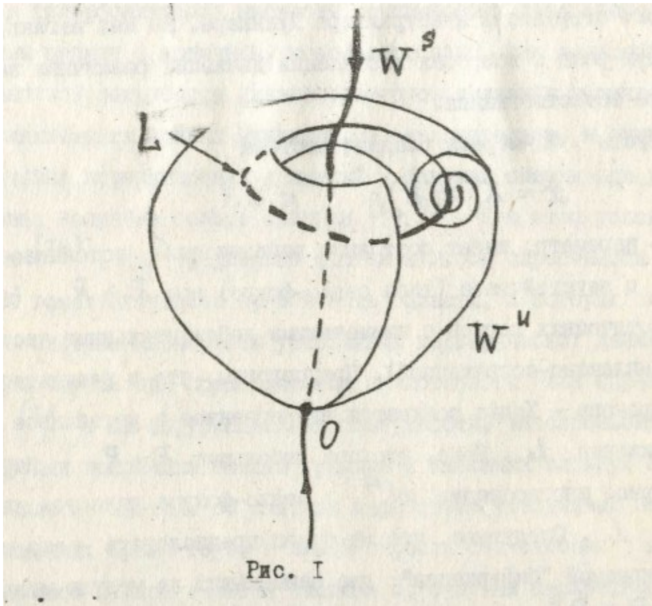


FIG. 2. Bifurcations leading to the occurrence of repellers inside chaotic attractors.

Scenario: (i) the saddle-repeller bifurcation of a particular unstable periodic orbit usually of low period, (ii) the appearance of a repelling node in the saddle-node bifurcation, after which the chaotic attractor becomes riddled, (iii) the absorption of the repeller originally located out of the attractor by the growing attractor.

Shilnikov’s discrete attractors



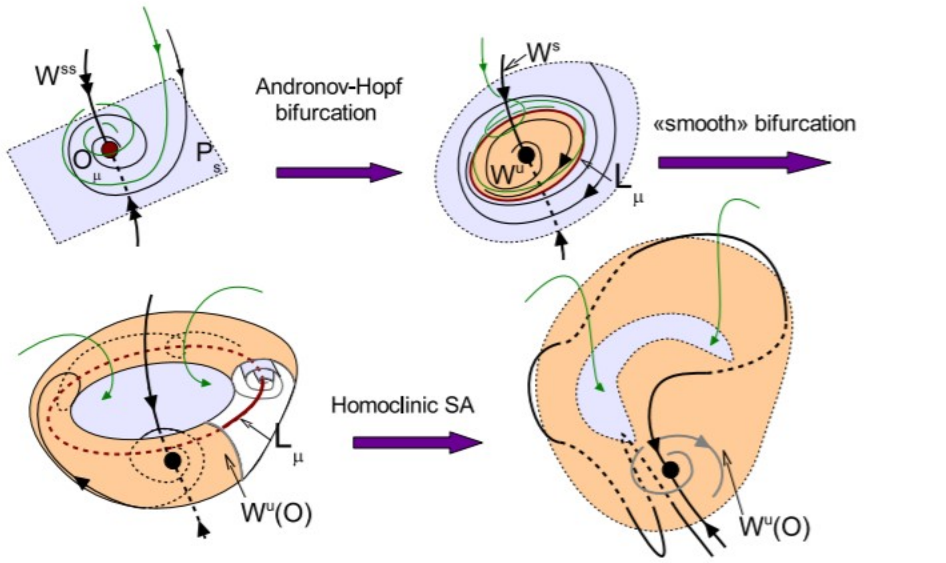


Figure 12: A sketch of scenario of the emergence of a discrete Shilnikov attractor.

[1] Шильников, Л. П. (1986). Теория бифуркаций и турбулентность. I. Избранные научные труды, 128. (in Russia)

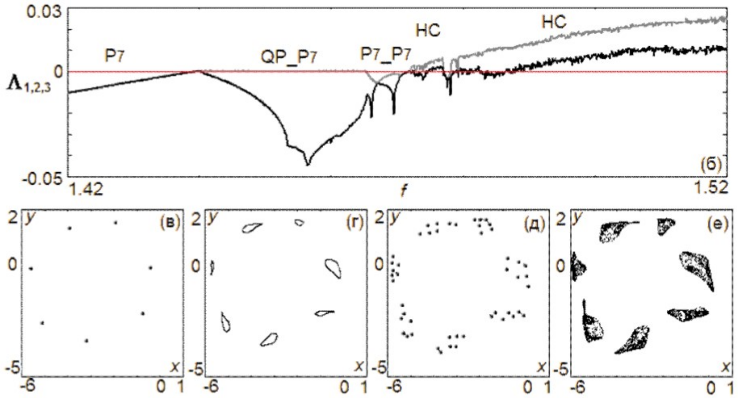
[2] Гонченко, А. С., Гонченко, С. В., & Шильников, Л. П. (2012). К вопросу о сценариях возникновения хаоса у трехмерных отображений. Нелинейная динамика, 8(1), 3-28.

[3] Gonchenko, A., Gonchenko, S., Kazakov, A., & Turaev, D. (2014). Simple scenarios of onset of chaos in three-dimensional maps. International Journal of Bifurcation and Chaos, 24(08), 1440005.

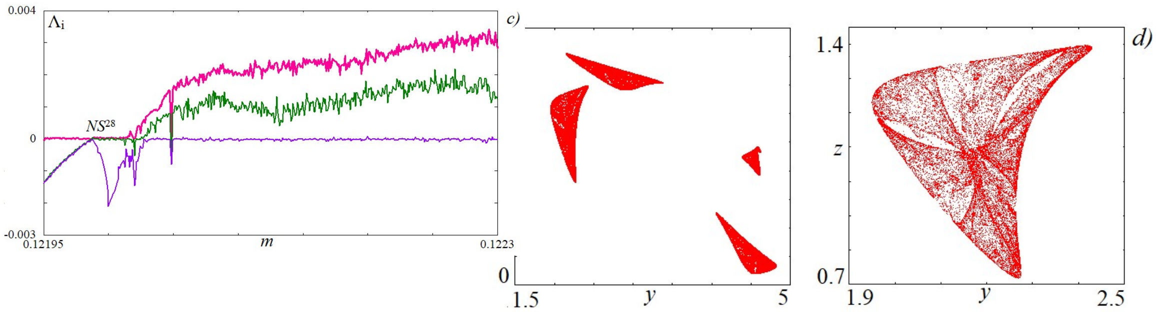
[4] Gonchenko, A. S., & Gonchenko, S. V. (2016). Variety of strange pseudohyperbolic attractors in three-dimensional generalized Hénon maps. *Physica D: Nonlinear Phenomena*, *337*, 43-57.

Scenarios of occerrence hyperchaos associated with Shilnikov'attractor

[1] Stankevich, N. V., Dvorak, A., Astakhov, V., Jaros, P., Kapitaniak, M., Perlikowski, P., & Kapitaniak, T. (2018). Chaos and Hyperchaos in Coupled Antiphase Driven Toda Oscillators. Regular and Chaotic Dynamics, 23(1), 120-126.



[2] Stankevich, N., Kuznetsov, A., Popova, E., & Seleznev, E. (2019). Chaos and hyperchaos via secondary Neimark–Sacker bifurcation in a model of radiophysical generator. Nonlinear Dynamics, 97(4), 2355-2370.



[3] Garashchuk, I. R., Sinelshchikov, D. I., Kazakov, A. O., & Kudryashov, N. A. (2019). Hyperchaos and multistability in the model of two interacting microbubble contrast agents. Chaos: An Interdisciplinary Journal of Nonlinear Science, 29(6), 063131.

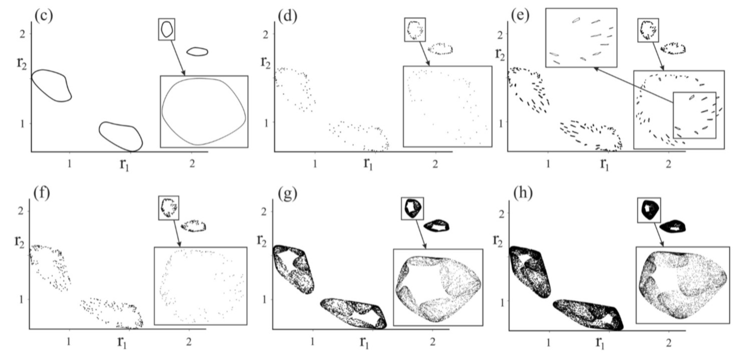


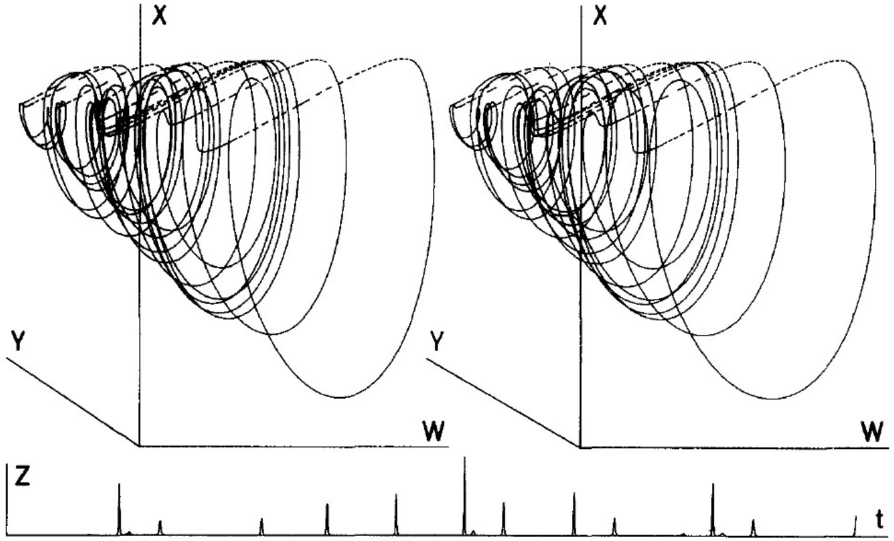
FIG. 9. The implementation of the proposed scenario of the onset of a hyperchaotic attractor along the path EF: $P_{ac} = 1.2$, $13 < d/R_0 < 25$. (a) and (b) Bifurcation tree and the graph of two largest Lyapunov exponents associated with this path with the enlarged area for $17.5 < d/R_0 < 19$; projections of the Poincaré maps for different

Простейший гиперхаос

Первый пример динамической системы с гиперхаосом – 1979 год

$$\dot{x} = -y - z,$$
$$\dot{y} = x + py$$
$$\dot{z} = q + (x - r)z$$

$$\dot{x} = -y - z\,, \quad \dot{y} = x + 0.25y + w\,,$$
$$\dot{z} = 3 + xz\,, \quad \dot{w} = -0.5z + 0.05w\,.$$



Rossler, O. E. (1979). An equation for hyperchaos. *Physics Letters A*, *71*(2-3), 155-157.

Гиперхаос в системе Ресслера

Обобщенная модель Ресслера

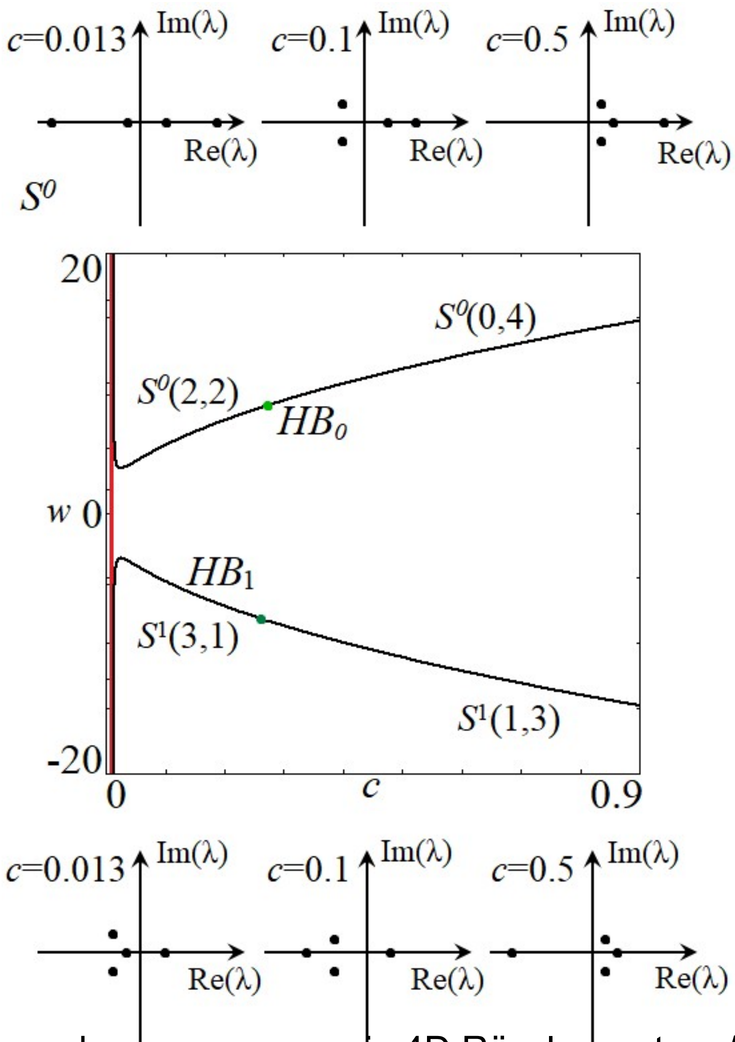
$$\dot{x} = -y - z,$$
$$\dot{y} = x + ay + w,$$
$$\dot{z} = b + xz,$$
$$\dot{w} = -cz + dw$$

a=0.25, b=3, c=0.5, d=0.05

Состояния равновесия

$$S^0(-x_0, -y_0, -z_0, -w_0)$$
$$S^1(x_0, y_0, z_0, w_0)$$

$$x_0 = \sqrt{\frac{b(c-ad)}{d}}, y_0 = \sqrt{\frac{bd}{c-ad}},$$
$$z_0 = \sqrt{\frac{bd}{c-ad}}, w_0 = c\sqrt{\frac{b}{d(c-ad)}}.$$

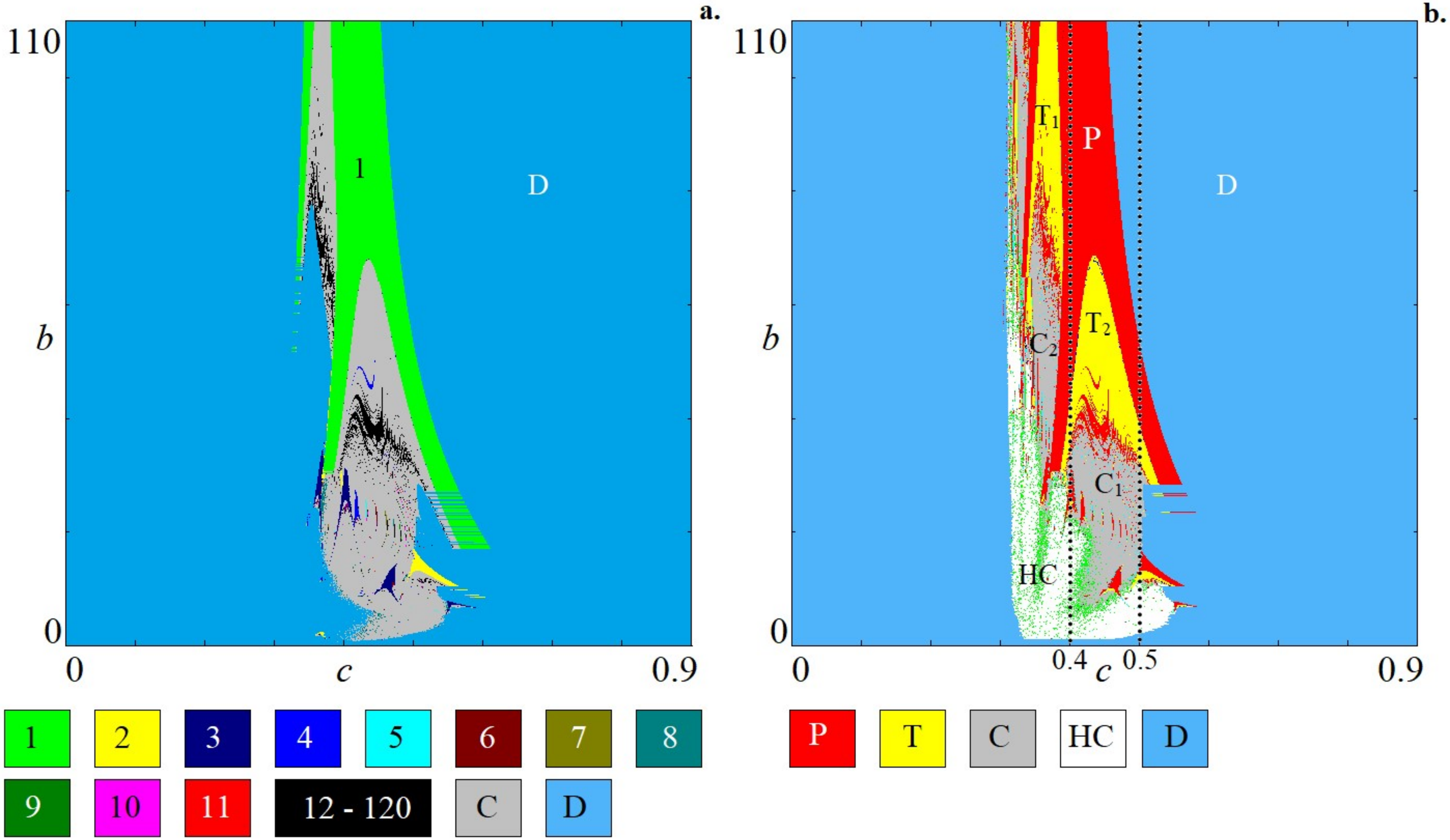


Stankevich N.V., Kazakov A.O., Gonchenko S.V. Scenarios of hyperchaos occurrence in 4D Rössler system // CHAOS. 2020. (accepted)

Гиперхаос в системе Ресслера

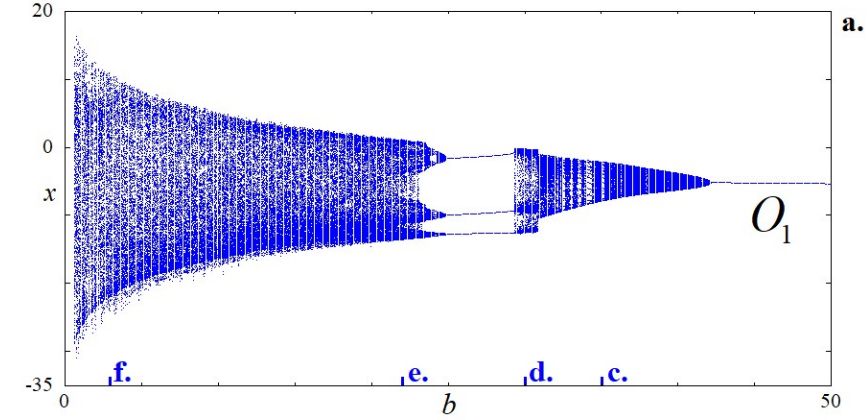
Карты режимов

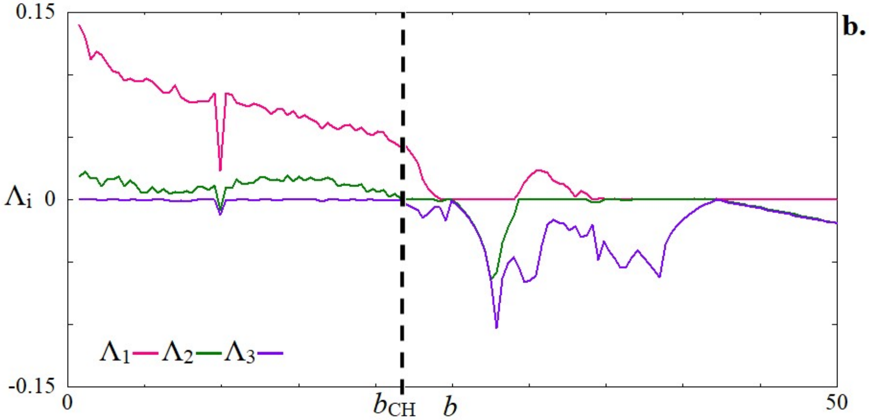
a=0.25, d=0.05

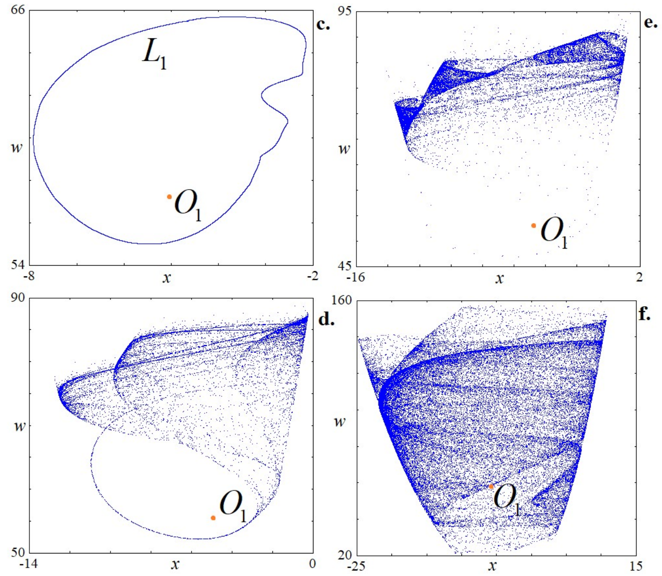


Гиперхаос в системе Ресслера

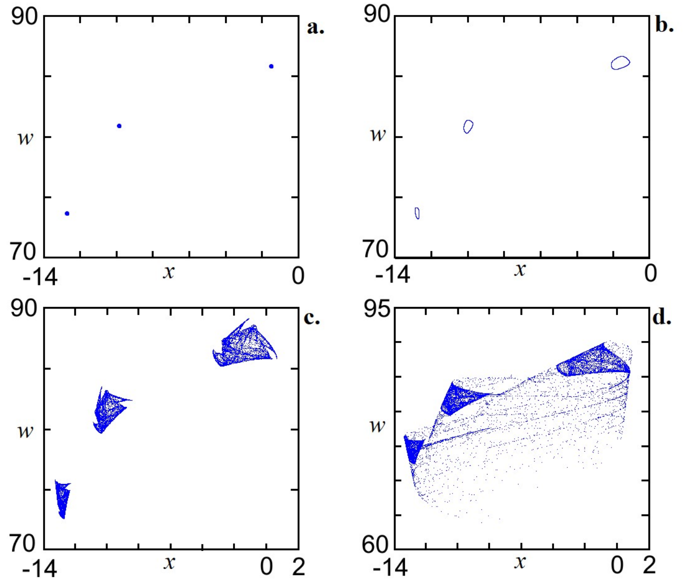
a=0.25, c=0.4, d=0.05







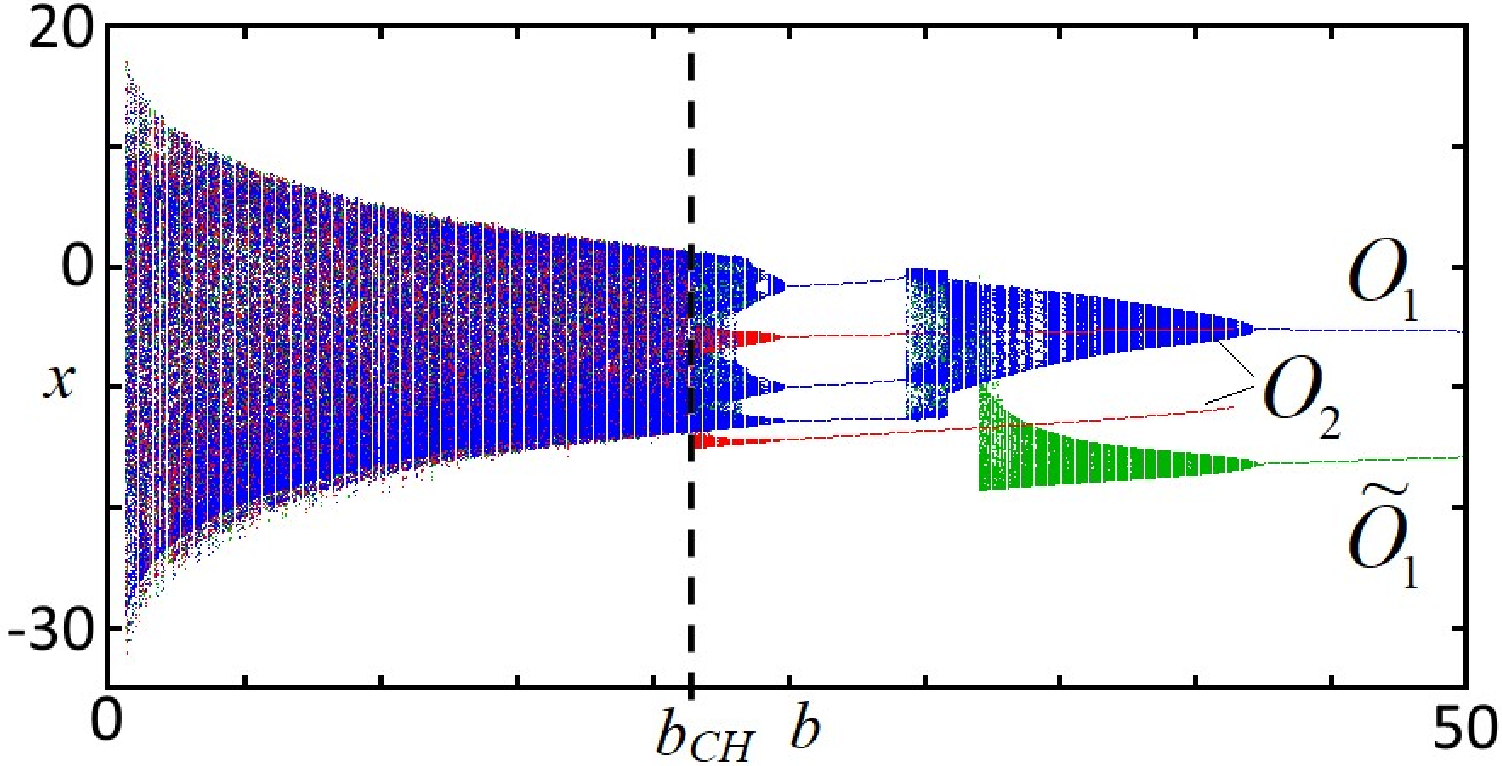
b=35
b=30
b=22
b=3



b=26
b=24.7
b=23.5
b=23

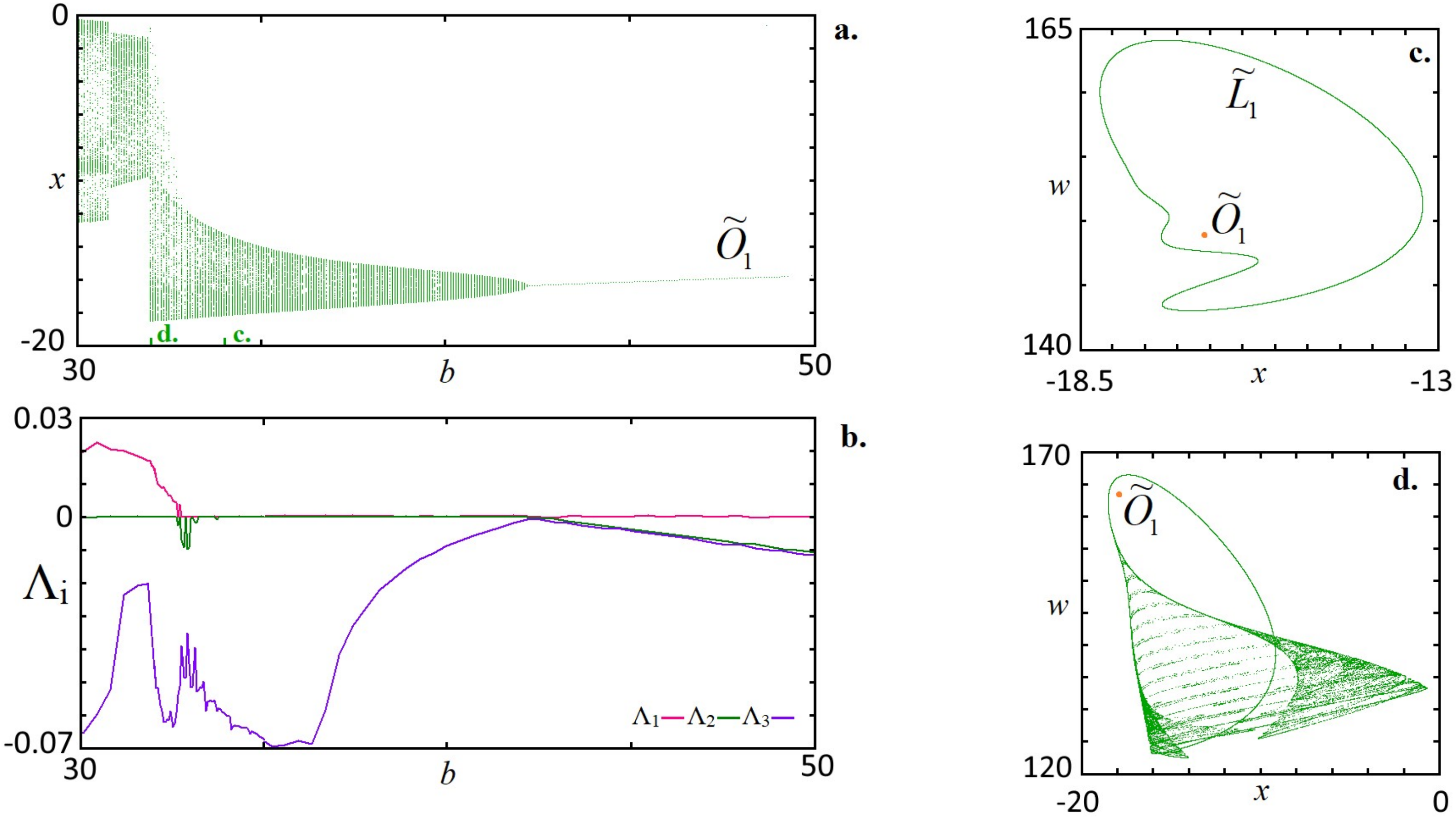
Гиперхаос в системе Ресслера

a=0.25, c=0.4, d=0.05



Гиперхаос в системе Ресслера

a=0.25, *c*=0.4, *d*=0.05

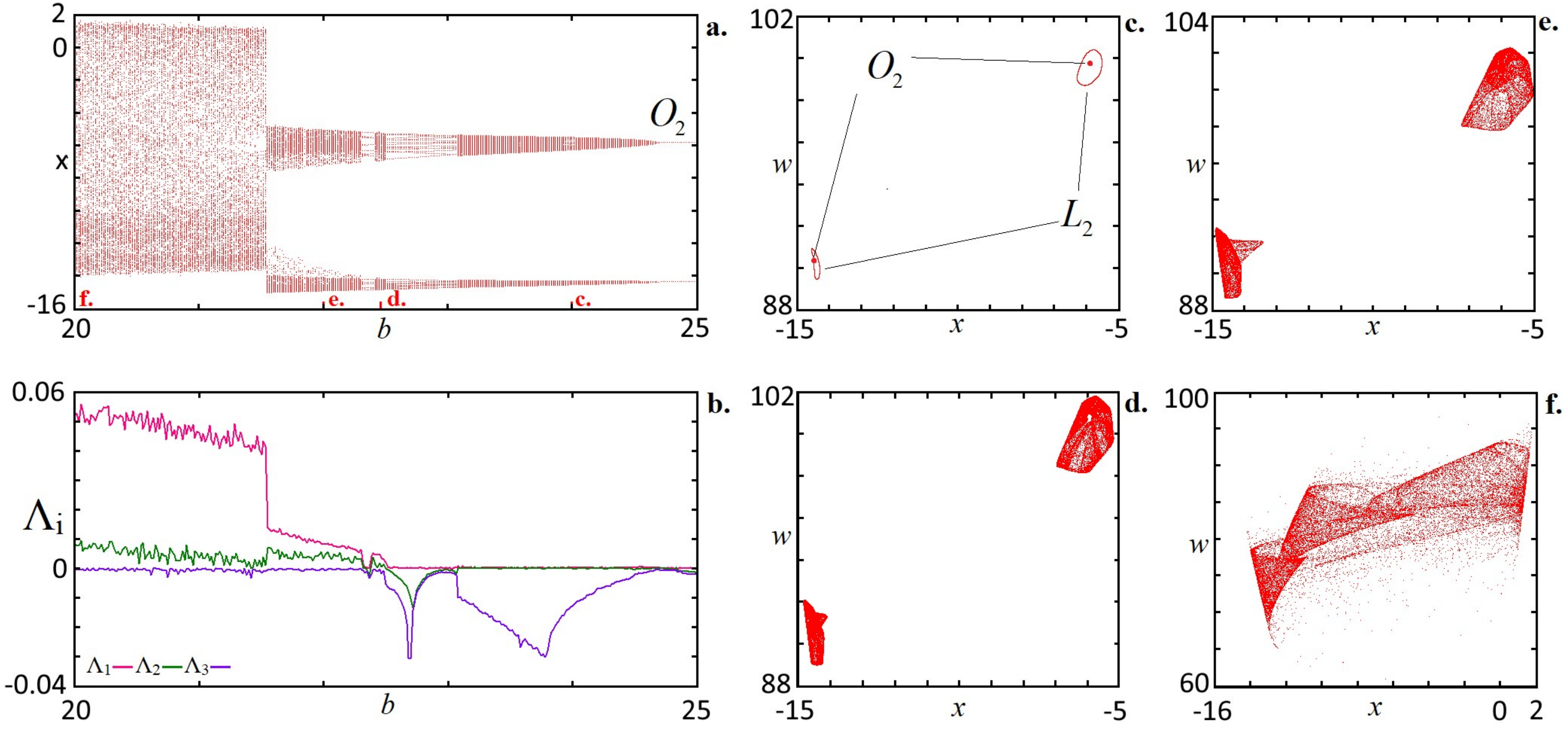


b=34

b=32

Гиперхаос в системе Ресслера

a=0.25, *c*=0.4, *d*=0.05

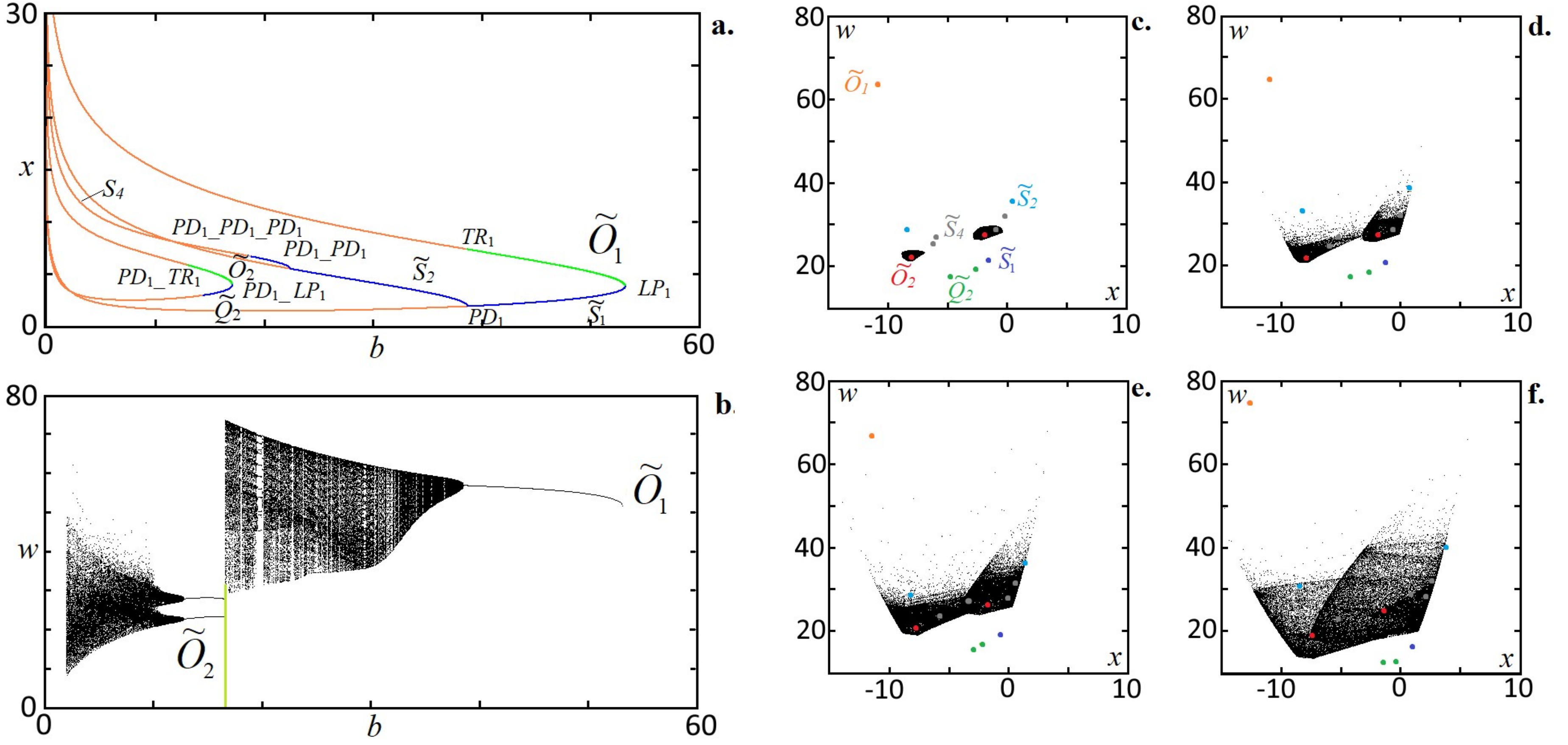


b=24
b=22.45

b=22
b=20

Гиперхаос в системе Ресслера

a=0.25, c=0.5, d=0.05

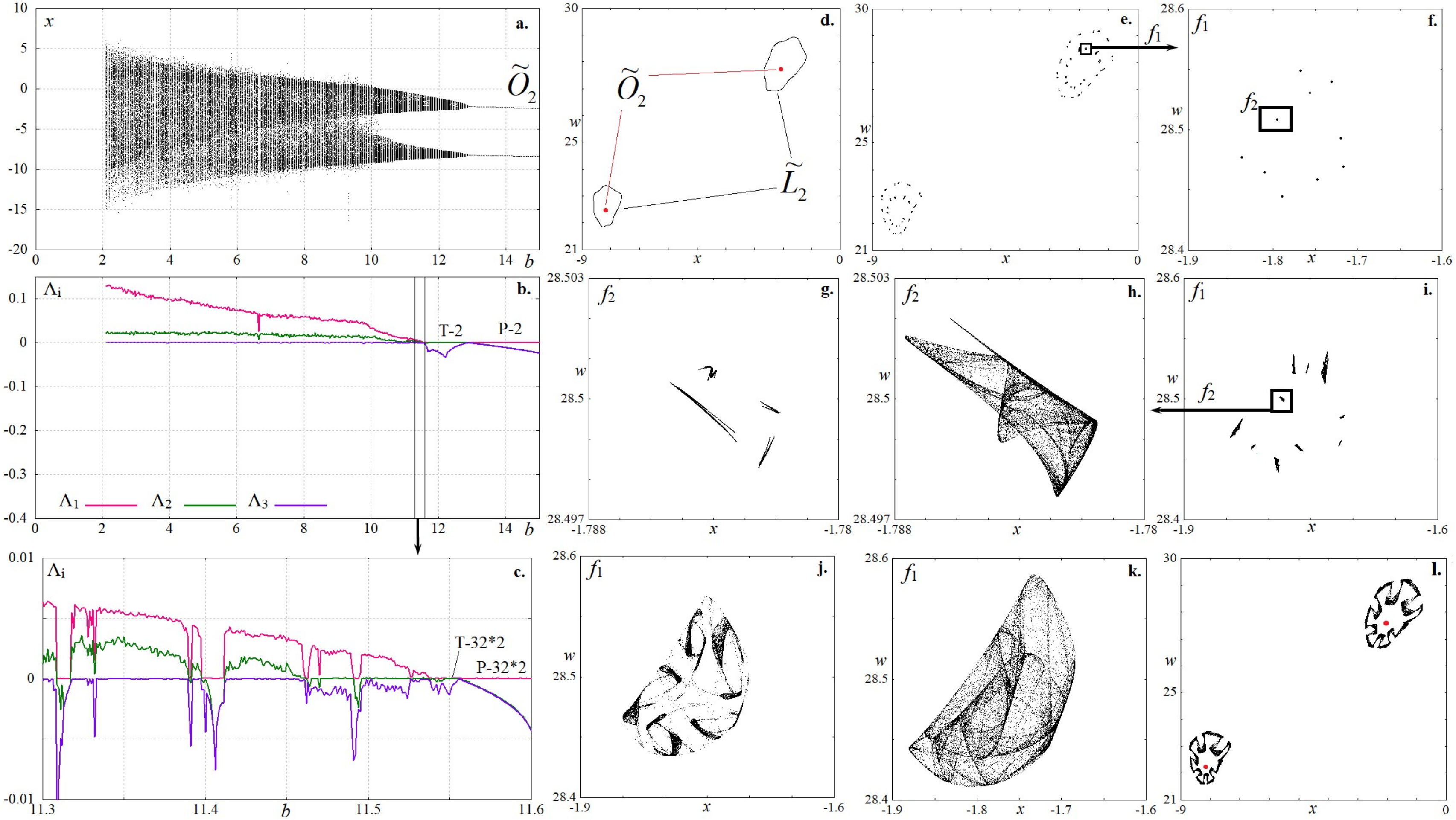


b=11
b=10

b=8
b=4

Гиперхаос в системе Ресслера

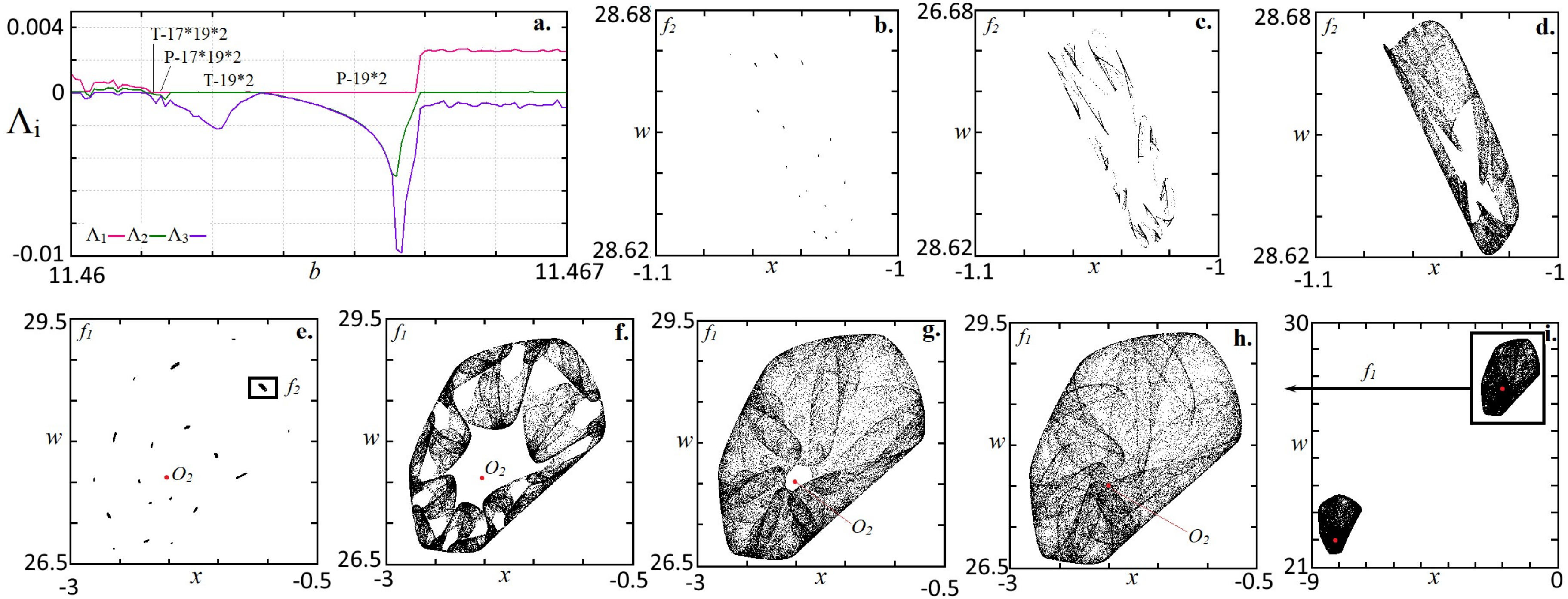
a=0.25, *c*=0.5, *d*=0.05



12
11.55
11.54
11.53714
11.5371
11.537
11.53
11.5

Гиперхаос в системеРесслера

a=0.25, *c*=0.5, *d*=0.05



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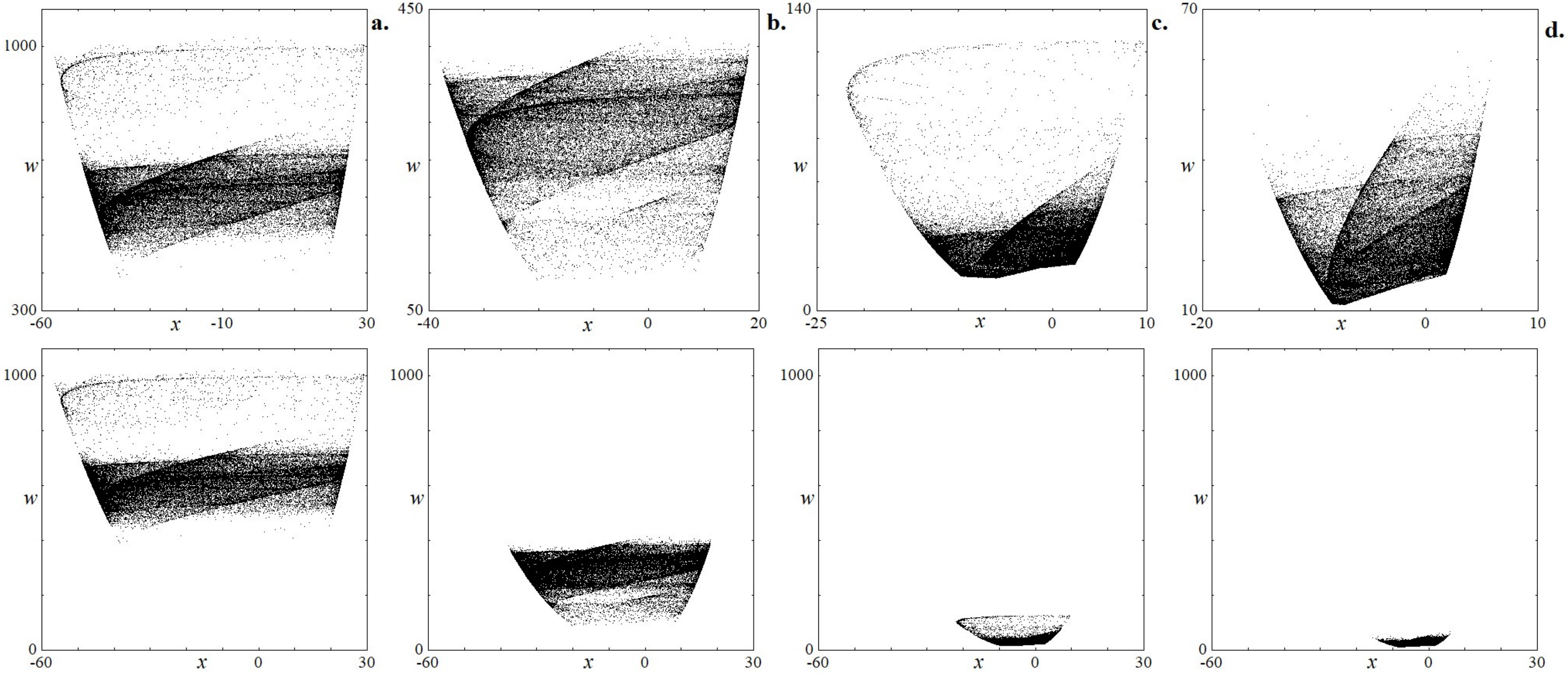
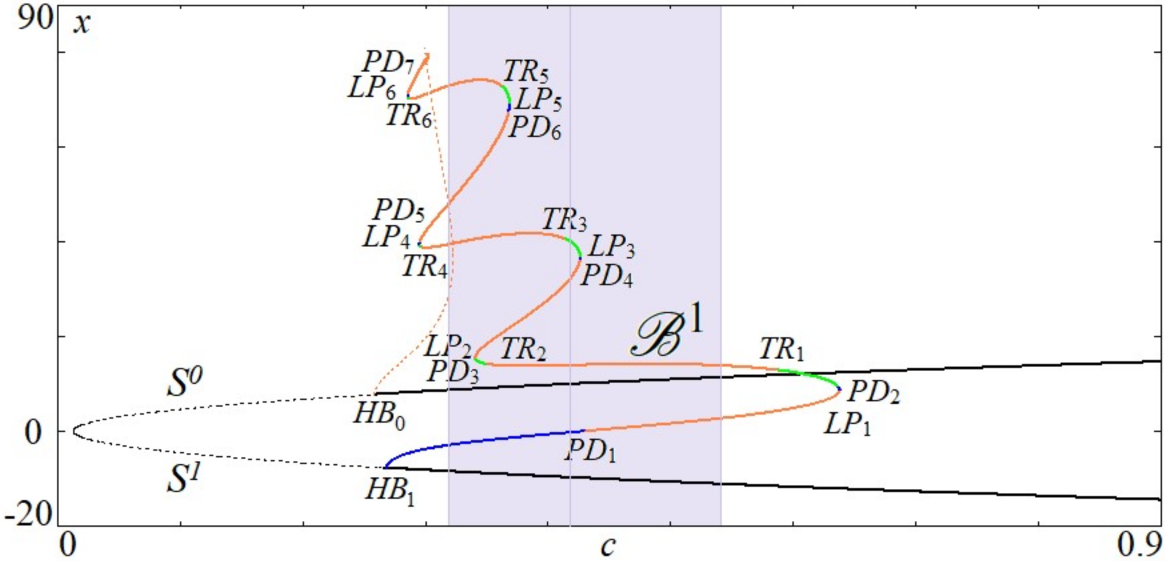


c=0.32 *c*=0.34 *c*=0.45 *c*=0.5

c	Λ_1	Λ_2	Λ_3	Λ_4
0.32	0.0925	0.0188	0.0	-657.6230
0.34	0.0993	0.0174	0.0	-334.7749
0.45	0.1132	0.0187	0.0	-37.7681
0.5	0.1123	0.0208	0.0	-24.6006

a=0.25, b=3, d=0.05

Гиперхаос в системе Ресслера



a=0.25, b=12, d=0.05

$$\xrightarrow{HB_1} C_1^1(2,1) \xrightarrow{PD_1} C_1^1(1,2)$$

$$\xrightarrow{LP_1} C_2^1(2,1) \xrightarrow{PD_2} C_2^1(3,0) \xrightarrow{TR_1} C_2^1(1,2) \xrightarrow{TR_2} C_2^1(3,0) \xrightarrow{PD_3} C_2^1(2,1)$$

$$\xrightarrow{LP_2} C_3^1(1,2)$$

$$\xrightarrow{LP_2} C_3^1(1,2) \xrightarrow{PD_4} C_3^1(2,1)$$

$$\xrightarrow{LP_3} C_4^1(3,0) \xrightarrow{TR_3} C_4^1(1,2) \xrightarrow{TR_4} C_4^1(3,0)$$

$$\xrightarrow{LP_4} C_5^1(2,1)$$

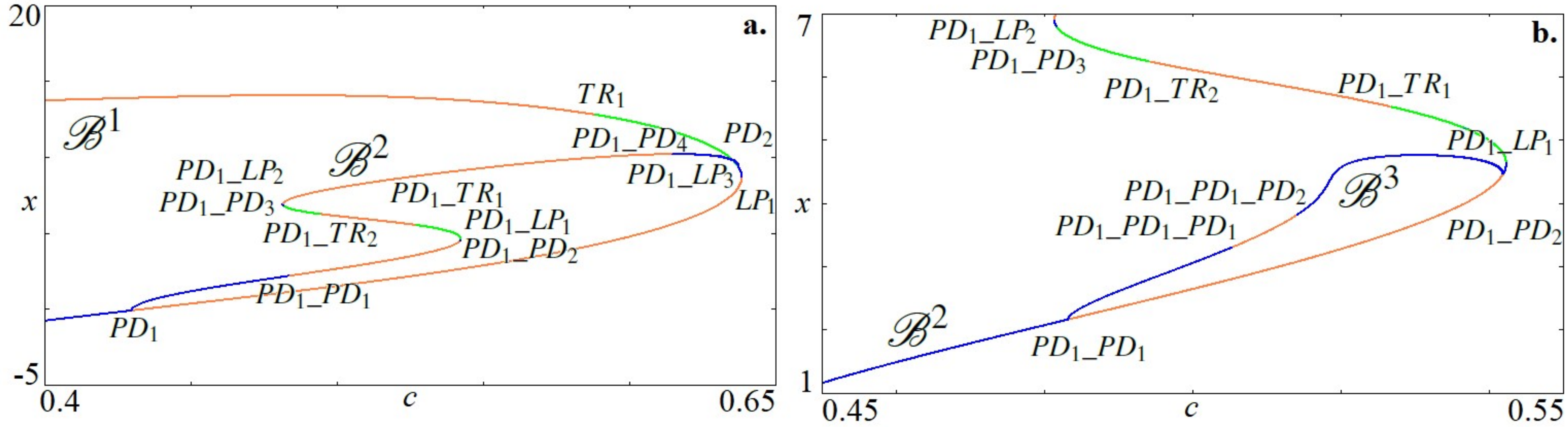
$$\xrightarrow{LP_4} C_5^1(2,1) \xrightarrow{PD_5} C_5^1(1,2) \xrightarrow{PD_6} C_5^1(2,1)$$

$$\xrightarrow{LP_5} C_6^1(3,0) \xrightarrow{TR_5} C_6^1(1,2) \xrightarrow{TR_6} C_6^1(3,0)$$

$$\xrightarrow{LP_6} C_7^1(2,1)$$

$$\xrightarrow{LP_6} C_7^1(2,1) \xrightarrow{PD_7} C_7^1(1,2)$$

Гиперхаос в системе Ресслера



$$\xrightarrow{PD_1} C_1^2(2,1) \xrightarrow{PD_1\_PD_1} C_1^2(1,2) \xrightarrow{PD_1\_PD_2} C_1^2(2,1)$$

$$\xrightarrow{PD_1\_LP_1} C_2^2(3,0)$$

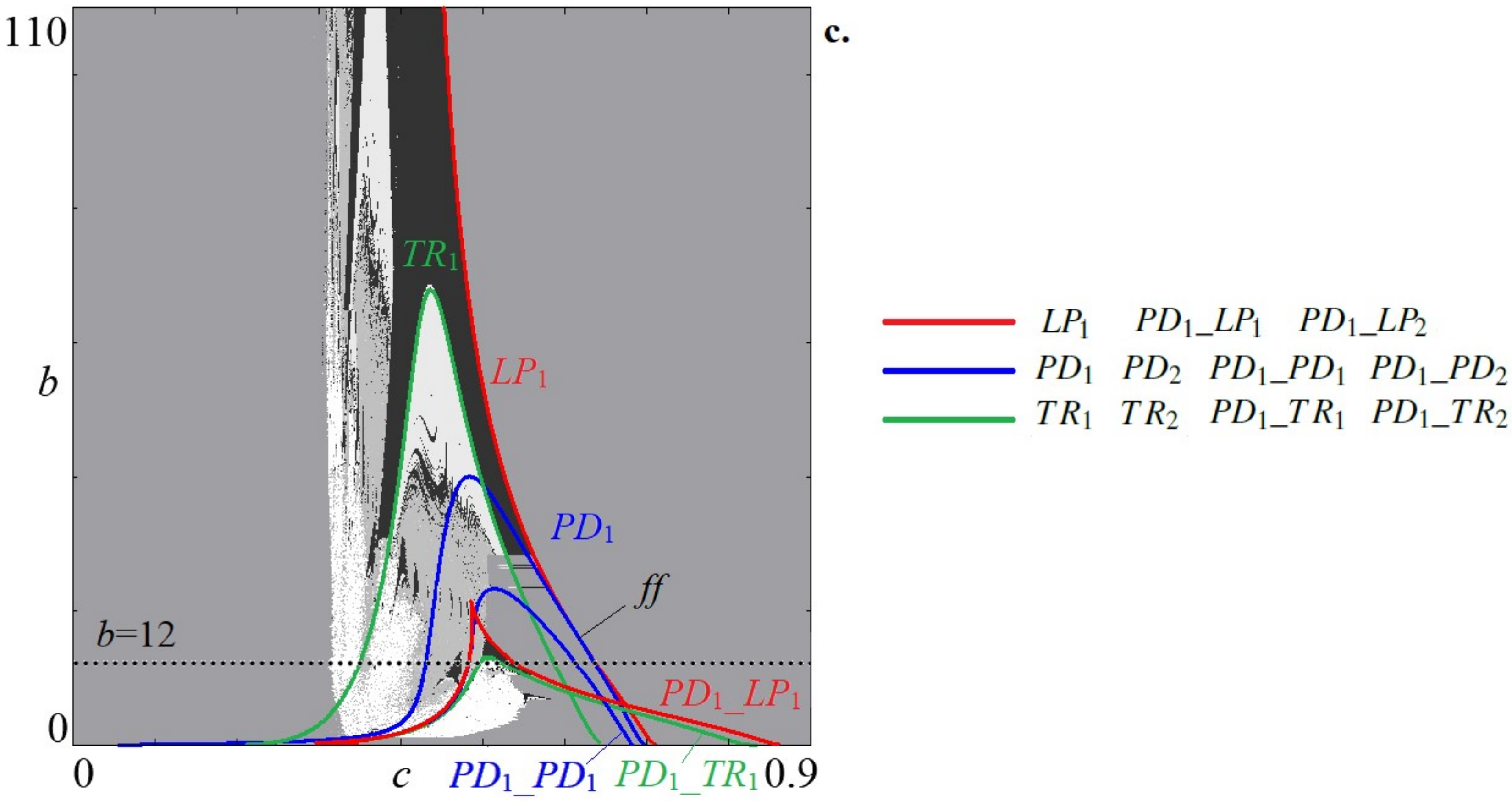
$$\xrightarrow{PD_1\_LP_1} C_2^2(3,0) \xrightarrow{PD_1\_TR_1} C_2^2(1,2) \xrightarrow{PD_1\_TR_2} C_2^2(3,0) \xrightarrow{PD_1\_PD_3} C_2^2(2,1)$$

$$\xrightarrow{PD_1\_LP_2} C_3^2(1,2)$$

$$\xrightarrow{PD_1\_LP_2} C_3^2(1,2) \xrightarrow{PD_1\_PD_4} C_3^2(2,1) \xrightarrow{PD_1\_LP_3 = PD_2}$$

$$\xrightarrow{PD_1\_PD_1} C_1^3(2,1) \xrightarrow{PD_1\_PD_1\_PD_1} C_1^3(1,2) \xrightarrow{PD_1\_PD_1\_PD_2} C_1^3(2,1) \quad PD_1\_LP_1$$

Гиперхаос в системе Ресслера



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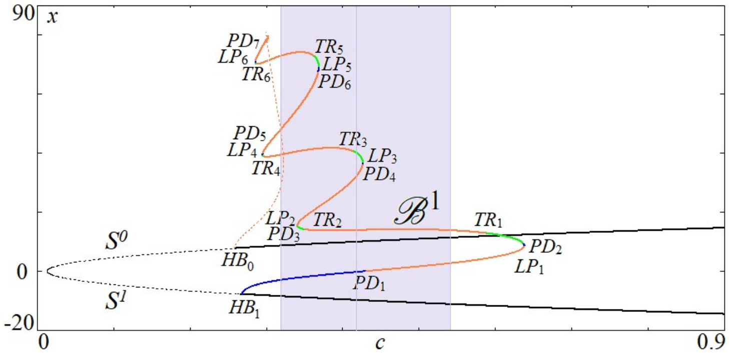
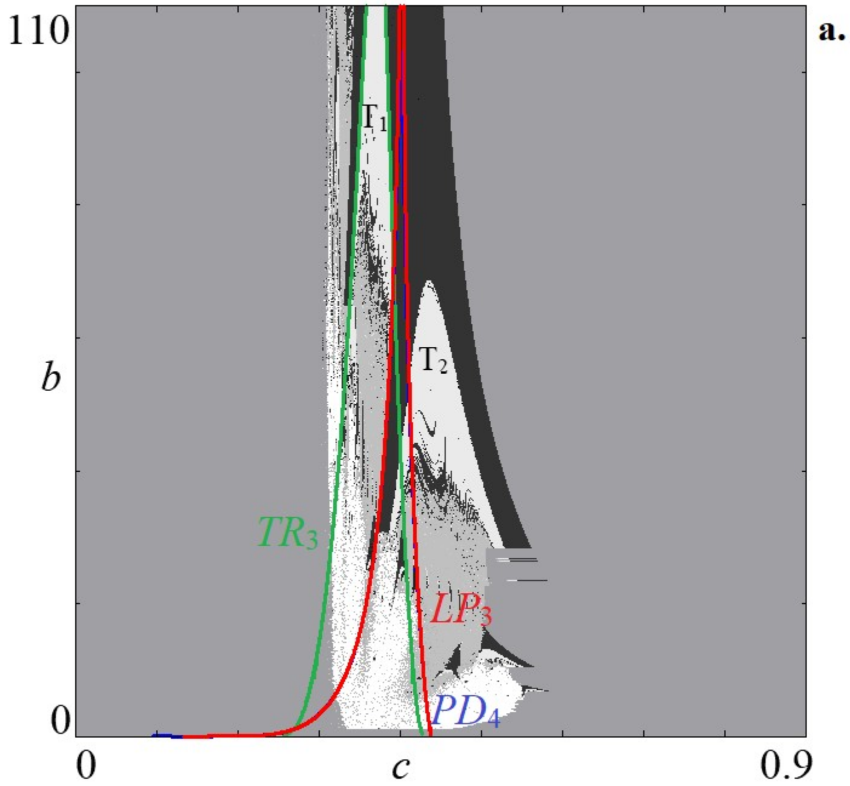
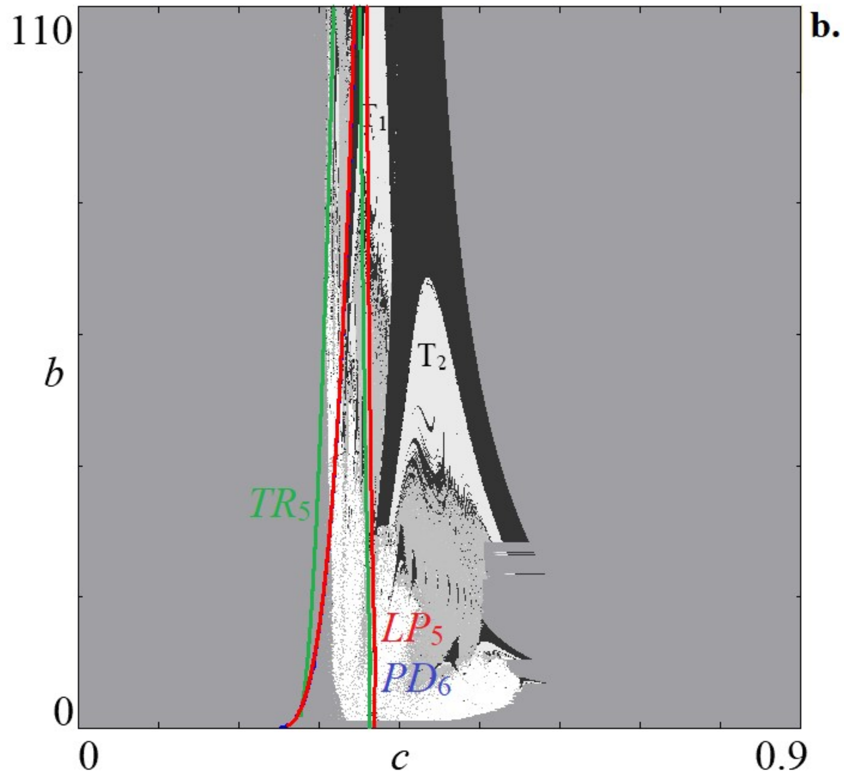






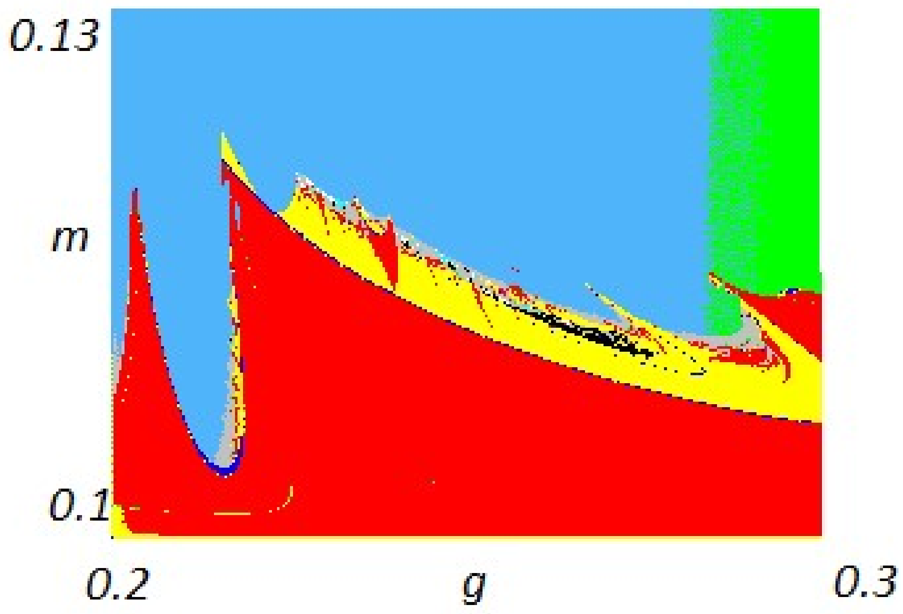
Гиперхаос в модифицированном генераторе Анищенко-Астахова

$$\dot{x} = mx + y - x\phi - dx^3,$$
$$\dot{y} = -y,$$
$$z = \varphi,$$
$$\phi = -\gamma\varphi + \gamma\Phi(x) - gz,$$

$$\Phi(x) = I(x)x^2, \quad I(x) = \begin{cases} 1, & x > 0, \\ 0, & x \le 0 \end{cases}$$

D	P	T	C	HC	C0

Режим	
бесконечность	D
периодический	P
квазипериодический	T
хаос	C
гиперхаос	HC
хаос с дополнительным нулевым показателем	C0

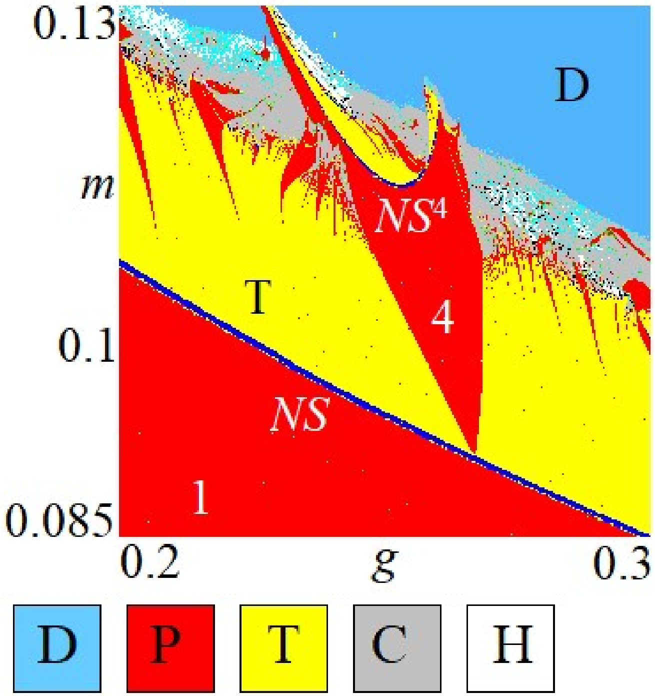


Карта показателей Ляпунова модели (1), γ=0.001, d=0.2

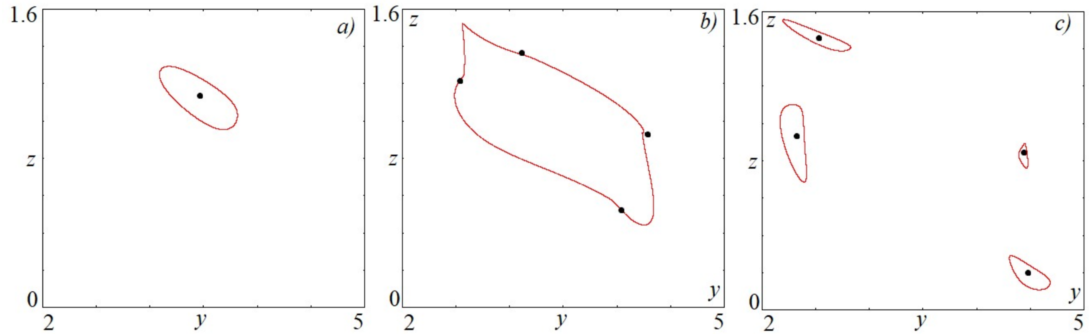
$$\ddot{y} - (m - z)\dot{y} + y = 0$$

Sataev I.R., Stankevich N.V. Cascade of torus birth bifurcations and inverse cascade of Shilnikov attractors merging at the threshold of hyperchaos.

Модифицированный генератор Анищенко-Астахова



Regime	Symbol	Spectrum of Lyapunov exponents
periodic	P	$\Lambda_1 = 0, \Lambda_4 < \Lambda_3 < \Lambda_2 < 0$
quasiperiodic	T	$\Lambda_1 = 0, \Lambda_2 = 0, \Lambda_4 < \Lambda_3 < 0$
chaos	C	$\Lambda_1 > 0, \Lambda_2 = 0, \Lambda_4 < \Lambda_3 < 0$
hyperchaos	CH	$\Lambda_2 > \Lambda_1 > 0, \Lambda_3 = 0, \Lambda_4 < 0$
divergency	D	



Two-dimensional projections of phase portraits in the Poincaré section by surface x=0 demonstrating main bifurcations in the system (1) at g = 0.25: a) Neimark-Sacker bifurcation, m = 0.095 (black dot), m = 0.097 (red invariant curve); b) synchronization on the torus, m = 0.106 (red invariant curve), m = 0.107 (black dots); c) secondary Neimark-Sacker bifurcation, m = 0.115 (black dots), m = 0.116 (red invariant curves).

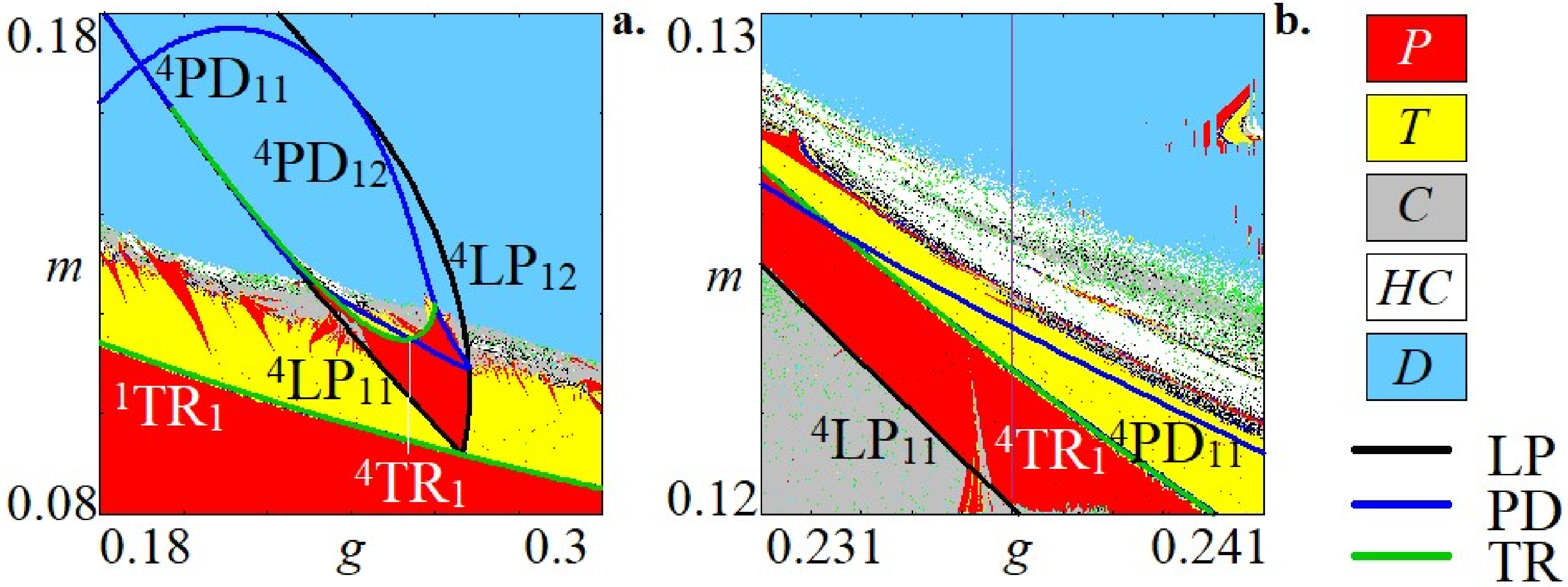
Hierarchy of the limit cycles and tori

Cycle number: P4:T1:P1

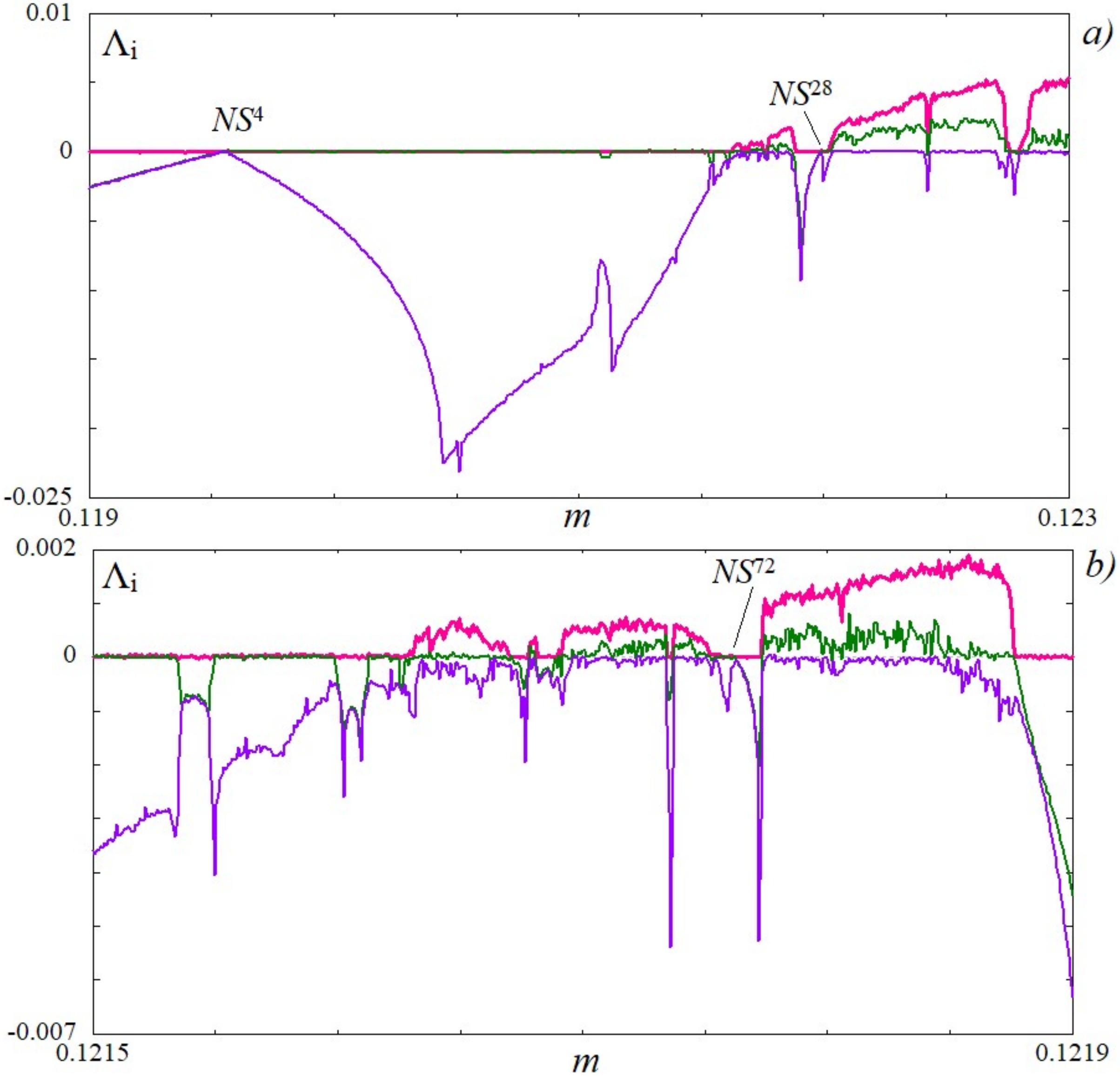
Torus number: T4:P4:T1:P1

Гиперхаос в модифицированном генераторе Анищенко-Астахова

γ=0.2, d=0.001

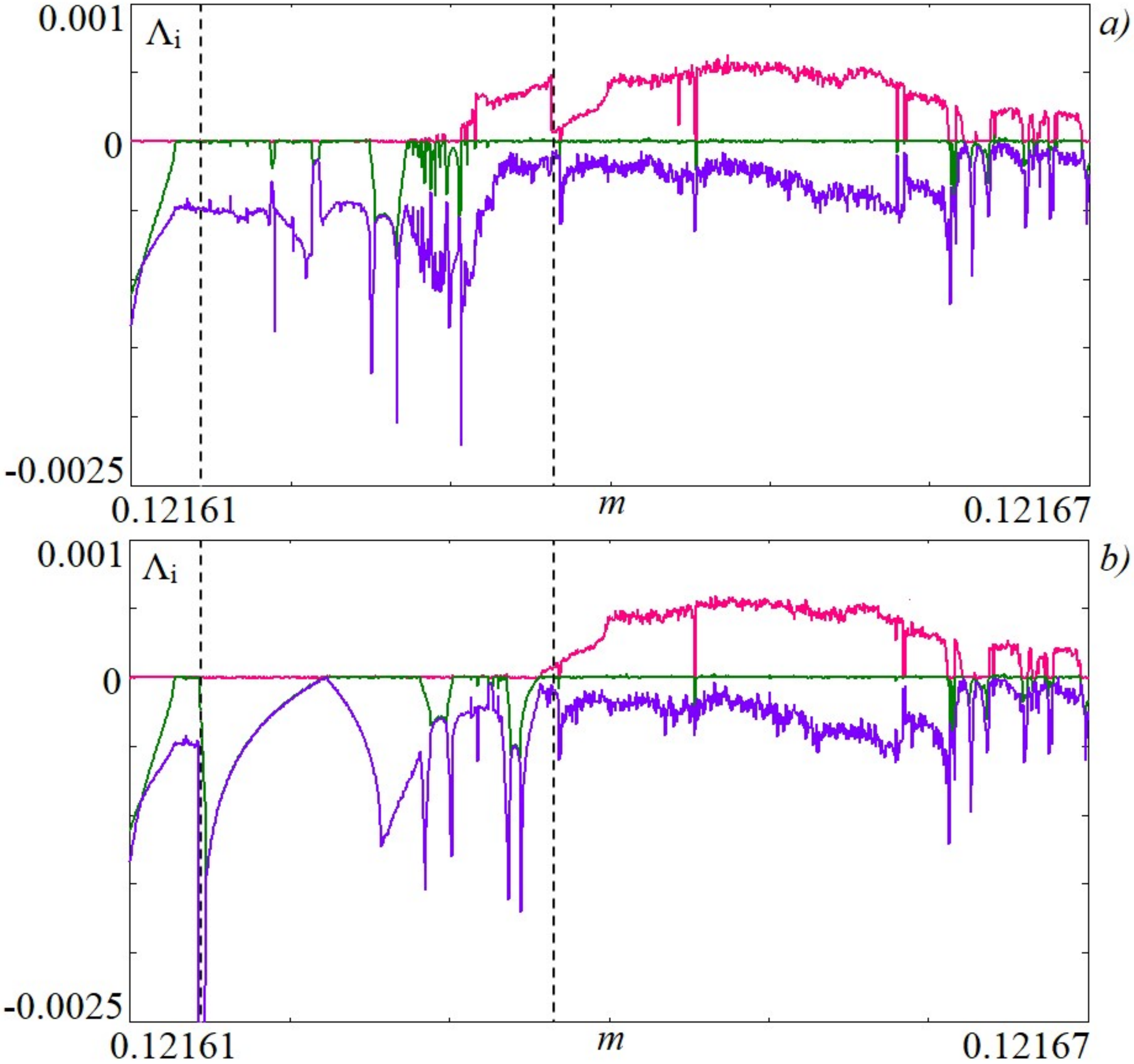


Модифицированный генератор Анищенко-Астахова



The dependence of the three largest Lyapunov exponents on the parameter m in the transition from torus to hyperchaos at g = 0.2407

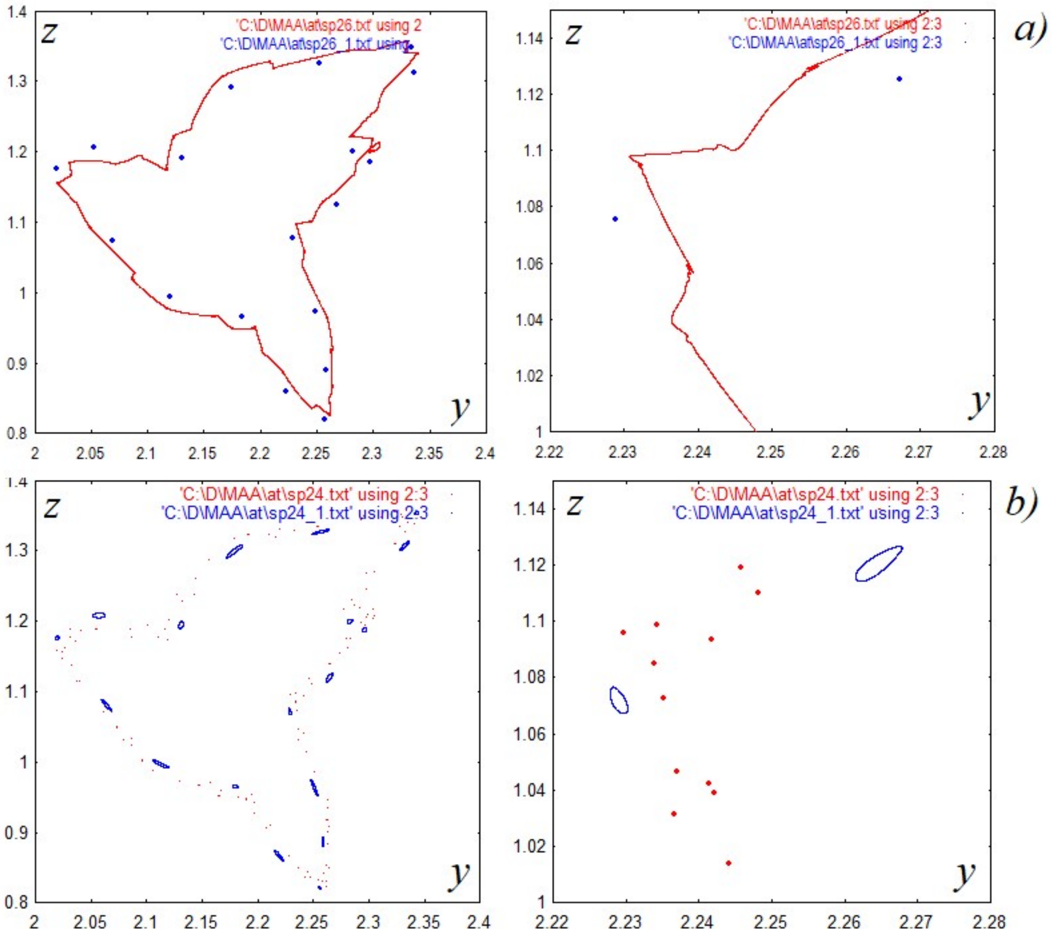
Модифицированный генератор Анищенко-Астахова



The dependence of the three largest Lyapunov exponents on the parameter m in the transition from torus to hyperchaos at $g = 0.2407$

Модифицированный генератор Анищенко-Астахова

T4:P4:T1:P1   P18:T4:P4:T1:P1



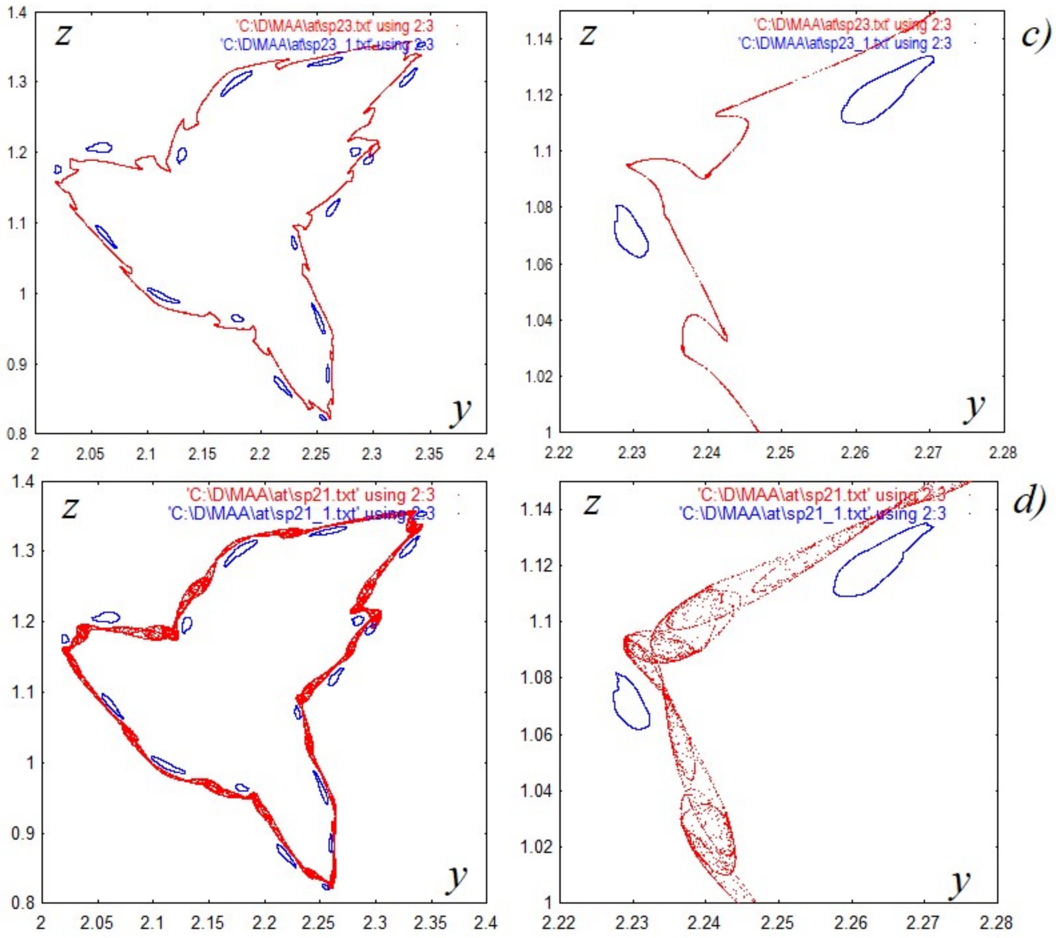
P177:T4:P4:T1:P1   T18:P18:T4:P4:T1:P1

Two-dimensional projections of phase portraits in the Poincare section by surface $x = 0$ demonstrating multistability and transition to chaos, $g = 0.2407$; a) m=0.121615; b) m=0.121625

m	Regime	Λ_1	Λ_2	Λ_3	Λ_4
0.121615	T^4	0.0000	0.0000	-0.0005	-0.0883
0.121615	P^{72}	0.0000	-0.0012	-0.0013	-0.0863
0.121625	P^{XX}	0.0000	-0.0001	-0.0012	-0.0875
0.121625	T^{72}	0.0000	0.0000	-0.0006	-0.0882
0.121631	C	0.0001	0.0000	-0.0010	-0.0879
0.121631	T^{72}	0.0000	0.0000	-0.0003	-0.0885
0.121632	C	0.0003	0.0000	-0.0004	-0.0887
0.121632	T^{72}	0.0000	0.0000	-0.0002	-0.0886
0.121636	C	0.0004	0.0000	-0.0002	-0.0890
0.121636	C	0.0001	0.0000	-0.0001	-0.0887
0.121640	C	0.0004	0.0000	-0.0002	-0.0890

Модифицированный генератор Анищенко-Астахова

T4:P4:T1:P1 T18:P18:T4:P4:T1:P1

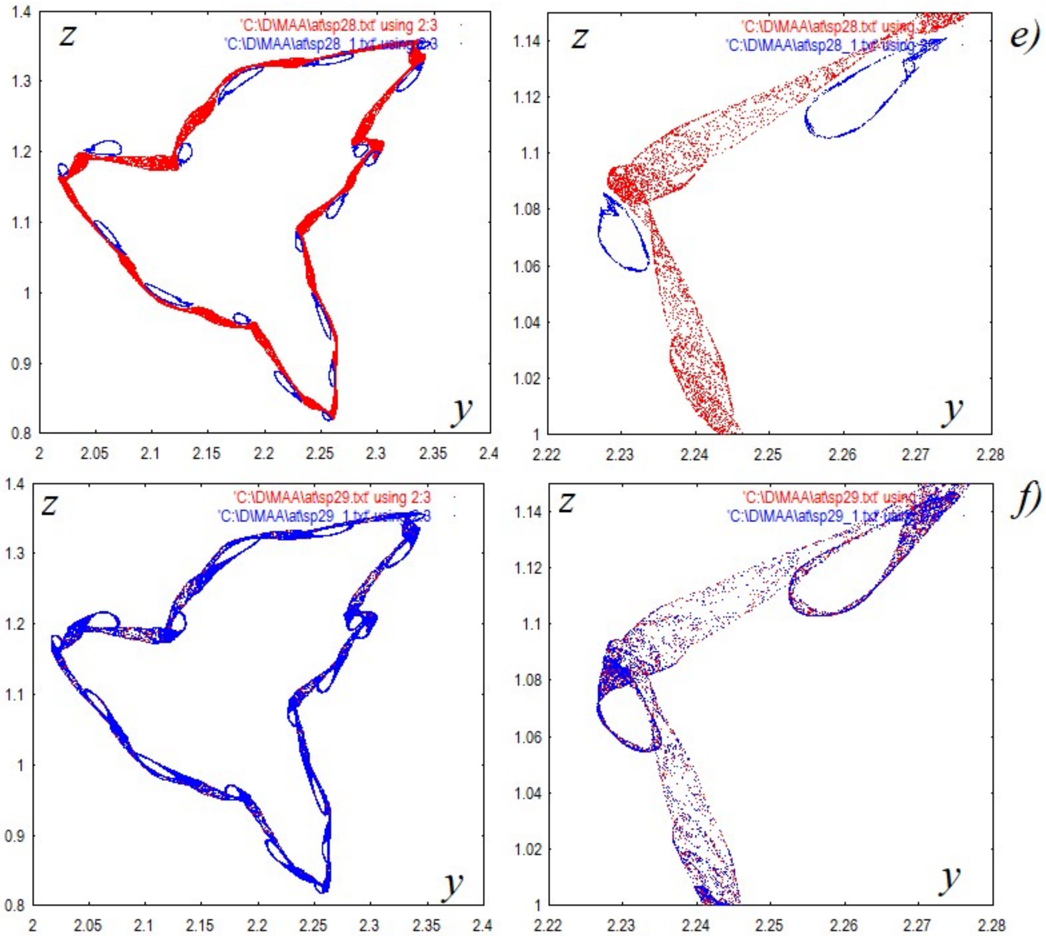


T18:P18:T4:P4:T1:P1

Two-dimensional projections of phase portraits in the Poincare section by surface $x = 0$ demonstrating multistability and transition to chaos, $g = 0.2407$; c) m=0.121631; d) m=0.121632

m	Regime	Λ_1	Λ_2	Λ_3	Λ_4
0.121615	T^4	0.0000	0.0000	-0.0005	-0.0883
0.121615	P^{72}	0.0000	-0.0012	-0.0013	-0.0863
0.121625	P^{XX}	0.0000	-0.0001	-0.0012	-0.0875
0.121625	T^{72}	0.0000	0.0000	-0.0006	-0.0882
0.121631	C	0.0001	0.0000	-0.0010	-0.0879
0.121631	T^{72}	0.0000	0.0000	-0.0003	-0.0885
0.121632	C	0.0003	0.0000	-0.0004	-0.0887
0.121632	T^{72}	0.0000	0.0000	-0.0002	-0.0886
0.121636	C	0.0004	0.0000	-0.0002	-0.0890
0.121636	C	0.0001	0.0000	-0.0001	-0.0887
0.121640	C	0.0004	0.0000	-0.0002	-0.0890

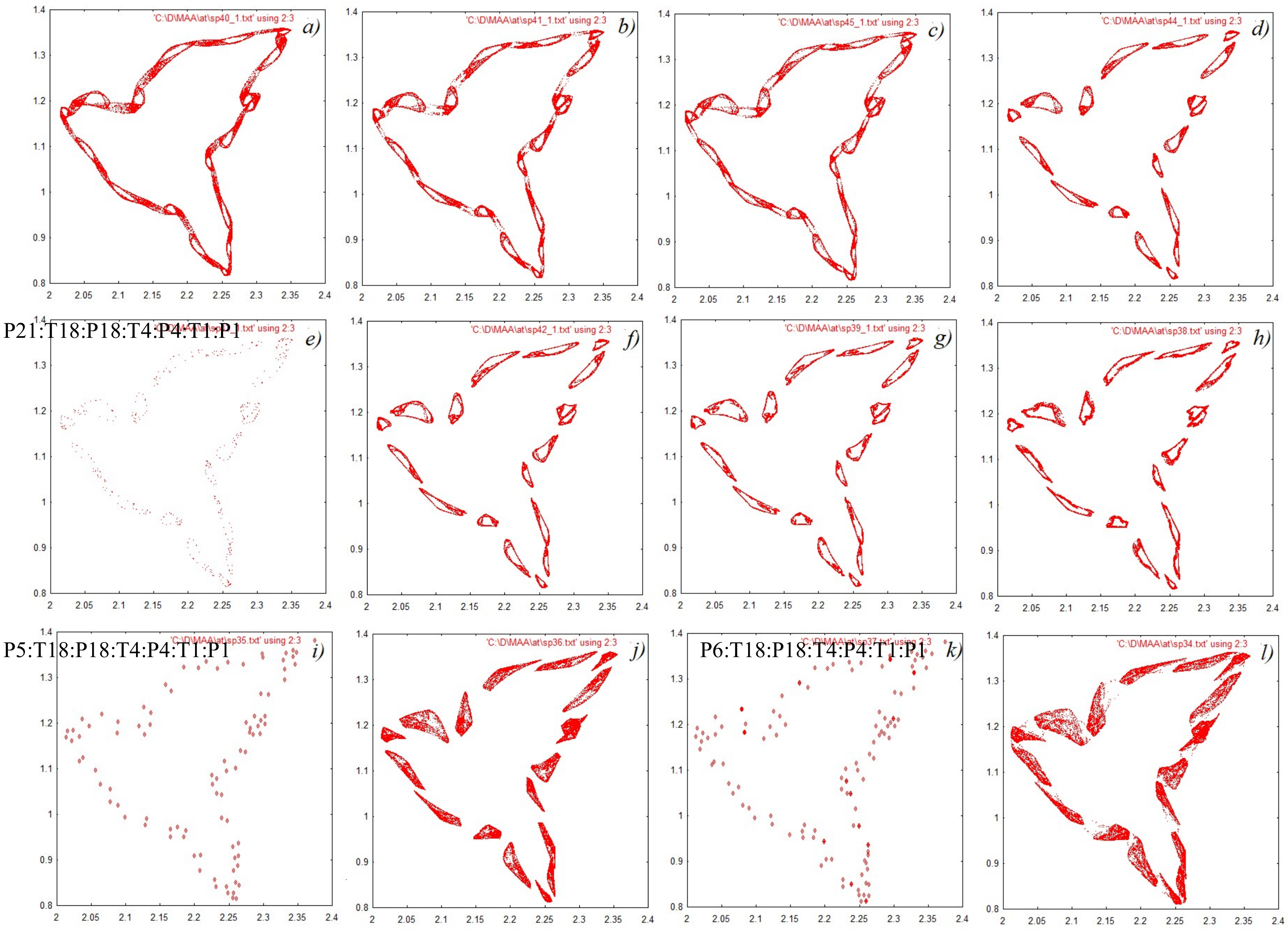
Модифицированный генератор Анищенко-Астахова



Two-dimensional projections of phase portraits in the Poincare section by surface $x = 0$ demonstrating multistability and transition to chaos, $g = 0.2407$; e) m=0.121636; f) m=0.121640

m	Regime	Λ_1	Λ_2	Λ_3	Λ_4
0.121615	T^4	0.0000	0.0000	-0.0005	-0.0883
0.121615	P^{72}	0.0000	-0.0012	-0.0013	-0.0863
0.121625	P^{XX}	0.0000	-0.0001	-0.0012	-0.0875
0.121625	T^{72}	0.0000	0.0000	-0.0006	-0.0882
0.121631	C	0.0001	0.0000	-0.0010	-0.0879
0.121631	T^{72}	0.0000	0.0000	-0.0003	-0.0885
0.121632	C	0.0003	0.0000	-0.0004	-0.0887
0.121632	T^{72}	0.0000	0.0000	-0.0002	-0.0886
0.121636	C	0.0004	0.0000	-0.0002	-0.0890
0.121636	C	0.0001	0.0000	-0.0001	-0.0887
0.121640	C	0.0004	0.0000	-0.0002	-0.0890

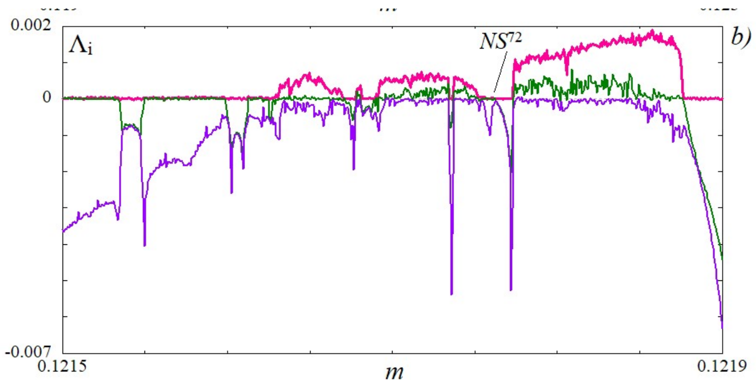
Модифицированный генератор Анищенко-Астахова



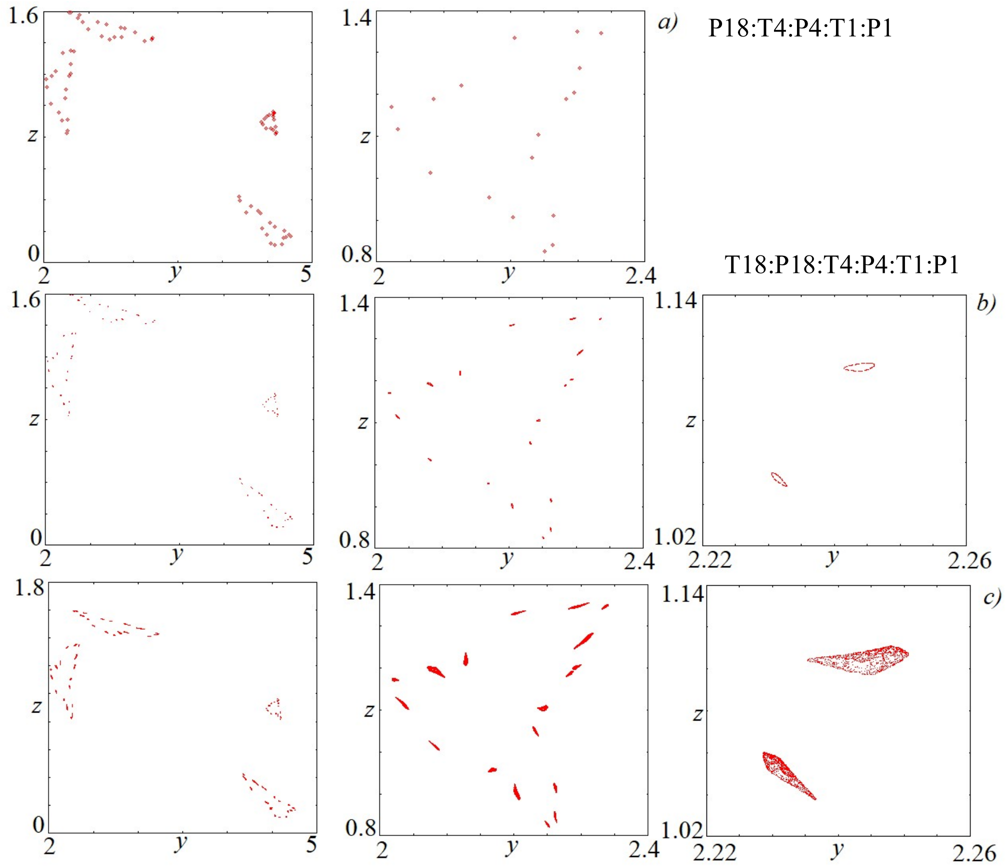
Two-dimensional projections of phase portraits in the Poincare section by surface $x = 0$

a) m=0.12165;
b) m=0.121655;
c) m=0.121656;
d) m=0.121657;
e) m=0.121658;
f) m=0.121659;
g) m=0.12166;
h) m=0.121655;
i) m=0.121675;
j) m=0.12168;
Λ1=0.0003,
Λ2=0.0001,
Λ3=0,
Λ4=-0.0891
k) m=0.121685;
l) m=0.1217.

Модифицированный генератор Анищенко-Астахова



The dependence of the three largest Lyapunov exponents on the parameter m in the transition from torus to hyperchaos at g = 0.2407

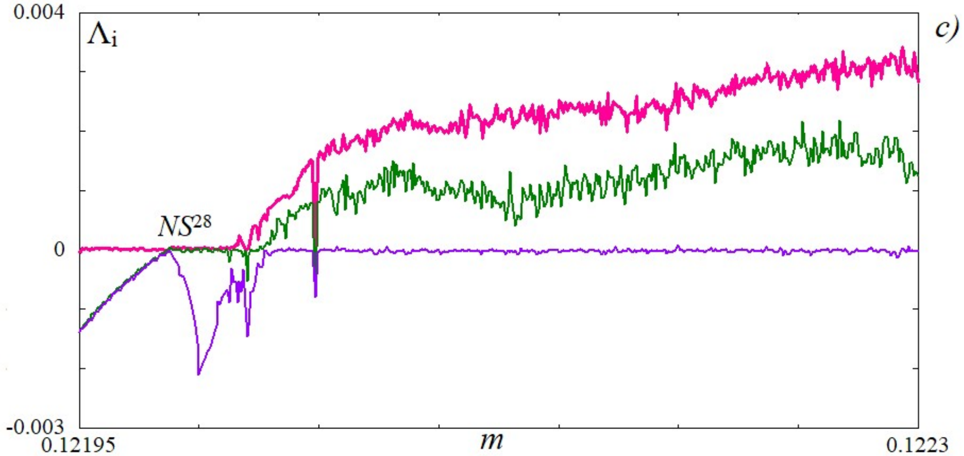


m	regime	Λ_1	Λ_2	Λ_3	Λ_4
0.12177	P-72	0.0000	-0.0010	-0.0011	-0.0866
0.12176	T	0.0000	0.0000	-0.0004	-0.0882
0.12175	HC	0.0002	0.0001	0.0000	-0.0890

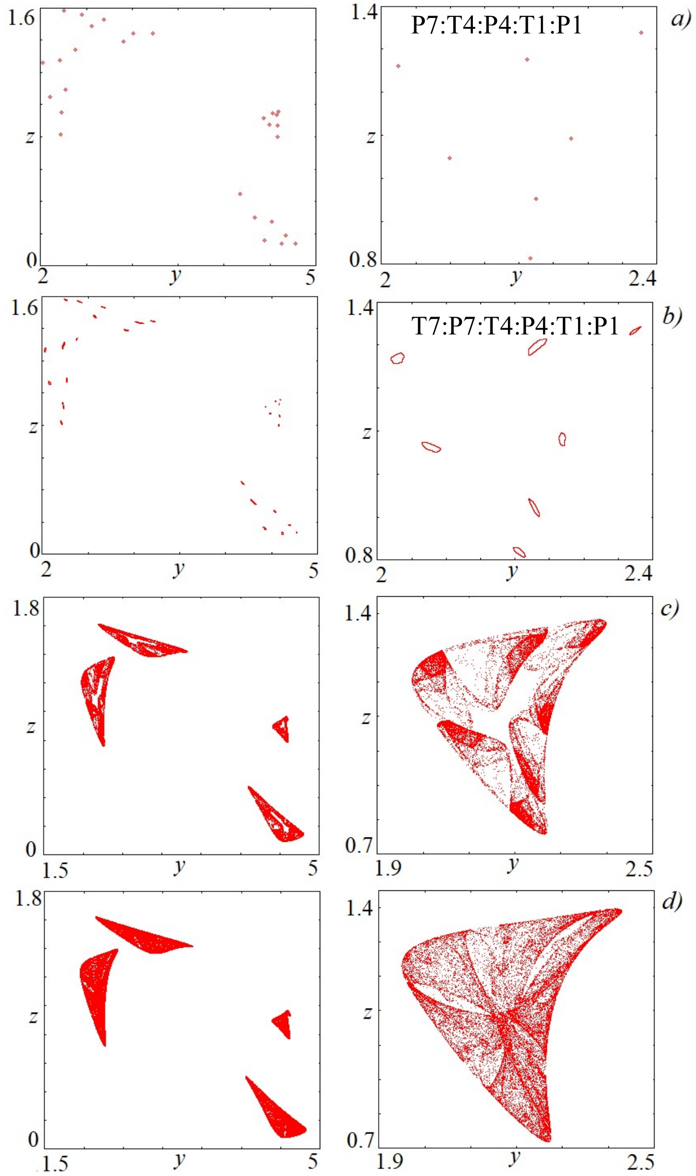
Two-dimensional projections of phase portraits in the Poincare section by surface x = 0, g = 0.2407; a) m=0.12177; b) m=0.12176; c) m=0.12175

Модифицированный генератор Анищенко-Астахова

The dependence of the three largest Lyapunov exponents on the parameter m in the transition from torus to hyperchaos at g=0.2407

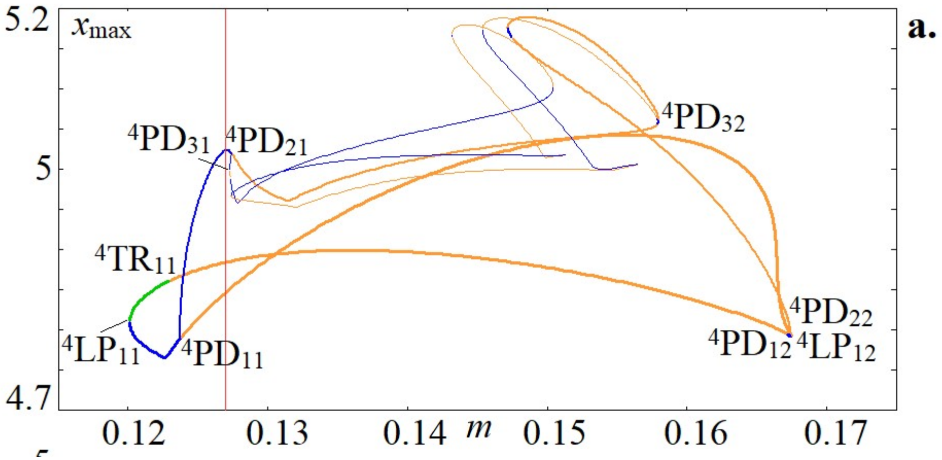


m	regime	Λ_1	Λ_2	Λ_3	Λ_4
0.1219	P-28	0.0000	-0.0044	-0.0062	-0.0779
0.122	T	0.0000	0.0000	-0.0021	-0.0863
0.12205	HC	0.0016	0.0008	0.0000	-0.0907
0.1223	HC	0.0031	0.0014	0.0000	-0.0925

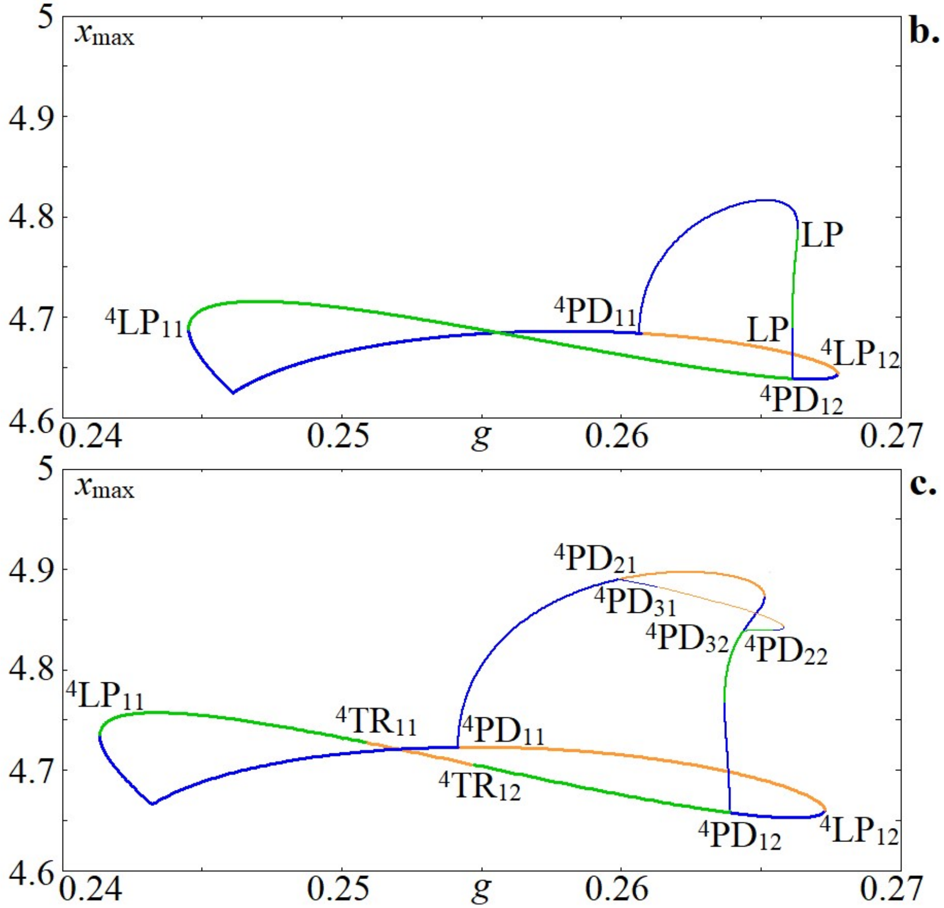


Two-dimensional projections of phase portraits in the Poincare section by surface $x = 0$, $g = 0.2407$;
a) m=0.1219; b) m=0.122; c) m=0.12205; d) m=0.1223

Гиперхаос в модифицированном генераторе Анищенко-Астахова



g=*0.236, γ=0.2, d*=0.001

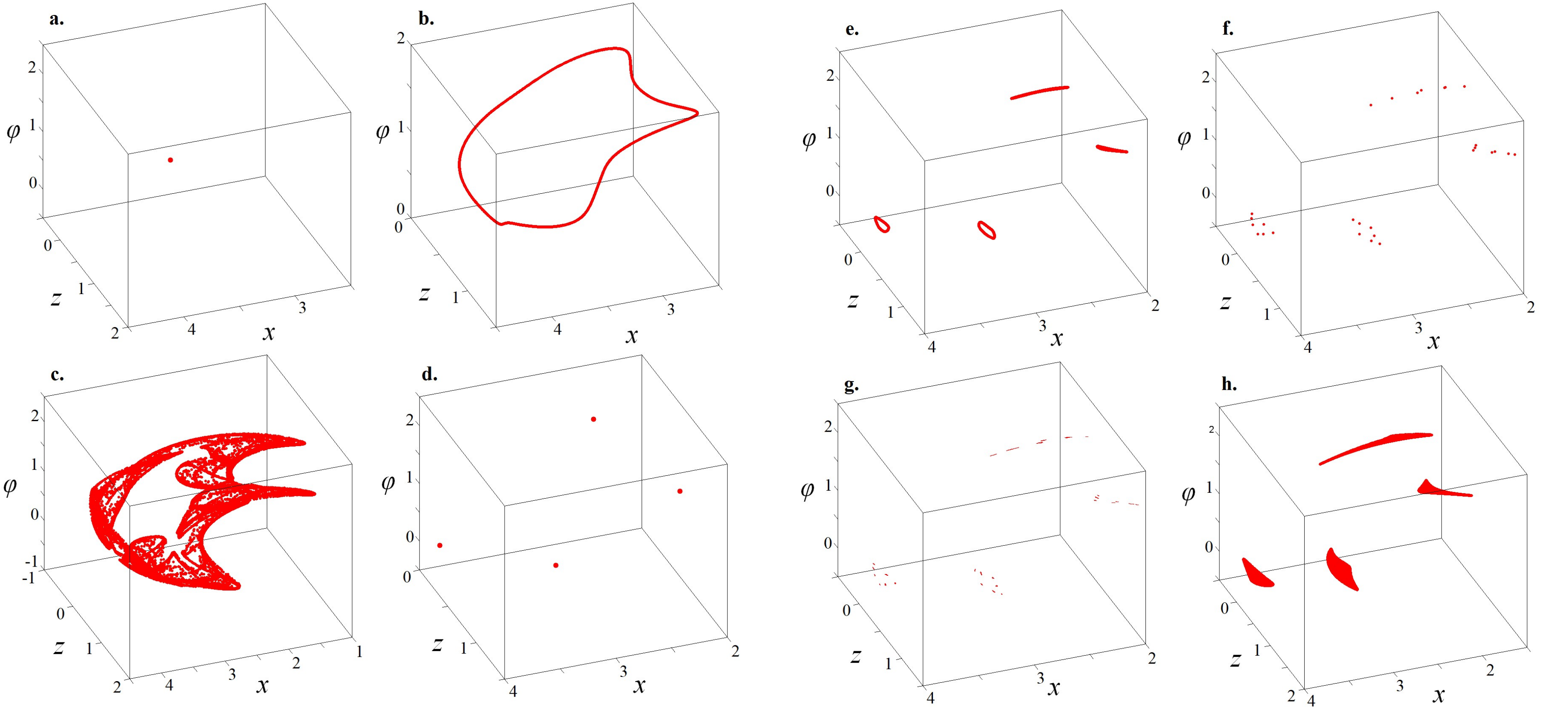


m=0.112,

m=0.115, γ=0.2, d=0.001

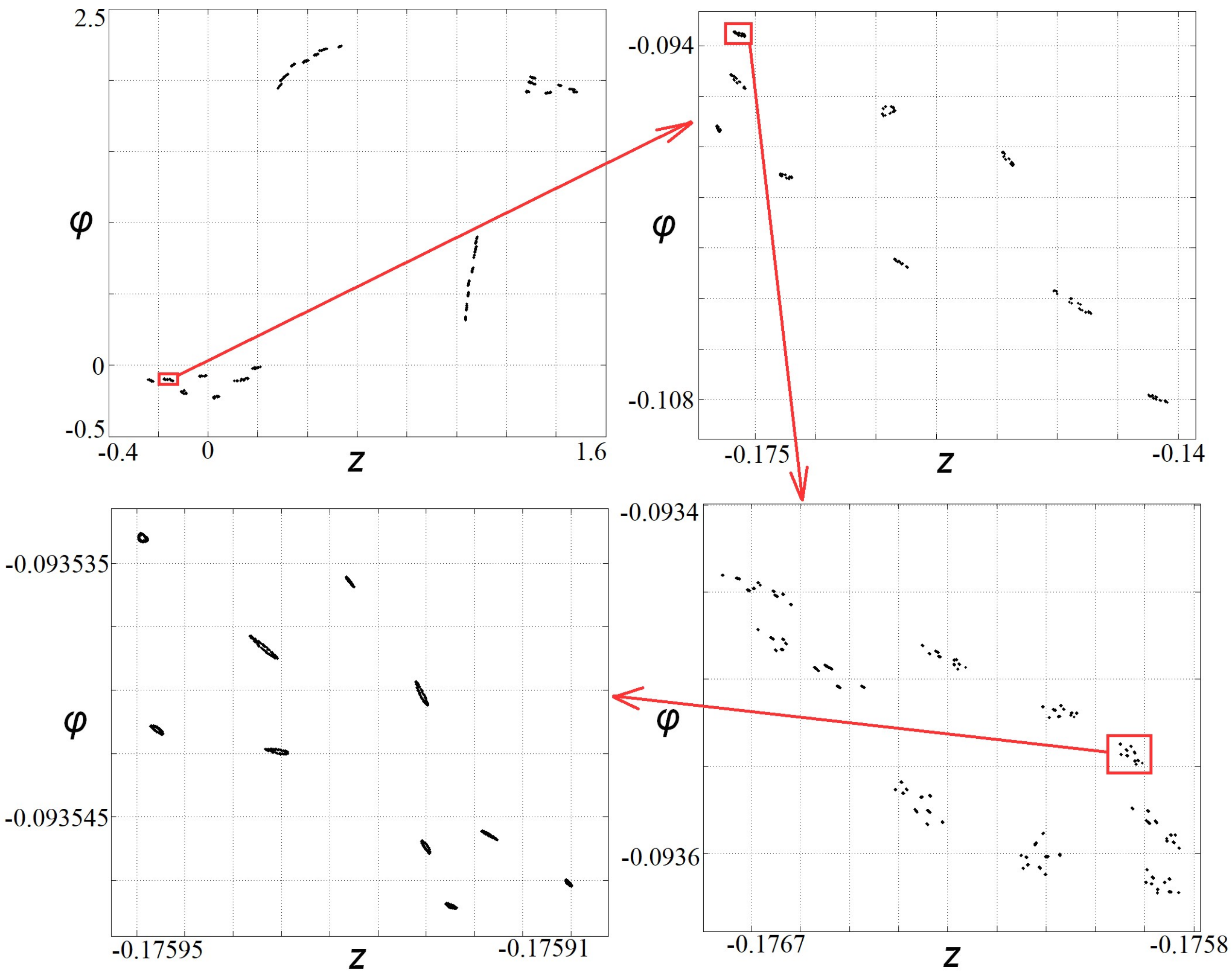
Гиперхаос в модифицированном генераторе Анищенко-Астахова

g=0.236, *γ*=0.2, *d*=0.001



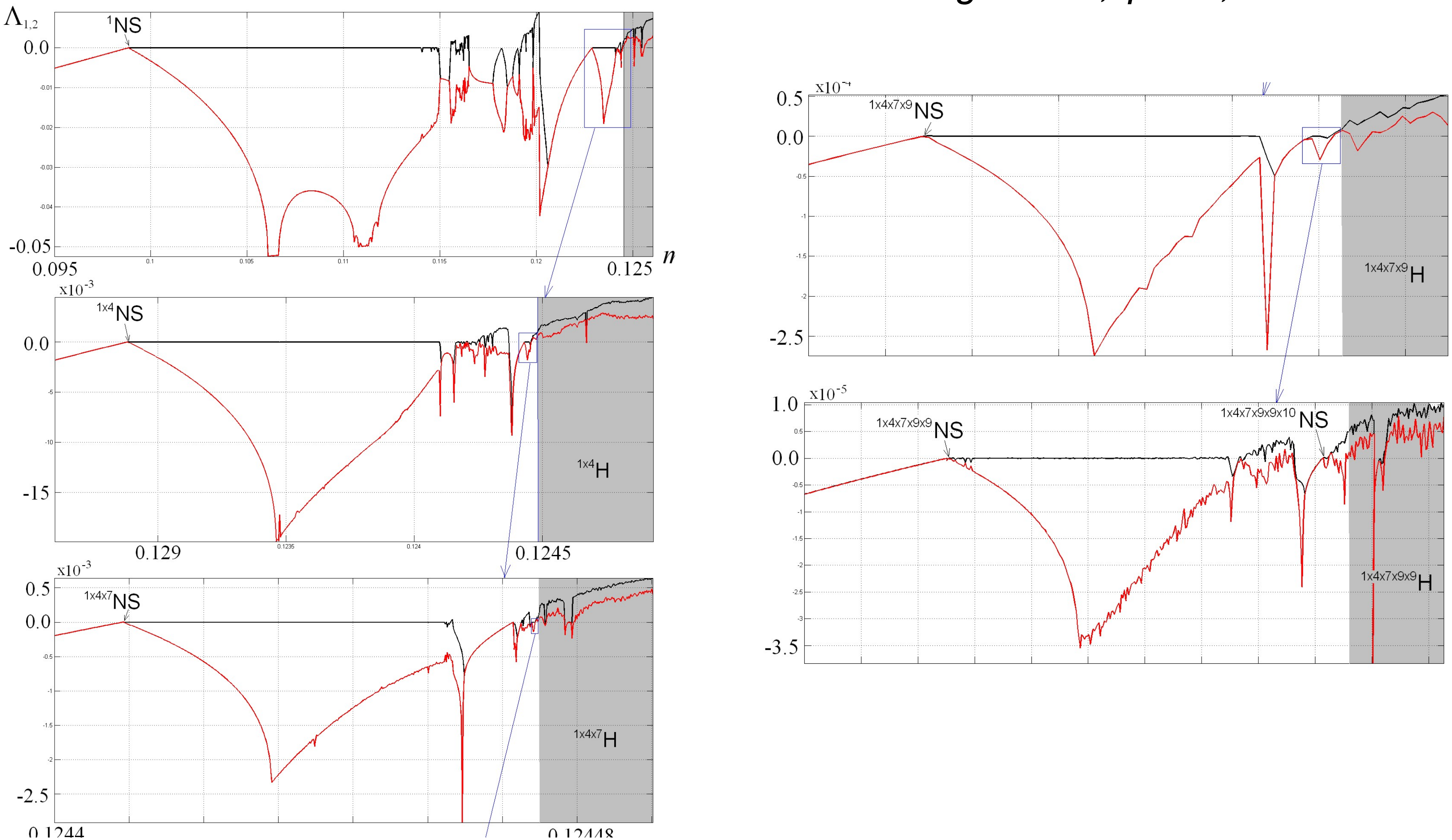
Гиперхаос в модифицированном генераторе Анищенко-Астахова

Phase portrait of the torus attractor $^{1*4*7*9*9*10}T$, m=0.124457208289, 0.236, d = 0.2, γ = 0.001.

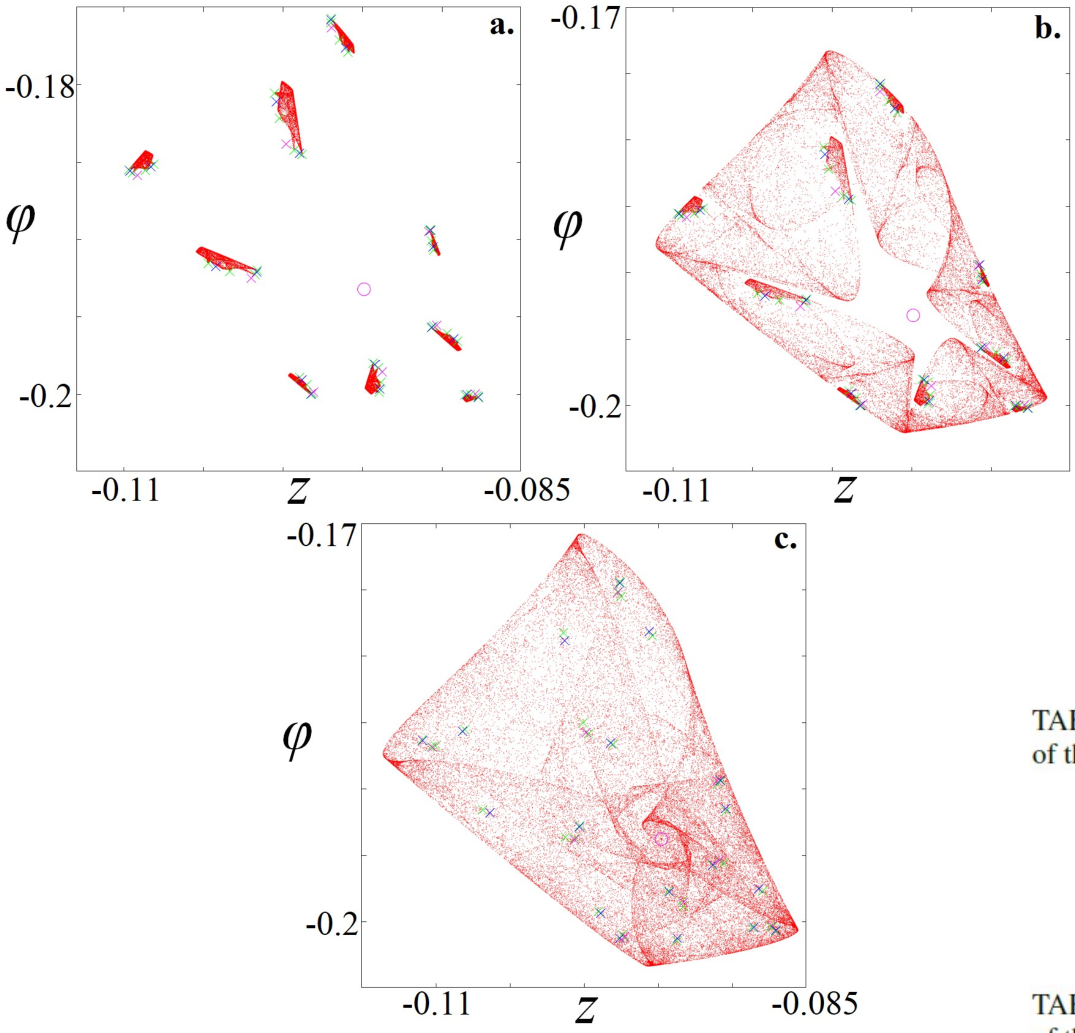


Гиперхаос в модифицированном генераторе Анищенко-Астахова

g=0.236, *γ*=0.2, *d*=0.001



Гиперхаос в модифицированном генераторе Анищенко-Астахова



g=*0.236*, *γ*=*0.2*, *d*=0.001
252-component ^{1*4*7*9}H
28-component ^{1*4*7}H
a) m = 0.124457465;
b) m = 0.12445748;
c) m = 0.1244620.
purple - period 252, blue - 504, green - 1008. purple is saddle-focus of period 28.

Cycle period	*m*
28	0.124370616
56	0.124427167
112	0.124450152
224	0.124455662

TABLE II. Bifurcation values of the parameter *m* for the first steps of the cascade for unstable cycle $^{1\times4\times7}S$.

Cycle period	*m*
252	0.1244564989
504	0.1244571880
1008	0.1244574263

TABLE III. Bifurcation values of the parameter *m* for the first steps of the cascade for unstable cycle $^{1\times4\times7\times9}S$.

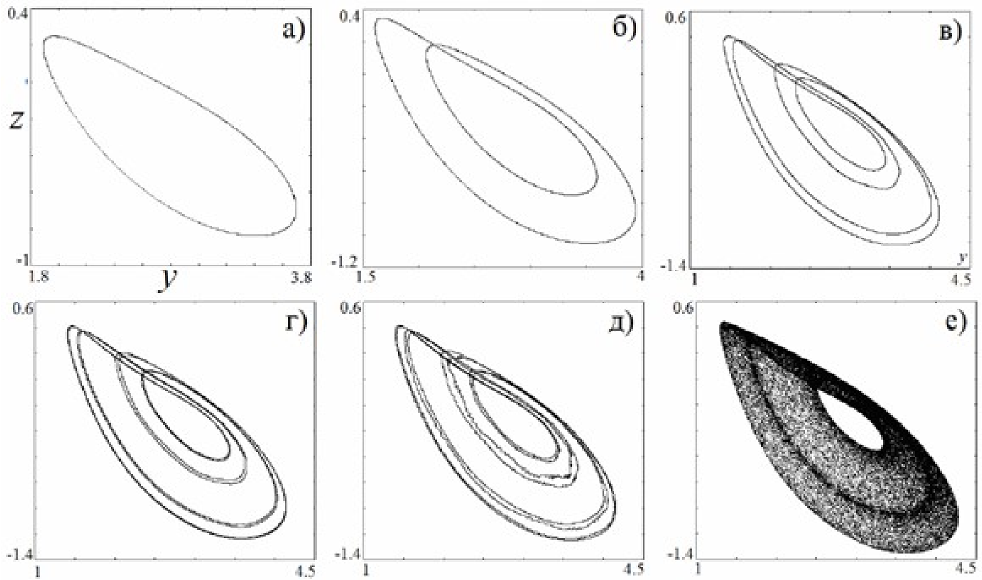
Формирование гиперхаотического множетсва

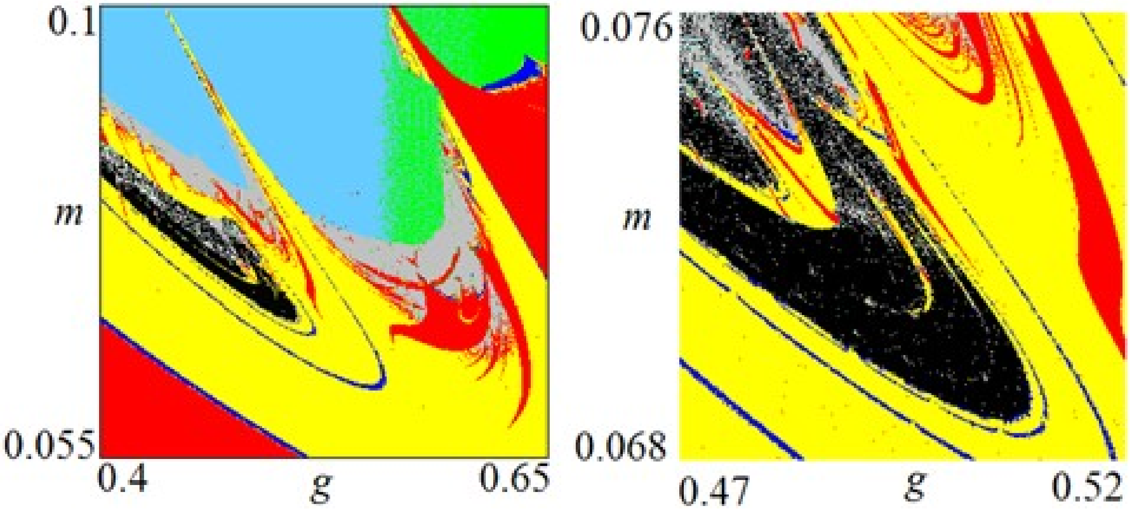
- седло-узловая бифуркация рождения пары седловых циклов (2,1) и (1,2);

- бифуркация удвоения периода седлового цикла (2,1);

- бифуркация рождения тора.

Модифицированный генератор Анищенко-Астахова





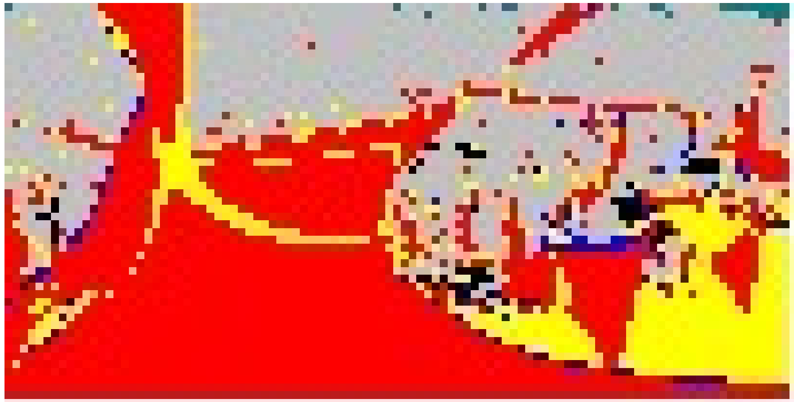
D P T C HC C0

Режим	
бесконечность	D
периодический	P
квазипериодический	T
хаос	C
гиперхаос	HC
хаос с дополнительным нулевым показателем	C0

Неавтономный генератор Анищенко-Астахова

Model

$$\dot{x} = mx + y - xz,$$
$$\dot{y} = -x$$
$$\dot{z} = -gz + l(x)x^2,$$
$$l(x) = \begin{cases} 0, x \le 0, \\ 1, x > 0 \end{cases}$$



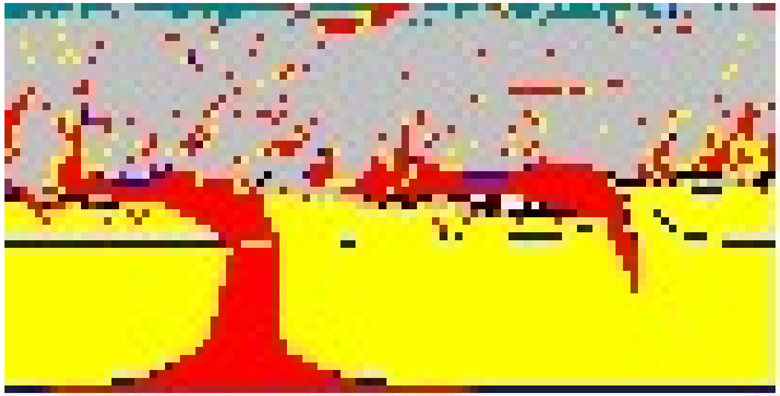


Chart of dynamical regimes