International Online Conference
One-Parameter Semigroups of Operators

BOOK OF ABSTRACTS

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Nizhny Novgorod

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Preface

International online conference "One-Parameter Semigroups of Operators" (OPSO 2023), 27 February – 3 March 2023, is organized by the International laboratory of dynamical systems and applications, and research group Evolution semigroups and their new applications, both located in Russia, Nizhny Novgorod city, and hosted by the National research university Higher School of Economics (HSE).

Website of the Laboratory: https://nnov.hse.ru/en/bipm/dsa/

HSE is a young university (established in 1992) which rapidly become one of the leading Russian universities according to international ratings. In 2023 HSE is a large university focused not only on economics. There are departments of Economics (including Finance, Statistics etc), Law, Mathematics, Computer Science, Media and Design, Physics, Chemistry, Biotechnology, Geography and Geoinformation Technologies, Foreign Languages and some other.

Website of the HSE: https://www.hse.ru/en/

The conference covered the following topics:

1. One-parameter (semi)groups of linear operators, their applications and generalizations;
2. Nonlinear (semi)flows: ergodicity, chaos and other dynamical phenomena;
3. Interplay between linear infinite-dimensional systems and nonlinear finite-dimensional systems;
4. Quantum physics, quantum information and quantum dynamical semigroups;
5. Infinite-dimensional analysis, probability, stochastic processes and financial mathematics;
6. Related topics;

Website of the OPSO 2023 conference: https://nnov.hse.ru/bipm/dsa/opso2023

Organizing Committee: Ivan Remizov (chairman), Oleg Galkin (vice chairman), Ksenia Dragunova, Anna Ivanova, Denis Mineev, Polina Panteleeva, Anna Smirnova, Alexander Vedenin, HSE University (Russia, Nizhny Novgorod City)

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Contents

Section 1. One-parameter (semi)groups of linear operators, their applications and generalizations

Eventual positivity of delay semigroups

P. Anuragi [1], S. Rastogi [2]. S. Srivastava [3]

Keywords: One-parameter semigroups of linear operators; semigroups on Banach lattices; Delay semigroups; eventually positive semigroups; perturbation theory.

MSC2020 codes: 47D06; 47B65; 34G10

Introduction. In a series of papers see [1, 2, 3] Daniel Daners, Jochen Glück and James B. Kennedy initiated the study of eventually positive C_0 semigroups of linear operators on Banach lattices, that is, of semigroups for which, for every positive initial value, the solution of the corresponding Cauchy problem becomes positive for large times.They introduced several notions of eventual positivity such as an individual and a uniform one and also gave characterisations of such semigroups by means of spectral and resolvent properties of the corresponding generators. In the paper [4] Daners and Glück studied the eventual positivity of semigroups under bounded perturbations of the generators and concretely demonstrated that the perturbation theory is much more subtle for eventally positive semigroups than it is for positive semigroups. They demonstrated that, in sharp contrast to the case of positive semigroups, eventual positivity of a semigroup is in general lost, if we perturb its generator by a positive operator of large norm. They also showed that individual eventual positivity is not even stable with respect to small positive perturbations.
We study the eventual positivity of semigroups under unbounded perturbations of generators. We prove the eventual positivity of the perturbed semigroups under the unbounded perturbations of generators of analytic, norm continuous and eventually norm continuous semigroups and apply the same to deduce the eventual positivity of Delay semigroups.

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Local Solutions of Quasilinear Equations with Gerasimov — Caputo Derivatives. Sectorial Case.

K. V. Boyko [4], V. E. Fedorov [5]

Keywords: Gerasimov — Caputo derivative; fractional differential equation; analytic resolving family of operators; multi-term fractional equation; Cauchy problem; initial boundary value problem.

MSC2020 codes: 35R11, 34A08.

Introduction. Over the past few decades, there has been a sharp increase in the interest of researchers in fractional differential equations, primarily due to their increasing importance in modeling various phenomena that arise in physics, chemistry, mathematical biology, and engineering [1,2].

The unique solvability issues for initial problems to some types of equations in Banach spaces with the Gerasimov — Caputo time-fractional derivative were researched in the works [3–7].

In this paper, we study the Cauchy problem $z^{(l)}(t_0) = z_l$, $l = 0, 1, \dots, m-1$, for a differential equation with several fractional derivatives in the linear and nonlinear parts

$$D^{\alpha} z(t) = \sum_{k=1}^{n} D^{\alpha_k} A_k z(t) + B(t, D^{\beta_1} z(t), D^{\beta_2} z(t), \dots, D^{\beta_r} z(t)). \qquad (1)$$

Here D^{α} is the Gerasimov — Caputo derivative of the order $\alpha > 0$, or the Riemann — Liouville integral of the order $-\alpha$ in the case $\alpha \le 0$, $m-1 < \alpha \le m \in \mathbb{N}$, $n, r \in \mathbb{N} \cup \{0\}$, $\alpha_1 < \alpha_2 < \dots < \alpha_n < \alpha$, $\beta_1 < \beta_2 < \dots < \beta_r < \alpha$, Z — Banach space, A_k, $k = 1, 2, \dots, n$, are linear closed operators with domains $D_{A_k} \subset Z$, the non-linear mapping $B : [t_0, T] \times Z^r \to D := \bigcap\limits_{k=1}^{n} D_{A_k}$ is continuous in the norm $\|\cdot\|_D = \|\cdot\|_Z + \sum\limits_{k=1}^{n} \|A_k \cdot\|_Z$. The unique solvability of the Cauchy problem for the linear inhomogeneous equation (1) ($B = f(t)$) in the case when the operators A_k are bounded, $k = 1, 2, \dots, n$, was proved in [7]. In the case when the set of unbounded operators $(A_1, A_2, \dots, A_n)$ belongs to the class $\mathcal{A}^n_{\alpha,G}$, the unique solvability of the Cauchy problem for linear inhomogeneous equation (1) was studied in [8, 9]. Under the condition that the nonlinear operator B is locally Lipschitz, we obtain a theorem on the local unique solvability of the Cauchy problem for quasilinear equation (1). For this, the fixed point theorem in a specially constructed metric space is used.

Local solution. Denote $D := \bigcap_{k=1}^{n} D_{A_k}$, $R_{\lambda} := \left(\lambda^{\alpha} I - \sum\limits_{k=1}^{n} \lambda^{\alpha_k} A_k\right)^{-1} : Z \to D$. We endow the set D with the norm $\|\cdot\|_D = \|\cdot\|_Z + \sum\limits_{k=1}^{n} \|A_k \cdot\|_Z$, with respect to which D is a Banach space, since it is the intersection of the Banach spaces $D_{A_1}, D_{A_2}, \dots, D_{A_n}$ with the corresponding graph norms.

Denote $n_l := \min\{k \in \{1, 2, \dots, n\} : l \le m_k - 1\}$ for $l = 0, 1, \dots, m-1$. If the set $\{k \in \{1, 2, \dots, n\} : l \le m_k - 1\}$ is empty for some $l \in \{0, 1, \dots, m-1\}$ (this holds exactly when $\alpha_n \le m-1$), then we set $n_l := n + 1$.

Definition 1. The set of operators $(A_1, A_2, \dots, A_n)$ belongs to the class $\mathcal{A}^n_{\alpha,G}(\theta_0, a_0)$ for some $\theta_0 \in (\pi/2, \pi)$, $a_0 \ge 0$, if

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(i) D is dense in Z;

(ii) for all $\lambda \in S_{\theta_0,a_0} := \{\mu \in \mathbb{C} : |\arg(\mu - a_0)| < \theta_0,\ \mu \neq a_0\}$, $l = 0,1,\dots,m-1$ we have

$$R_{\lambda,l}\left(\lambda^{\alpha} I - \sum_{k=n_l}^{n} \lambda^{\alpha_k} A_k\right) \in \mathcal{L}(Z);$$

(iii) for any $\theta \in (\pi/2, \theta_0)$, $a > a_0$ there exists $K(\theta, a) > 0$, such that for all $\lambda \in S_{\theta,a}$, $l = 0,1,\dots,m-1$

$$\|R_{\lambda,l}\|_{\mathcal{L}(Z)} \le \frac{K(\theta,a)}{|\lambda - a||\lambda|^{\alpha-1}}, \quad \left\|R_{\lambda,l}\left(\lambda^{\alpha} I - \sum_{k=n_l}^{n} \lambda^{\alpha_k} A_k\right)\right\|_{\mathcal{L}(Z)} \le \frac{K(\theta,a)}{|\lambda - a||\lambda|^{\alpha-1}}.$$

Let $\beta_1 < \beta_2 < \dots < \beta_r < \alpha$, $r_i - 1 < \beta_i \le r_i \in \mathbb{Z}$, $i = 1,2,\dots,r$, U be an open set in $\mathbb{R} \times Z^r$; $B : U \to Z$. Consider the Cauchy problem

$$z^{(l)}(t_0) = z_l, \quad l = 0,1,\dots,m-1, \tag{2}$$

$$D^{\alpha} z(t) = \sum_{k=1}^{n} D^{\alpha_k} A_k z(t) + B(t, D^{\beta_1} z(t), D^{\beta_2} z(t), \dots, D^{\beta_r} z(t)). \tag{3}$$

A solution of problem (2), (3) on a segment $[t_0,t_1]$ is a function $z \in C((t_0,t_1];D) \cap C^{m-1}([t_0,t_1];Z)$ for which $D^{\alpha} z \in C((t_0,t_1];Z)$, $D^{\alpha_k} A_k z \in C((t_0,t_1];Z)$, $k = 1,2,\dots,n$, $D^{\beta_i} z \in C([t_0,t_1];Z)$, $i = 1,2,\dots,r$, the inclusion $(t, D^{\beta_1} z(t), D^{\beta_2} z(t), \dots, D^{\beta_r} z(t)) \in U$ for $t \in [t_0,t_1]$ and the equality (2) for all $t \in (t_0,t_1]$, as well as the conditions (3) are satisfied.

Denote $x := (x_1, x_2, \dots, x_r) \in Z^r$, $S_\delta(x) = \{y \in Z^r : \|y_l - x_l\|_Z \le \delta, l = 1,2,\dots,r\}$. A mapping $B : U \to Z$ is called locally Lipschitz in x, if for any $(t,x) \in U$ there exist $\delta > 0, q > 0$ such that $[t-\delta, t+\delta] \times S_\delta(x) \subset U$ and for any $(s,y),(s,v) \in [t-\delta, t+\delta] \times S_\delta(x)$ the inequality $\|B(s,y) - B(s,v)\|_Z \le q \sum_{i=1}^{r} \|y_i - v_i\|_Z$ holds.

Using the initial data $z_0, z_1, \dots, z_{m-1}$, we define the polynomial

$$\tilde{z}(t) = z_0 + (t - t_0) z_1 + \frac{(t-t_0)^2}{2!} z_2 + \dots + \frac{(t-t_0)^{m-1}}{(m-1)!} z_{m-1}$$

and vectors $\tilde{z}_i = D^{\beta_i}|_{t=t_0} \tilde{z}(t)$, $i = 1,2,\dots,r$. Note that $\tilde{z}_i = 0$ if $\beta_i \notin \{0,1,\dots,m-1\}$. In the case $\beta_i \in \{0,1,\dots,m-1\}$ we have $\tilde{z}_i = z_{\beta_i}$. Thus, the value of the argument of the nonlinear operator B at the initial moment of time is $(t_0, \tilde{z}_1, \tilde{z}_2, \dots, \tilde{z}_r)$.

Lemma 1. [10]. Let $l - 1 < \beta \le l \in \mathbb{N}$. Then

$$\exists C > 0 \quad \forall h \in C^l([t_0,t_1];Z) \quad \|D_t^{\beta} h\|_{C([t_0,t_1];Z)} \le C \|h\|_{C^l([t_0,t_1];Z)}.$$

Lemma 2. Let $\alpha_1 < \alpha_2 < \dots < \alpha_n < \alpha$, $\beta_1 < \dots < \beta_r < \alpha$; $m - 1 < \alpha \le m \in \mathbb{N}$; $(A_1, A_2, \dots, A_n) \in \mathcal{A}^n_{\alpha,G}(\theta_0, a_0)$ for some $\theta_0 \in (\pi/2, \pi)$, $a_0 \ge 0$, $z_l \in D$, $l = 0,1,\dots,m-1$, U be an open set in $\mathbb{R} \times Z^r$, $B \in C(U;D)$, $(t_0, \tilde{z}_1, \tilde{z}_2, \dots, \tilde{z}_r) \in U$. Then the function z is a solution to the problem (2), (3) on the segment $[t_0,t_1]$ if and only if $z \in C^{m-1}([t_0,t_1];Z)$, $D^{\beta_i} z \in C([t_0,t_1];Z)$, $i = 1,2,\dots,r$, and for all $t \in [t_0,t_1]$ the inclusion $(t, D^{\beta_1} z(t), D^{\beta_2} z(t), \dots, D^{\beta_r} z(t)) \in U$ and equality

$$z(t) = \sum_{l=0}^{m-1} Z_l(t - t_0) z_l + \int_{t_0}^{t} Z(t-s) B(s, D^{\beta_1} z(s), D^{\beta_2} z(s), \dots, D^{\beta_r} z(s))\, ds \tag{4}$$

are valid.

Denote $i^* := \min\{i \in \{1,2,\dots,r\} : \beta_i > m-1\}$ if the set $\{i \in \{1,2,\dots,r\} : \beta_i > m-1\}$ is not empty, otherwise $i^* := r+1$. For $t_1 > t_0$ we define the space $C^{m-1,\{\beta_i\}}([t_0,t_1];Z) := \{z \in C^{m-1}([t_0,t_1];Z) : D^{\beta_i} z \in C([t_0,t_1];Z),\ i = i^*, i^*+1,\dots,r\}$ and equip this space with the norm

$$\|z\|_{C^{m-1,\{\beta_i\}}([t_0,t_1];Z)} = \|z\|_{C^{m-1}([t_0,t_1];Z)} + \sum_{i=i^*}^{r} \|D^{\beta_i} z\|_{C([t_0,t_1];Z)}.$$

Remark 1. For the function $z \in C^{m-1}([t_0,t_1];Z)$, by Lemma 1 $D^{\beta_i} z \in C([t_0,t_1];Z)$, $i = 1,2,\dots,i^*-1$. Therefore, functions from $C^{m-1}([t_0,t_1];Z)$ for which $D^{\beta_i} z \in C([t_0,t_1];Z)$, $i = 1,2,\dots,r$ referred to in the Lemma 2 are exactly functions from $C^{m-1,\{\beta_i\}}([t_0,t_1];Z)$.

Lemma 3. $C^{m-1,\{\beta_i\}}([t_0,t_1];Z)$ is a Banach space.

Theorem 1. Let $\alpha_1 < \alpha_2 < \dots < \alpha_n < \alpha$, $\beta_1 < \dots < \beta_r < \alpha$; $m-1 < \alpha \le m \in \mathbb{N}$; $(A_1,A_2,\dots,A_n) \in \mathcal{A}^n_{\alpha,G}(\theta_0,a_0)$ for some $\theta_0 \in (\pi/2,\pi)$, $a_0 \ge 0$, $z_l \in D$, $l = 0,1,\dots,m-1$, U be an open set in $\mathbb{R} \times Z^r$, $B \in C(U;D)$ be locally Lipschitz in x, $(t_0,\tilde z_1,\tilde z_2,\dots,\tilde z_r) \in U$. Then for some $t_1 > t_0$ problem (2), (3) has a unique solution on the interval $[t_0,t_1]$.

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On Trotter–Kato type inductive limits in the category of C_0-semigroups

C. Budde [6]

Keywords: C_0-semigroups; Banach inductive limits;Trotter–Kato conditions.

MSC2020 codes: 47D06, 46M10, 46M15, 46M40

We will show that the category of C_0-semigroups possesses inductive limits under certain Trotter–Kato type conditions. Recently, the theory of C_0-semigroups firstly has been approached by A. Ng by means of category theory [2]. We want to jump on the bandwagon and continue the study of this approach. In particular, we want to study a specific construction the category theory of C_0-semigroups, the so-called inductive limits. We will, see that the typical Trotter–Kato approximation conditions appear naturally when constructing the desired limit. Evolution equations in their own right in connection with category already appeared earlier in the work of Liu [1]. Category theory in the framework of functional analysis appears in different areas and also with different perspectives, see for example [2,4].

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A class of fractional quasilinear equations in the sectorial case

V. E. Fedorov [7]

Keywords: Riemann — Liouville fractional derivative, Riemann — Liouville fractional integral, quasilinear equation, Cauchy type problem, defect of Cauchy type problem, analytic resolving operators family, complex power of operator.

MSC2020 codes: 34G20,34R11,47D99,34A08

Introduction. In the operator semigroup theory [1] the introduction of fractional powers A for a continuously invertible generator A of an analytic resolving semigroup and of spaces Z as the domains of A with the graph norm allows to study the solvability issues of partial differential equations with nonlinearity, which depends on lower order derivatives with respect to spatial variables. In this work we consider complex powers of an operator A, such that A generates an analytic resolving family of operators of a fractional order equation $D z(t) + Az(t) = 0$, and use them for a quasilinear equation

$$D z(t) + Az(t) = B\left(D^{1} z(t); D^{2} z(t); \dots; D^{n} z(t); D^{m\ r} z(t); \dots; D^{1} z(t)\right);$$

where $m\ 1 < \ m \in \mathbb{N}$, $r \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, $n \in \mathbb{N}$, ${}_1 < {}_2 < \dots < {}_n < 1$, $m_k\ 1 < {}_k\ m_k \in \mathbb{Z}$, ${}_k\ m_k \neq m$, $k = 1; 2; \dots; n$, D is the fractional Riemann — Liouville derivative of an order > 0, or the fractional Riemann — Liouville integral of an order , if 0, operator B is locally Lipschitzian with respect to the norm in Z, $\in (0; 1)$. Abstract result we apply to the study of an initial boundary value problem with a nonlinear part, containing partial derivatives in spatial variables in the nonlinear part.

Fractional sectorial operators and their complex powers. Denote by (A) the resolvent set of an operator A, $R(A) := (I\ A)^{1}$, $S_{0;a_0} := \{ \in \mathbb{C} : |\arg(\ a_0)| < {}_0; \neq a_0\}$; $:= \{ \in \mathbb{C} : |\arg| < '; \neq 0\}$.

Let ${}_0 \in (=2;)$, $a_0\ 0$, denote by $A(_0; a_0)$ a class of linear, closed and densely defined in Z operators A, acting into Z, such that the following conditions are satisfied [2]:

(i) for every $\in S_{0;a_0}$ the inclusion $\in (A)$ is valid;

(ii) for any $\in (=2; {}_0)$, $a\ a_0$ there exists $K = K(; a) > 0$, such that

$$\forall \in S_{;a}\ \|R(A)\|_{L(Z)} \ \frac{K(;a)}{|^{1}(\ a)|}:$$

If > 0, $A \in A(_0; a_0)$, $\in \mathbb{R}$, then the operators

$$Z(t) := \frac{1}{2i}\int^{1+} R(A)e^{t} d; \quad t \in \mathbb{R}_+;$$

are defined and analytically extendable on ${}_0\ {}_{=2}$, where $:= {}_+ \cup \cup {}_0$, $:= \{ \in \mathbb{C} : = a + re^{i}; r \in (; 1)\}$, ${}_0 := \{ \in \mathbb{C} : = a + e^{i'}; ' \in (;)\}$ for > 0, $a > a_0$, $\in (=2; {}_0)$ (see [3]). The estimates

$$\|Z(t)\|_{L(Z)}\ C(;a)e^{at}(t^{1} + a); \quad t > 0; \quad 0;$$

$$\|Z(t)\|_{L(Z)}\ C(;a)e^{at}t; \quad t > 0; \quad < 0;$$

hold for every $a > a_0$ [3].

Theorem 1. Let > 0, $A \in A(_0; a_0)$. Then for all < 1, < 1, $t; s > 0$

$$Z(s)Z(t) = \frac{1}{}Z_{+}(s+t) + \frac{t}{2i}\int^{1+} R(A)E_{;1}(\ t)e^{s}d +$$

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$$+\frac{s}{2 i}\int \quad ^{1+} R \; (\; A)E_{\;;1} \; (\; s \;)e^{\;t} d \; .$$

It is known that for $=1$ $\{Z_0(t) \in L(Z) : t \in \mathbb{R}_+\}$ is an analytic semigroup of operators [1]. Take in Theorem 1 $=1$, $=\;=0$ and obtain the semigroup property $Z_0(t)Z_0(s) = Z_0(t+s)$, $t, s > 0$. Thus, Theorem 1 gives some generalization of the semigroup property for resolving families of operators, which are generated by an operator from the class $A\;(\;_0, a_0)$.

As in [1] complex powers A , $\in \mathbb{C}$, of such A can be defined.

Theorem 2. Let >0, $A \in A\;(\;_0, 0)$, $0 \in \;(A)$. Then

(i) for $\in \mathbb{C}$ A is a closed operator;

(ii) if $\mathrm{Re} \; > \mathrm{Re} \quad 0$, then $D_A \quad D_A$;

(iii) $\overline{D_A} = Z$ for every $\mathrm{Re} \quad 0$;

(iv) if $\;,\; \in \mathbb{C}$, then $A^{\;+}z = A\;A\;z$ for every $z \in D_A \cap D_A \cap D_{A^{\;+}}$;

(v) $Z\;(t) : Z \to D(A\;)$ for all $\in \mathbb{R}$, $\mathrm{Re} \in [0, 1)$, $t > 0$;

(vi) $Z\;(t)A\;z = A\;Z\;(t)z$ for $\in \mathbb{R}$, $\in \mathbb{C}$, $z \in D(A\;)$;

(vii) for $\in \mathbb{R}$, $\mathrm{Re} \; < 1$, $t > 0$ the operator $A\;Z\;(t)$ is bounded;

(viii) for <1, $\mathrm{Re} \in (0, 1)$

$$A \quad = \frac{\sin}{\sin(\;(\;+\;))\;(\;+\;)}\int_0^{\infty} t^{\;+\;1} Z\;(t)dt;$$

(ix) for $\in \mathbb{R}$, $t > 0$ $\|AZ\;(t)\|_{L(Z)} \quad Ct$;

(x) for $\in (\;\mathrm{Re}\;, 1)$, $\mathrm{Re} \in (0, 1)$, $t > 0$ $\|A\;Z\;(t)\|_{L(Z)} \quad C\;t^{\;\mathrm{Re}}$;

(xi) for <1, $\mathrm{Re} \in (0, 1)$, $z \in D(A\;)$

$$\|D\;Z\;(t)z \quad z\|_Z \quad C\;t^{\;\mathrm{Re}}\|A\;z\|_Z.$$

Incomplete Cauchy type problem for a quasilinear equation. Consider a quasilinear equation

$$D\;z(t) + Az(t) = B\left(D^{\;1}z(t), D^{\;2}z(t), \ldots, D^{\;n}z(t), D^{\;m\;r}z(t), \ldots, D^{\;1}z(t)\right), \qquad (1)$$

where $m \; 1 < \quad m \in \mathbb{N}$, $r \in \mathbb{N}_0$, $n \in \mathbb{N}$, $_1 < \;_2 < \; < \;_n < \; 1$, $m_k \; 1 < \;_k \; m_k \in \mathbb{Z}$, $_k \; m_k \neq \; m$, $k = 1, 2, \ldots, n$. Some of $_k$ may be negative. As in [4] denote $\underline{\;} := \max\{\;_k : \;_k \; m_k < \; m, k = 1, 2, \ldots, n\}$, $\underline{m} := \lceil \underline{\;} \rceil$, $\overline{\;} := \max\{\;_k : \;_k \; m_k > \; m, k = 1, 2, \ldots, n\}$, $\overline{m} := \lceil \overline{\;} \rceil$, $m \; := \max\{\underline{m} \; 1, \overline{m}\}$, $m \; := \max\{m \; + 1, 0\}$. For the study of an initial problem to (1) we need the existence of finite limits $\lim_{t \to t_0} D^{\;l}z(t) := D^{\;l}z(t_0)$, $l = 1, 2, \ldots, n$, therefore, as it follows from results of [4], problem

$$D^{\;m+k}z(t_0) = z_k, \quad k = m\;, m\; + 1, \ldots, m \; 1, \qquad (2)$$

will be considered with the necessary condition $D^{\;m+k}z(t_0) = 0$, $k = 0, 1, \ldots, m\;$. Since $_n < \; 1$, we have $m \; m \; 2$, $m \; m \; 1$, therefore, (2) contains one condition at least.

Let $\in (0, 1)$, $Z \; := D_A$ is a Banach space with the norm $\|\cdot\| := \|A\;\cdot\|_Z$, since A is a continuously invertible closed operator. Let U be an open subset of $\mathbb{R} \times Z^{n+m+r}$, a mapping $B : U \to Z$ is given, for every $(t, x_1, x_2, \ldots, x_{n+m+r}) \in U$ there exists a neighbourhood $V \subset U$, $C > 0$, $\in (0, 1]$ such that for all $(t, y_1, y_2, \ldots, y_{n+m+r}), (s, v_1, v_2, \ldots, v_{n+m+r}) \in V$

$$\|B(t, y_1, y_2, \ldots, y_{n+m+r}) \; B(s, v_1, v_2, \ldots, v_{n+m+r})\|_Z \quad C\left(|t \; s| + \sum_{k=1}^{n+m+r}\|y_k \; v_k\|\right). \qquad (3)$$

A function $z \in C((t_0,t_1];D_A)$, such that $J^{\ \ m}z \in C^{m\ 1}([t_0,t_1];Z) \cap C^m((t_0,t_1];Z)$, $D^{\ 1}z$, $D^{\ 2}z,\ldots,D^{\ n}z \in C([t_0,t_1];Z)$, is called a solution of Cauchy type problem (1), (2) on a segment $[t_0,t_1]$, if it satisfies conditions (2), for all $t \in (t_0,t_1]$ $(D^{\ 1}z(t),D^{\ 2}z(t),\ldots,D^{\ \ 1}z(t)) \in U$ and (1) holds.

Theorem 3. Let $\ > 0$, $\ _1 < \ _2 < \ \ < \ _n < \ \ \ 1$, $A \in \mathcal{A}\ (\ _0,0)$, $0 \in \ (A)$, a map $B : U \to Z$ satisfy condition (3), $\ > 1\ \ 1/\ $, $(t_0,0,\ldots,0,z_m\ ,z_{m\ +1},\ldots,z_{m\ 1}) \in U$, $z_k \in Z_{1+}$, $k = m\ ,m\ +1,\ldots,m\ \ 1$. Then for some $t_1 > t_0$ there exists a unique solution of problem (1), (2) on $[t_0,t_1]$.

Application. Let $\ \ \mathbb{R}^3$ be a bounded region with a smooth boundary $\partial\ $, $\ \in (1,2)$, then $m\ \ = 0$, or $m\ \ = 1$. Consider the initial boundary value problem

$$D\ \ ^{m+k}v(\ ,t_0) = v_k(\ ),\quad k = m\ ,1,\quad \ \in\ ,\tag{4}$$

$$v(\ ,t) = 0,\quad \ \in \partial\ ,\ t > t_0,\tag{5}$$

for an equation

$$D_t\, v(\ ,t) = \ \ v(\ ,t) + \sum_{l=1}^{n} D_t^{\ l}v(\ ,t)\sum_{i=1}^{3}\frac{\partial}{\partial\ _i}D_t^{\ l}v(\ ,t)+$$

$$+\sum_{k=\ r}^{m\ \ 1} D_t^{\ \ m+k}v(\ ,t)\sum_{i=1}^{3}\frac{\partial}{\partial\ _i}D_t^{\ \ m+k}v(\ ,t),\quad \ \in\ ,\ t > t_0,\tag{6}$$

where $D_t\, v$ are the partial fractional derivatives for $\ > 0$ or integrals for $\ \ \ 0$ with respect to t. Take $Z = L_2(\ )$, $A = \ \ $, $D_A = H^2(\ ) \cap H_0^1(\ )$, then $A \in \mathcal{A}\ (\ _0,0)$ for any $\ _0 \in (\ /2,\ )$, since $\ \in (1,2)$ (see Theorem 4 in [5] for $n = 0$, $P_0\ \ \ 1$, $p = 1$, $Q_1(\ ) = \ \ $). Reasoning as in [1, §8.8.3], obtain that the nonlinear operators of the form $f_l(v) = D_t^{\ l}v\sum_{i=1}^{3}\frac{\partial}{\partial\ _i}D_t^{\ l}v$ satisfy the conditions of Theorem 3 at $\ > 3/4$. Therefore, for all $v_k \in D_{A^{1+}}$, $k = m\ ,1$, there exists a unique solution of problem (4)–(6) in $\ \ [t_0,t_1]$ with some $t_1 > t_0$.

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Unique solvability of equations with a distributed fractional derivative given by the Stieltjes Integral

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Keywords: distributed fractional derivative; fractional differential equation; inhomogeneous equation; Cauchy problem

MSC2020 codes: 34G10,35R11,34A08,47D99

Let $\mathcal{L}(\mathcal{Z})$ be the Banach space of all linear continuous operators on a Banach space $\mathcal{Z}$, denote by $\mathcal{Cl}(\mathcal{Z})$ the set of all linear closed operators, densely defined in $\mathcal{Z}$, acting in the space $\mathcal{Z}$. Introduce the notations $S_{\theta,a} := \{\mu \in \mathbb{C} : |\arg(\mu - a)| < \theta, \mu \neq a\}$ for $\theta \in [\pi/2, \pi]$, $a \in \mathbb{R}$.

Let $b, c \in \mathbb{R}$, $b < c$, $\omega : [b, c] \to \mathbb{C}$ is a function with a bounded variation. Introduce the notations of the complex-valued function $W(\lambda) := \int\limits_b^c \lambda^\alpha d\omega(\alpha)$. Here the integral is understood in the sense of Riemann — Stieltjes.

We define a class $\mathcal{A}_W(\theta_0, a_0)$ as the set of all operators $A \in \mathcal{Cl}(\mathcal{Z})$ satisfying the following conditions:

(i) there exist $\theta_0 \in (\pi/2, \pi]$, $a_0 \geq 0$, such that $W(\lambda) \in \rho(A)$ for every $\lambda \in S_{\theta_0,a_0}$;

(ii) for every $\theta \in (\pi/2, \theta_0)$, $a > a_0$ there exists $K(\theta, a) > 0$, such that for all $\lambda \in S_{\theta,a}$

$$\|(W(\lambda)I - A)^{-1}\|_{\mathcal{L}(\mathcal{Z})} \leq \frac{|\lambda| K(\theta, a)}{|W(\lambda)||\lambda - a|}.$$

Theorem 1. [1]. Let $b, c \in \mathbb{R}$, $b < c$, $m - 1 < c \leq m \in \mathbb{N}$, $\omega : [b, c] \to \mathbb{C}$ is a function with a bounded variation, c be a variation point of the measure $d\omega(t)$, $\theta_0 \in (\pi/2, \pi]$, $a_0 \geq 0$, $A \in \mathcal{A}_W(\theta_0, a_0)$, $g \in C([0, T]; D_A) \cup C^\gamma([0, T]; \mathcal{Z})$, $\gamma \in (0, 1]$, $z_k \in D_A$, $k = 0, 1, \ldots, m - 1$. Then there exists a unique solution of problem

$$z^{(k)}(0) = z_k, \quad k = 0, 1, \ldots, m - 1, \tag{1}$$

for the inhomogeneous equation

$$\int\limits_b^c D^\alpha z(t) d\omega(\alpha) = Az(t) + g(t), \quad t \in (0, T]. \tag{2}$$

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Error of Chernoff approximations based on Chernoff function with a given coefficient at t^2

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Keywords: Chernoff product formula, approxition of C_0-semigroup, speed of convergence

MSC2010 codes: 47D03, 47D06, 35A35, 41A25

This talk is devoted to the error of Chernoff approximations [1,2,3] to strongly continuous one-parameter semigroups [4, 5] in the case when Chernoff function has a coefficient at t^2 which is known.

Let $(X, \|\cdot\|)$ be any Banach space and $\mathscr{L}(X)$ denotes the set of all bounded linear operators on X.

Definition 1 (see, for example, Engel, Nagel [5]). The family $\{G(t)\}_{t\geqslant 0}$ of bounded linear operators on the Banach space X is called the *strongly continuous (one-parameter) semigroup (and also the C_0–semigroup)*, if it is strongly continuous, $G(0) = I$ and for all $t, s \geqslant 0$ the equality $G(t+s) = G(t)G(s)$ is true.

Definition 2 (see, for example: Engel, Nagel [5]). *Generator of a strongly continuous semigroup $\{G(t)\}_{t\geqslant 0}$ on the Banach space X* is the operator $A\colon D(A) \to X$, defined by the equality $Ax = \lim\limits_{t\to +0}(G(t)x - x)/t$ for all x from the domain $D(A)$, where

$$D(A) = \{\, x \in X \mid \lim_{t\to +0}(G(t)x - x)/t \text{ exists} \}.$$

In 1968 Paul Chernoff proved the following theorem:

Theorem 1 (Chernoff [6]). Let X be a Banach space, $F(t)$ be a strongly continuous function from $[0, \infty)$ to a subset of the compressing operators from $\mathscr{L}(X)$, with $F(0) = I$. Suppose that the closure A of the strong derivative $F'(0)$ is the generator of the contracting C_0–semigroup $\{e^{tA}\}_{t\geqslant 0}$. Then $[F(t/n)]^n$ converges to e^{tA} in a strong operator topology.

Let us note that this theorem does not contain an estimate of the rate of convergence, that is, an estimate of the form

$$\|[F(t/n)]^n x - e^{tA}x\| \leqslant C(t, x, n) \to 0 \quad (n \to \infty).$$

In 2022 was published the theorem that provides such estimate under certain conditions:

Theorem 2 (Galkin, Remizov [3]). Suppose that:

1) $T > 0$, $M_1 \geqslant 1$, $w \geqslant 0$. $(A, D(A))$ is generator of C_0-semigroup $(e^{tA})_{t\geqslant 0}$ in a Banach space X, such that $\|e^{tA}\| \leqslant M_1 e^{wt}$ for $t \in [0, T]$.

2) There are a mapping $F\colon (0, T] \to \mathscr{L}(X)$ and constant $M_2 \geqslant 1$ such that we have $\|(F(t))^k\| \leqslant M_2 e^{kwt}$ for all $t \in (0, T]$ and all $k \in \mathbb{N} = \{1, 2, 3, \ldots\}$.

3) $m \in \mathbb{N} \cup \{0\}$, $p \in \mathbb{N}$, subspace $\mathcal{D} \subset D(A^{m+p})$ is $(e^{tA})_{t\geqslant 0}$-invariant.

4) There exist such functions $K_j\colon (0, T] \to [0, +\infty)$, $j = 0, 1, \ldots, m+p$ that for all $t \in (0, T]$ and all $f \in \mathcal{D}$ we have

$$\left\| F(t)f - \sum_{k=0}^{m} \frac{t^k A^k f}{k!} \right\| \leqslant t^{m+1} \sum_{j=0}^{m+p} K_j(t)\|A^j f\|.$$

Then: for all $t > 0$, all integer $n \geqslant t/T$ and all $f \in \mathcal{D}$ we have

$$\|(F(t/n))^n f - e^{tA} f\| \leqslant \frac{M_1 M_2 t^{m+1} e^{wt}}{n^m} \sum_{j=0}^{m+p} C_j(t/n)\|A^j f\|,$$

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$C_{m+1}(t) = K_{m+1}(t)e^{-wt} + M_1/(m+1)!$, $C_j(t) = K_j(t)e^{-wt}$ $(j \neq m+1)$.

Let us consider particular example. *Example 1.* Suppose that $\|e^{tA}\| \le M_1 e^{wt}$, $\|F(t)\| \le M_2 e^{wt}$, where $w \ge 0$,

$$\|F(t)x - x - tAx\| \le K_2 t^2 \|A^2 x\|$$

for all $x \in D(A^2)$ and $t \in (0,1]$. Then $m = 1$, $K_0(t) = K_1(t) = 0$ for any $t \in (0,1]$. So theorem 2 states that for any fixed $t > 0$, all $x \in D(A^2)$ and all integer $n \ge t$ the following estimate is true, having the following asymptotic behaviour as $n \to \infty$:

$$\|(F(t/n))^n x - e^{tA}x\| \le \frac{M_1 M_2 t^2 e^{wt}}{n}\left(K_2 e^{-wt/n} + \frac{M_1}{2}\right)\|A^2 x\| \le M_1 M_2 (K_2 + M_1/2)\frac{t^2 e^{wt}}{n}\|A^2 x\|.$$

So the question arises: what is the lower estimate of the error $\|(F(t/n))^n x - e^{tA}x\|$?

In 2018, Ivan Remizov formulated the following conjecture:

Conjecture 1 (Remizov [7]). Let $(e^{tA})_{t \ge 0}$ be a C_0-semigroup in a Banach space X, and F is a Chernoff function for operator A (recall that this implies $F(0) = I$ and $F'(0) = A$ but says nothing about $F''(0)$) and number $T > 0$ is fixed. Suppose that vector x is from intersection of domains of operators $F'(t)$, $F''(t)$, $F'''(t)$, $F''''(t)$, $F'(t)F''(t)$, $(F'(t))^2 F''(t)$, $(F''(t))^2$ for each $t \in [0,T]$, and suppose that if $Z(t)$ is any of these operators then function $t \to Z(t)x$ is continuous for each $t \in [0,T]$. Then there exists such a number $C_x > 0$, that for each $t \in [0,T)$ and each $n \in \mathbb{N}$ the following inequality holds, where $B = F''(0)$:

$$\left\|(F(t/n))^n x - e^{tA}x - \frac{t^2}{2n}e^{tA}(B - A^2)x\right\| \leqslant \frac{C_x}{n^2}.$$

Unfortunately, this hypothesis can only be true if the operators A and B commute. We prove the following theorem:

Theorem 3. Suppose that:

1) C_0-semigroup $(e^{tA})_{t>0}$ in a Banach space X has bounded generator $A \in \mathcal{L}(X)$.

2) $T > 0$ and there are a mapping $F\colon [0,T] \to \mathcal{L}(X)$ and constants $M \ge 1$, $w \ge 0$ such that $\|(F(t))^k\| \leqslant M e^{kwt}$ for all $t \in [0,T]$, $k \in \mathbb{N}$.

3) There exist such bounded operator $B \in \mathcal{L}(X)$ and constant $K \ge 0$ that for all $t \in [0,T]$ we have

$$\left\|F(t) - I - tA - \frac{t^2}{2}B\right\| \leqslant K t^3.$$

Then: there exists such a number $C > 0$, that for each $t \in [0,T]$ and each $n \in \mathbb{N}$ the following inequality holds:

$$\left\|(F(t/n))^n - e^{tA} - \frac{t^2}{2n}\int_0^1 e^{tsA}(B - A^2)e^{t(1-s)A}ds\right\| \le \frac{C}{n^2}.$$

If A and B commute then:

$$\left\|(F(t/n))^n - e^{tA} - \frac{t^2}{2n}e^{tA}(B - A^2)\right\| \le \frac{C}{n^2}.$$

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Local heat kernel: construction and properties

A. V. Ivanov [12]

Keywords: local heat kernel; Riemannian manifold; Seeley–DeWitt coe□cient.

MSC2020 codes: 35K08, 47B93, 53B20, 81T13, 81T20

Heat kernels play crucial roles [1–3] in modern theoretical physics and mathematics, for example, in the Atiyah–Patodi–Singer theorem or in the renormalization of quantum field models. Their explicit construction is possible only in some special cases, so investigation of asymptotic expansions is an important task.

In this talk I am going to discuss a local heat kernel [4] on a smooth Riemannian manifold M, which actually is the main part of the standard heat kernel. In our case the locality means that we work in some smooth open convex domain U M, and the new object does not depend on information from M n U and any boundary conditions.

This presentation contains a definition of the local heat kernel, its asymptotic expansion, properties of the Seeley–DeWitt coefficients, construction of some useful special functions, and discussion of open questions.

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On the stability of solutions to the stochastic Ho equation

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Keywords: the Nelson-Gliklikh derivative, stochastic Sobolev-type equations, invariant manifolds.

MSC2020 codes: 35S10, 60G99

The Hoff equation

$$(\quad + \quad)\underline{u} = \quad u + \quad u^3 \tag{3}$$

is a model of buckling of an I-beam from the equilibrium position. Consider the stochastic analogue of the equation (3). The operators L, M and N are defined by formulas

$$L: \quad ! \ (\quad + \quad) \ ; \quad 2\ \mathbf{U}_{W\mathbf{K}}\mathbf{L}_2; \ M: \quad ! \qquad ; \ N: \quad ! \qquad ^3; \quad 2\ \mathbf{U}_{\mathbf{K}}\mathbf{L}_2: \tag{4}$$

Then the stochastic analogue of the Hoff equation (3) is represented as an equation

$$L\ \overset{o}{\ } = M \quad + N(\ ): \tag{5}$$

This work is a continuation [1], [2] on the study of local stability of a semilinear stochastic equation.

Theorem 1. Let $\ ; \ ; \quad 2\ \mathrm{R}_+:$

(i) If $\qquad\quad {}_1$ then the equation (5) has only a stable invariant manifold that coincides with $\mathbf{M}_{\mathbf{K}}\mathbf{L}_2$;

(ii) If $\quad {}_1 < \quad$ then there are a finite-dimensional unstable invariant the manifold $\mathbf{M}^{+}_{\mathbf{K}}\mathbf{L}_2$ and an infinite-dimensional stable invariant manifold $\mathbf{M}_{\mathbf{K}}\mathbf{L}_2$ of the equation (5) in the neighborhood of point zero.

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A unified approach to degenerate problems in the half-space

L. Negro [15], G. Metafune [16], C. Spina [17]

Keywords: degenerate elliptic operators, boundary degeneracy, vector-valued harmonic analysis, maximal regularity.

MSC2020 codes: 35K67, 35B45, 47D07, 35J70, 35J75

Abstract. We study elliptic and parabolic problems governed by the singular elliptic operators

$$L = y^{\alpha_1}\Delta_x + y^{\alpha_2}\left(D_{yy} + \frac{c}{y}D_y - \frac{b}{y^2}\right), \qquad \alpha_1, \alpha_2 \in \mathbb{R}$$

in the half-space $\mathbb{R}^{N+1}_+ = \{(x,y) : x \in \mathbb{R}^N, y > 0\}$. We prove elliptic and parabolic L^p-estimates and solvability for the associated problems. In the language of semigroup theory, we prove that L generates an analytic semigroup, characterize its domain as a weighted Sobolev space and show that it has maximal regularity.

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Improved L^2-approximations in homogenization of parabolic equations with account of correctors

S. E. Pastukhova [18]

Keywords: homogenization; error estimates; correctors.

MSC2020 codes: 35K15

We consider the Cauchy problem for a second order parabolic equation

$$L_\varepsilon u^\varepsilon = f \text{ in } \mathbb{R}^d \times (0,T), \quad u^\varepsilon(x,0) = 0 \text{ for } x \in \mathbb{R}^d.$$

Here, $L_\varepsilon = \partial_t - div_x\, a(x/\varepsilon)\nabla_x$, the measurable real-valued coefficient matrix $a(x/\varepsilon)$ is ε-periodic and is not necessarily symmetric; ε is a small positive parameter tending to zero; and the right-hand side function f belongs to $L^2(\mathbb{R}^d \times (0,T))$. We find approximations for the solution u^ε in the norm $\|\cdot\|_{L^2(\mathbb{R}^d \times (0,T))}$ with the remainder term of order ε^2. These approximations are of the form

$$u^\varepsilon(x,t) = u(x,t) + \varepsilon U(x, x/\varepsilon, t) + O(\varepsilon^2),$$

where the main term $u(x,t)$ is the solution to the well-known homogenized problem

$$L_0 u = f \text{ in } \mathbb{R}^d \times (0,T), \quad u(x,0) = 0 \text{ for } x \in \mathbb{R}^d,$$

with the parabolic operator $L_0 = \partial_t - div_x\, a^0 \nabla_x$ having the constant coefficient matrix a^0 defined via solutions to the auxiliary problems on the periodicity cell which is the unit cube in $\mathbb{R}^d$; the corrector $U(x, x/\varepsilon, t)$, generally, has the three-part structure, that is, $U(x, x/\varepsilon, t) = U_1(x, x/\varepsilon, t) + U_2(x, x/\varepsilon, t) + U_3(x,t)$, and the each part of it is defined with the help of the solutions to the aforementioned cell problems. In the selfadjoint case, the corrector U becomes simpler, because its third term vanishes: $U_3 = 0$.

The above asymptotic for the solution $u^\varepsilon(x,t)$ admits the operator formulation in terms of the resolvent operators L_ε^{-1}, L_0^{-1} and the corresponding correcting operator, which can be restored in accordance with the above corrector $U(x, x/\varepsilon, t)$. Namely,

$$\|L_\varepsilon^{-1} f - L_0^{-1} f - \varepsilon K_\varepsilon f\|_{L^2(\mathbb{R}^d \times (0,T))} \le C\varepsilon^2 \|f\|_{L^2(\mathbb{R}^d \times (0,T))},$$

where the constant C depends only on the dimension d, the ellipticity constants of the matrix $a(\cdot)$ and the value T.

To obtain these results, we use the so-called shift method proposed firstly in [1,2] and applied to the parabolic homogenization earlier in [3,4].

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The order of convergence for fractional equations

S. I. Piskarev [19]

Keywords: Cauchy problem; Caputo derivative; Banach space; finite difference scheme; stability; accuracy estimate; graded mesh; full discretization.

MSC2020 codes: 65N06

In this talk we study a well–posed Cauchy problem with a fractional Caputo derivative of the order $2\ (0;1)$ in time in a Banach space E :

$$D\ u(t) = Au(t) + f(t); \quad u(0) = u^0: \tag{1}$$

It is well–known [1] that the order of convergence in the approximation by a difference scheme with uniform grid of such equations has an order controlled by the exponent $:$ Here we first investigate the well-posedness of (1) on a Holder class of functions [2] and the second we consider the non-uniform grid of the scheme. The stability and accuracy estimates for a proposed finite difference scheme [3] are obtained.

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Chernoff approximations for resolvents of generators of C_0-semigroups

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Keywords: operator semigroups; resolvent operator; linear ODE with variable coefficients; representation of the solution; Chernoff approximations

MSC2020 codes: 47A10, 47D06, 34A05

Summary of the talk. The method of Chernoff approximation [1] is an extremely effective tool for expressing e^{tL} in terms of variable coefficients of operator L. The talk shows that this method can be also be used for expressing $(\lambda I - L)^{-1}$ in terms of variable coefficients of operator L, and for finding the solution of the corresponding differential equation $\lambda f - Lf = g$. We demonstrate this on the second order differential operator L. As a corollary, we obtain two new representations of the solution of an inhomogeneous second order linear ordinary differential equation in terms of functions that are the coefficients of this equation playing the role of parameters for the problem. This reasoning also works in the multi-dimensional situation, where we have an elliptic PDE instead of ODE. Full proofs are available in the preprint [2].

Let us recall the Chernoff theorem.

Chernoff theorem, one of the wordings. Suppose that the following three conditions are met:
1. C_0-semigroup $(e^{tL})_{t\geq 0}$ with generator $(L, D(L))$ in Banach space F is given, such that for some $w \geq 0$ the inequality $\|e^{tL}\| \leq e^{wt}$ holds for all $t \geq 0$.

2. There exists a strongly continuous mapping $S\colon [0, +\infty) \to \mathscr{L}(F)$ such that $S(0) = I$ and the inequality $\|S(t)\| \leq e^{wt}$ holds for all $t \geq 0$.

3. There exists a dense linear subspace $D \subset F$ such that for all $f \in D$ there exists a limit $S'(0)f := \lim_{t\to+0}(S(t)f - f)/t$. Moreover, $S'(0)$ on D has a closure that coincides with the generator $(L, D(L))$. Then the following statement holds:

(C) For every $f \in F$, as $n \to \infty$ we have $S(t/n)^n f \to e^{tL}f$ locally uniformly with respect to $t \geq 0$, i.e. for each $T > 0$ and each $f \in F$ we have $\lim_{n\to\infty} \sup_{t\in[0,T]} \|S(t/n)^n f - e^{tL}f\| = 0.$

Remark 1. Above $S(t/n)^n = \underbrace{S(t/n) \circ \cdots \circ S(t/n)}_{n}$ is the composition of n copies of linear bounded operator $S(t/n)$ defined everywhere on F.

Definition 1. Let C_0-semigroup $(e^{tL})_{t\geq 0}$ with generator L in Banach space F be given. The mapping $S\colon [0, +\infty) \to \mathscr{L}(F)$ is called a *Chernoff function for operator* L iff it satisfies the condition (C) of Chernoff theorem above. In this case expressions $S(t/n)^n$ are called *Chernoff approximations to the semigroup* e^{tL}.

Main idea of the talk. Thanks to Chernoff theorem we have $e^{tL}f = \lim_{n\to\infty} S(t/n)^n f$ for all vectors f and for properly selected operator-valued function S. Also, there is a standard fact that for λ with $Re\,\lambda$ large enough for the resolvent of L we have the followng representation: $(\lambda I - L)^{-1}f = \int_0^\infty e^{-\lambda t}e^{tL}f\,dt$, so we can substitute e^{tL} by $S(t/n)^n$ and get approximations for the resolvent:

$$(\lambda I - L)^{-1}f = \int_0^\infty e^{-\lambda t}e^{tL}f\,dt = \int_0^\infty e^{-\lambda t}\lim_{n\to\infty} S(t/n)^n f\,dt = \lim_{n\to\infty}\int_0^\infty e^{-\lambda t}S(t/n)^n f\,dt.$$

Above the first (left) equality is a classical fact, the second inequality is due to Chernoff theorem, and the last (the right) equality is the main idea of all results that follow below.

Theorem 1. Let F be real or complex Banach space, and let $\mathscr{L}(F)$ be the set of all linear bounded operators in F. Suppose that linear operator $L\colon F \supset D(L) \to F$ generates C_0-semigroup $(e^{tL})_{t\geq 0}$ satisfying for some constants $M \geq 1$ and $\omega \geq 0$ inequality $\|e^{tL}\| \leq Me^{\omega t}$ for all $t \geq 0$. Suppose that function $S\colon [0, +\infty) \to \mathscr{L}(F)$ is given and $\|S(t)^k\| \leq Me^{\omega tk}$ for all $t \geq 0$

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and all $k = 1, 2, 3, \dots$ Let us denote the resolvent of $(L, D(L))$ by the symbol $R_\lambda = (\lambda I - L)^{-1}$ for all $\lambda \in \rho(L)$. Suppose that the number $\lambda \in \mathbb{C}$ is given and $Re\,\lambda > \omega$. Then $\lambda \in \rho(L)$ and:

1. If for all $T > 0$ we have $\lim_{n\to\infty} \sup_{t\in[0,T]} \left\| e^{tL} f - (S(t/n))^n f \right\| = 0$ for all $f \in F$, then for all $f \in F$ we have

$$\lim_{n\to\infty} \left\| R_\lambda f - \int_0^\infty e^{-\lambda t} (S(t/n))^n f \, dt \right\| = 0.$$

2. If for all $T > 0$ we have $\lim_{n\to\infty} \sup_{t\in[0,T]} \left\| e^{tL} - (S(t/n))^n \right\| = 0$, then we have

$$\lim_{n\to\infty} \left\| R_\lambda - \int_0^\infty e^{-\lambda t} (S(t/n))^n dt \right\| = 0.$$

Theorem 2. Consider second order ordinary differential equation for function $f : \mathbb{R} \to \mathbb{R}$

$$a(x) f''(x) + b(x) f'(x) + (c(x) - \lambda) f(x) = -g(x) \text{ for all } x \in \mathbb{R}; \tag{1}$$

where functions $a, b, c, g : \mathbb{R} \to \mathbb{R}$ are known parameters and number $\lambda \in \mathbb{C}$ is also a known parameter. Assume that there exists constant $a_0 > 0$ such that $a(x) > a_0$ for all $x \in \mathbb{R}$. Assume that there exists $\alpha \in (0, 1]$ such that function c is bounded and Hölder continuous with Hölder exponent α, and functions a, $x \mapsto 1/a(x)$, b are bounded and Hölder continuous with Hölder exponent α with derivatives of order one and two. Assume that function g is continuous and vanishes at infinity. Assume that $\mathbb{R} \ni \lambda > \max(0, \sup_{x\in\mathbb{R}} c(x))$.

Then for equation (1) there exists a unique continuous and vanishing at infinity solution f given for all $x_0 \in \mathbb{R}$ by the formula

$$f(x_0) = \lim_{n\to\infty} \int_0^\infty e^{-\lambda t} \Bigg[\underbrace{\int_{\mathbb{R}} \dots \int_{\mathbb{R}}}_{n} \exp\left(\frac{t}{n} \sum_{j=1}^n \left(c(x_{j-1}) - \frac{b(x_{j-1})^2}{2a(x_{j-1})} \right) \right)$$

$$\exp\left(\sum_{j=1}^n \frac{b(x_{j-1})(x_j - x_{j-1})}{a(x_{j-1})} \right) p_a(t/n, x_0, x_1) \dots p_a(t/n, x_{n-1}, x_n) g(x_n) dx_1 \dots dx_n \Bigg] dt;$$

where the limit $\lim\limits_{n\to\infty}$ exists uniformly in $x_0 \in \mathbb{R}$, and we denoted

$$p_a(t, x, y) = \frac{1}{\sqrt{2\pi t a(x)}} \exp\left(-\frac{(x-y)^2}{2ta(x)} \right) \text{ for all } x, y \in \mathbb{R}, t > 0.$$

Some notation. Let us use symbol $UC_b(\mathbb{R})$ to denote Banach space of all bounded and uniformly continuous functions $f : \mathbb{R} \to \mathbb{R}$ with the uniform norm $\|f\| = \sup_{x\in\mathbb{R}} |f(x)|$. Let us use symbol $C_b^\infty(\mathbb{R})$ for the subspace of $UC_b(\mathbb{R})$ consisting of all infinitely differentible functions that are bounded and have bounded derivatives of all orders.

Theorem 3. Suppose that functions $a, b, c \in UC_b(\mathbb{R})$ are bounded with their derivatives up to order 3, and there exists such a constant $a_0 > 0$ that estimate $\inf_{x\in\mathbb{R}} a(x) \geq a_0 > 0$ is satisfied for all $x \in \mathbb{R}$. For each function $\varphi \in C_b^\infty(\mathbb{R}) = D(A)$ define $A\varphi = a\varphi'' + b\varphi' + c\varphi$. For each $t \geq 0$, each $x \in \mathbb{R}$ and each $f \in UC_b(\mathbb{R})$ define

$$(S(t)f)(x) = \frac{1}{4} f\left(x + 2\sqrt{a(x)t}\right) + \frac{1}{4} f\left(x - 2\sqrt{a(x)t}\right) + \frac{1}{2} f\left(x + 2b(x)t\right) + tc(x)f(x). \tag{2}$$

Assume also that $\mathbb{R} \ni \lambda > \sup_{x\in\mathbb{R}} |c(x)| = \|c\|$. Then:

1. Closure $\overline{A}$ of operator A generates a C_0-semigroup in $UC_b(\mathbb{R})$.
2. For each $g \in UC_b(\mathbb{R})$ the solution $f : \mathbb{R} \to \mathbb{R}$ of the equation

$$a(x) f''(x) + b(x) f'(x) + (c(x) - \lambda) f(x) = -g(x) \text{ for all } x \in \mathbb{R};$$

exists, is unique in $UC_b(\mathbb{R})$ and is given for all $x \in \mathbb{R}$ by the formula

$$f(x) = \int_0^\infty e^{-\lambda t}\left(e^{t\overline{A}}g\right)(x)dt = \lim_{n\to\infty}\int_0^\infty e^{-\lambda t}((S(t/n))^n g)(x)dt; \tag{3}$$

where $S(t/n)$ is obtained by substitution of t with t/n in (2), and $(S(t/n))^n$ is the composition of n copies of linear bounded operator $S(t/n)$.

Suppose additionally that function g is bounded with derivatives up to order 5. Then:

3. There exist nonnegative constants $C_0, C_1, \ldots, C_4$ such that for all $t > 0$ and all $n \in \mathbb{N}$ the following inequality holds:

$$\|S(t/n)^n g - e^{t\overline{A}}g\| \le \frac{t^2 e^{\|c\|t}}{n}\left(C_0\|g\| + C_1\|g'\| + C_2\|g''\| + C_3\|g'''\| + C_4\|g^{(IV)}\|\right).$$

4. Error bound in (3) for all $n \in \mathbb{N}$ is given by inequality

$$\sup_{x\in\mathbb{R}}\left|f(x) - \int_0^\infty e^{-\lambda t}((S(t/n))^n g)(x)dt\right| \le \frac{2C_g}{n(\lambda-\|c\|)^3},$$

where $C_g = C_0\|g\| + C_1\|g'\| + C_2\|g''\| + C_3\|g'''\| + C_4\|g^{(IV)}\|$.

5. Integral in item 2 can be calculated over $[0, T]$ instead of $[0, \infty)$ with controlled level of error. This means that for each $\varepsilon > 0$ there exists $T = \max\left(0, \frac{1}{\lambda-\|c\|}\ln\frac{2}{(\lambda-\|c\|)\varepsilon}\right)$ such that for all $n \in \mathbb{N}$ we have

$$\sup_{x\in\mathbb{R}}\left|f(x) - \int_0^T e^{-\lambda t}((S(t/n))^n g)(x)dt\right| \le \frac{2C_g}{n(\lambda-\|c\|)^3} + \varepsilon.$$

Remark 2. Independently of Chernoff function used (is it based on integral operators as in theorem 2 or on translation operators as in theorem 3), Chernoff approximations are allowing to calculate value of the solution in only one point of the domain of solution (in one point $x \in \mathbb{R}$ in our examples). Meanwhile methods based on a computational grid calculate values of the solution in all points of the computational grid. Moreover, values of Chernoff approximations at different points of the domain can be calculated in parallel, using multi-core processors and GPU which is an advantage of this approach.

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On the operator group generated by the one–dimensional Dirac system

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Keywords: Dirac operator, summable potential, operator group.

MSC2010 codes: 34L40

In our talk, we consider the well–known differential operator — the one–dimensional Dirac operator. We define this operator on a finite interval, adding arbitrary Birkhoff–regular boundary conditions. The difference from the classical theory is that the matrix potential of the operator is assumed to be non–smooth — we will require only Lebesgue summability of the potential on the entire interval. Our main goal is to define an operator exponent (operator group). The potential is assumed to be complex, and the boundary conditions may not be self–adjoint, so that the operator is not, in general, self–adjoint. So the question of the existence of a group turns out to be non–trivial. Nevertheless, the group exists, and not only in the space L_2, but also in the scale of Sobolev spaces, as well as in the spaces L_p. The issue of estimating the growth of this group for large values of time will be considered separately. It naturally leads to questions about the localization of the spectrum and estimates of the Riesz constant.

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Real Interpolation of functions on Banach spaces and Reiteration theorems

P. Sharma [23]

We study real interpolation, but instead of interpolating between Banach spaces (or actually norms on Banach spaces), we interpolate between general functions on Banach spaces taking values in $[0; 1]$: We also proved some general Reiteration theorems.
This is a joint work with my Ph.D. supervisors Prof. Ralph Chill from Institut fur Analysis, Fakult at Mathematik, TU Dresden, 01062 Dresden and Prof. Sachi Srivastava from Department of Mathematics, University of Delhi.

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Resolving families of operators and fractional multi-term quasilinear Equations

M. M. Turov [24], V. E. Fedorov [25]

Keywords: multi-term fractional differential equation, Riemann — Liouville fractional derivative, defect of Cauchy type problem, incomplete Cauchy type problem, analytic family of resolving operators, quasilinear equation, initial boundary value problem.

MSC2020 codes: 35R11, 34G10, 34G20, 34A08.

Let $A_1, A_2, \ldots, A_{m-1}, B_1, B_2, \ldots, B_n, C_1, C_2, \ldots, C_r$ be closed linear operators in a Banach space $\mathcal{Z}$ with domains $D_{A_1}, D_{A_2}, \ldots, D_{A_{m-1}}$, $D_{B_1}, D_{B_2}, \ldots, D_{B_n}$, $D_{C_1}, D_{C_2}, \ldots, D_{C_r}$ respectively, $m-1<\alpha\leq m\in\mathbb{N}$, $n, r, \varrho, q\in\mathbb{N}\cup\{0\}$, Z be an open subset in $\mathbb{R}\times\mathcal{Z}^{m+\varrho+q}$, $B\in C(Z;\mathcal{Z})$. Consider the quasilinear multi-term fractional equation

$$\begin{aligned} D_t^{\alpha} z(t) = \sum_{j=1}^{m-1} A_j D_t^{\alpha-m+j} z(t) + \sum_{l=1}^{n} B_l D_t^{\alpha_l} z(t) + \sum_{s=1}^{r} C_s J_t^{\beta_s} z(t) + \\ + F(t, D_t^{\alpha-m-\varrho} z(t), \ldots, D_t^{\alpha-1} z(t), D_t^{\gamma_1} z(t), D_t^{\gamma_2} z(t), \ldots, D_t^{\gamma_q} z(t)). \end{aligned} \tag{1}$$

Here D_t^{β} is the Riemann — Liouville derivative with $\beta>0$ and the Riemann — Liouville integral with $\beta<0$, $0<\alpha_1<\alpha_2<\cdots<\alpha_n<\alpha$, $m_l-1<\alpha_l\leq m_l\in\mathbb{Z}$, $\alpha_l-m_l\neq\alpha-m$, $l=1,2,\ldots,n$, $\gamma_1<\gamma_2<\cdots<\gamma_q<\alpha$, $n_i-1<\gamma_i\leq n_i\in\mathbb{Z}$, $\gamma_i-n_i\neq\alpha-m$, $i=1,2,\ldots,q$. Some γ_i may be negative. Let us define $\mu := m-\lambda(\alpha;\alpha_1,\alpha_2,\ldots,\alpha_n;\gamma_1+1,\gamma_2+1,\ldots,\gamma_q+1)$ (see [1]), $\mu_0:=\max\{\mu;0\}$, so for solving the Cauchy type problem

$$D_t^{\alpha-m+k} z(t_0) = z_k, \quad k=0,1,\ldots,m-1, \tag{2}$$

for equation (1) conditions are met

$$D_t^{\alpha-m+k} z(t_0) = 0, \quad k=-r, -r+1, \ldots, \mu_0-1,$$

$$D_t^{\alpha_l-m_l+k} z(t_0) = 0, \quad k=0,1,\ldots,m_l-1, \ l=1,2,\ldots,n,$$

$$D_t^{\gamma_i-n_i+k} z(t_0) = 0, \quad k=0,1,\ldots,n_i, \ i=1,2,\ldots,q.$$

Define by $\mathcal{L}(\mathcal{Z})$ the Banach space of all linear bounded operators on $\mathcal{Z}$,

$$D:=\bigcap_{j=1}^{m-1} D_{A_j}\cap\bigcap_{l=1}^{n} D_{B_l}\cap\bigcap_{s=1}^{r} D_{C_s}, \quad \|\cdot\|_D=\sum_{j=1}^{m-1}\|\cdot\|_{D_{A_j}}+\sum_{l=1}^{n}\|\cdot\|_{D_{B_l}}+\sum_{s=r}^{m-1}\|\cdot\|_{D_{C_s}}.$$

A solution to problem (1), (2) on $(t_0,t_1]$ is a function $z:(t_0,t_1]\to D$, such that $J_t^{m-\alpha} z\in C^m((t_0,t_1];\mathcal{Z})\cap C^{m-1}([t_0,t_1];\mathcal{Z})$, $D_t^{\alpha-m+j} z\in C((t_0,t_1];D_{A_j})$, $j=1,2,\ldots,m-1$, $D_t^{\alpha_l} z\in C((t_0,t_1];D_{B_l})$, $l=1,2,\ldots,n$, $D_t^{\gamma_i} z\in C([t_0,t_1];\mathcal{Z})$, $i=1,2,\ldots,q$, condition (2) are satisfied, inclusion $(t, D_t^{\alpha-m-\varrho} z(t), D_t^{\alpha-m-\varrho+1} z(t), \ldots, D_t^{\alpha-1} z(t), D_t^{\gamma_1} z(t), D_t^{\gamma_2} z(t), \ldots, D_t^{\gamma_q} z(t))\in Z$ for $t\in[t_0,t_1]$ and equality (1) for $t\in(t_0,t_1]$ hold.

Definition 1. A tuple of operators $(A_1, A_2, \ldots, A_{m-1}, B_1, B_2, \ldots, B_n, C_1, C_2, \ldots, C_r)$, which are linear and closed in a Banach space $\mathcal{Z}$, belongs to the class $\mathcal{A}^{n,r}(\theta_0, a_0)$ for some $\theta_0\in(\pi/2,\pi)$, $a_0\geq 0$, if

(i) D is dense in $\mathcal{Z}$;

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(ii) for all $\lambda \in S_{\theta_0,a_0} := \{\lambda \in \mathbb{C} : |\arg(\lambda - a_0)| < \theta_0\}$, $p = 0,1,\dots,m-1$ we have

$$R\left(I - \sum_{j=p+1}^{m-1} \lambda^{j-m} A_j\right) \in \mathcal{L}(Z);$$

(iii) for any $\theta \in (\pi/2, \theta_0)$, $a > a_0$, there exists such a $K(\theta,a)$, that for all $\lambda \in S_{\theta,a}$, $p = 0,1,\dots,m-1$ we have

$$\left\| R\left(I - \sum_{j=p+1}^{m-1} \lambda^{j-m} A_j\right)\right\|_{\mathcal{L}(Z)} \le \frac{K(\theta,a)}{|\lambda - a||\lambda|^{\alpha-1}}.$$

Definition 2. Let $p \in \{0,1,\dots,m-1\}$; a strongly continuous family of operators $\{S_p(t) \in \mathcal{L}(Z) : t > 0\}$ is called *p-resolving* for equation (1), if next conditions are satisfied:

(i) for $t > 0$ $S_p(t)[D_{A_j}] \subset D_{A_j}$, $S_p(t)A_jx = A_jS_p(t)x$ for all $x \in D_{A_j}$, $j = 1,2,\dots,m-1$; $S_p(t)[D_{B_l}] \subset D_{B_l}$, $S_p(t)B_lx = B_lS_p(t)x$ for all $x \in D_{B_l}$; $S_p(t)[D_{C_s}] \subset D_{C_s}$, $S_p(t)C_sx = C_sS_p(t)x$ for all $x \in D_{C_s}$;

(ii) for every $z_p \in D$ $S_p(t)z_p$ is a solution of linear ($B \equiv 0$) problem (1), (2) with $z_l = 0$ for every $l \in \{0,1,\dots,m-1\} \setminus \{p\}$.

A p-resolving family of operators for $p \in \{0,1,\dots,m-1\}$ is called *analytic*, if it has the analytic extension to a sector $\Sigma_{\psi_0} := \{t \in \mathbb{C} : |\arg t| < \psi_0, t \neq 0\}$ for some $\psi_0 \in (0,\pi/2]$. An analytic p-resolving family of operators $\{S_p(t) \in \mathcal{L}(Z) : t > 0\}$ has a type (ψ_0,a_0) for some $\psi_0 \in (0,\pi/2]$, $a_0 \in \mathbb{R}$, if for all $\psi \in (0,\psi_0)$, $a > a_0$ there exists such a $C(\psi,a)$, that for all $t \in \Sigma_\psi$ the inequality $\|S_p(t)\|_{\mathcal{L}(Z)} \le C(\psi,a)|t|^{\alpha-m+p}e^{a\,\mathrm{Re}\,t}$ is satisfied.

Theorem 1. Let $m-1 < \alpha \le m \in \mathbb{N}$, $\alpha_1 < \alpha_2 < \dots < \alpha_n < \alpha$, $m_l - 1 < \alpha_l \le m_l \in \mathbb{N}$, $\alpha_l - m_l \neq \alpha - m$, $m^* := m^*(\alpha, \alpha_1, \alpha_2, \dots, \alpha_n)$, $\beta_1 > \beta_2 > \dots > \beta_r \ge 0$, A_j, $j = 1,2,\dots,m-1$, B_l, $l = 1,2,\dots,n$, C_s, $s = 1,2,\dots,r$, are linear and closed operators, D dense Z. Then there are p-resolving families of operators $\{S_p(t) \in \mathcal{L}(Z) : t > 0\}$ of the type (θ_0,a_0) for equation (1) for all $p = m^*, m^*+1, \dots, m-1$, if and only if $(A_1,A_2,\dots,A_{m-1},B_1,B_2,\dots,B_n,C_1,C_2,\dots,C_r) \in \mathcal{A}^{n,r}(\theta_0,a_0)$. Moreover,

$$S_p(t) = Z_p(t) := \frac{1}{2\pi i}\int_{\Gamma} \lambda^{\alpha-m-1-p} R\left(I - \sum_{j=p+1}^{m-1}\lambda^{j-m}A_j\right)e^{\lambda t}d\lambda, \quad p = m^*, m^*+1, \dots, m-1;$$

where $\Gamma := \Gamma^+ \cup \Gamma^- \cup \Gamma^0$, $\Gamma^0 := \{\lambda \in \mathbb{C} : \lambda = a + r_0e^{i\varphi}, \varphi \in (-\theta,\theta)\}$, $\Gamma^{\pm} := \{\lambda \in \mathbb{C} : \lambda = a + r_0e^{\pm i\theta}, r \in [r_0,\infty)\}$, $\theta \in (\pi/2,\theta_0)$, $a > a_0$, $r_0 > 0$.

Theorem 2. Let $m-1 < \alpha \le m \in \mathbb{N}$, $0 < \alpha_1 < \alpha_2 < \dots < \alpha_n < \alpha$, $m_l - 1 < \alpha_l \le m_l \in \mathbb{N}$, $\alpha_l - m_l \neq \alpha - m$, $\gamma_1 < \gamma_2 < \dots < \gamma_q < \alpha - 1$, $n_i - 1 < \gamma_i \le n_i \in \mathbb{Z}$, $\gamma_i - n_i \neq \alpha - m$, $i = 1,2,\dots,q$, $(A_1,A_2,\dots,A_{m-1},B_1,B_2,\dots,B_n,C_1,C_2,\dots,C_r) \in \mathcal{A}^{n,r}(\theta_0,a_0)$, $z_k \in D$, $k = m_0, m_0+1, \dots, m-1$, Z be open in $\mathbb{R} \times Z^{m+\varkappa+q}$, $(t_0,0,0,\dots,0,z_{m_0},z_{m_0+1},\dots,z_{m-1},0,0,\dots,0) \in Z$; the mapping $B \in C(Z;D)$ be locally Lipschitz continuous with respect to the phase variables. Then there exists $t_1 > t_0$, such that problem (1), (2) has a unique solution on $(t_0,t_1]$.

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On divergence-free (form-bounded type) drifts

R. Vafadar [26]

We develop regularity theory for elliptic Kolmogorov operator with divergence-free drift in a large class. A key step in our proofs is new "Caccioppoli's iterations", used in addition to the classical De Giorgi's iterations and Moser's method.

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Inversion of the Pompeiu transform associated to spherical means

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Keywords: distributions; convolution equations; Pompeiu transform; inversion formulas.

MSC2010 codes: 44A35, 46F12, 53C35, 45E10

Let $n \geq 2$, $\mathcal{D}'(\mathbb{R}^n)$ be the space of distributions on $\mathbb{R}^n$, δ_r be the surface delta function concentrated on the sphere $S_r = \{x \in \mathbb{R}^n : |x| = r\}$. The problem of L. Zalcman on reconstruction of a distribution $f \in \mathcal{D}'(\mathbb{R}^n)$ by known convolutions $f * \delta_{r_1}$ and $f * \delta_{r_2}$ is studied (see [1], Sect. 8). The result obtained (see Theorem 2 below) significantly simplifies known procedures for recovering a function f from given spherical means $f * \delta_{r_1}$ and $f * \delta_{r_2}$.

Let $r > 0$ be fixed and λr be an arbitrary positive zero of the Bessel function J_0. Then, for any $k \in \mathbb{Z}$, the function $J_k(\lambda\rho)e^{ik\varphi}$ (ρ, φ are the polar coordinates in $\mathbb{R}^2$) has zero integrals over all circles of radius r in $\mathbb{R}^2$ (see [2], Sect. C). Similar examples related to the zeros of the Bessel function $J_{n/2-1}$ can also be constructed for spherical means in $\mathbb{R}^n$ for $n \geq 2$. This shown that knowing the averages of a function f over all spheres of the same radius is not enough to uniquely reconstruct f. Subsequently, the class of functions $f \in C(\mathbb{R}^n)$ that have zero integrals over all spheres of fixed radius in $\mathbb{R}^n$ was studied by many authors (see [3]–[6] and the references to these works). A well-known result in this direction is the following analogue of Delsarte's famous two-radius theorem for harmonic functions.

Theorem 1 ([1], [3]). Let $r_1, r_2 \in (0; +\infty)$, $\mathcal{N}_n = \{\nu_1; \nu_2; \ldots\}$ be the sequence of all positive zeros of the function $J_{n/2-1}$ numbered in ascending order, M_n be the set of numbers of the form α/β, where $\alpha; \beta \in \mathcal{N}_n$.

1) If $r_1/r_2 \notin M_n$, $f \in C(\mathbb{R}^n)$ and

$$\int_{|x-y|=r_1} f(x)d\omega(x) = \int_{|x-y|=r_2} f(x)d\omega(x) = 0; \quad \forall y \in \mathbb{R}^n \tag{1}$$

($d\omega$ is the area element), then $f = 0$.

2) If $r_1/r_2 \in M_n$, then there exists a nonzero real analytic function $f : \mathbb{R}^n \to \mathbb{C}$ satisfying the relations in (1).

In terms of convolutions Theorem 1 means that the Pompeiu transform

$$Pf = (f * \delta_{r_1}; f * \delta_{r_2}); \quad f \in C(\mathbb{R}^n)$$

is injective if and only if $r_1/r_2 \notin M_n$. Here we present a new inversion formula for the operator P under the condition $r_1/r_2 \notin M_n$.

Let $\mathcal{E}'(\mathbb{R}^n)$ be the space of compactly supported distributions on $\mathbb{R}^n$, $\mathcal{E}'_\natural(\mathbb{R}^n)$ be the space of radial (invariant under rotations of the space $\mathbb{R}^n$) distributions in $\mathcal{E}'(\mathbb{R}^n)$. If $T_1; T_2 \in \mathcal{D}'(\mathbb{R}^n)$ and at least one of these distributions has compact support then their convolution $T_1 * T_2$ is a distribution in $\mathcal{D}'(\mathbb{R}^n)$ acting according to the rule

$$\langle T_1 * T_2; \varphi\rangle = \langle T_2(y); \langle T_1(x); \varphi(x+y)\rangle\rangle; \quad \varphi \in \mathcal{D}(\mathbb{R}^n);$$

where $\mathcal{D}(\mathbb{R}^n)$ is the space of finite infinitely differentiable functions on $\mathbb{R}^n$. The spherical transform $\widetilde{T}$ of a distribution $T \in \mathcal{E}'_\natural(\mathbb{R}^n)$ is defined by

$$\widetilde{T}(z) = 2^{\frac{n}{2}-1}\Gamma\left(\frac{n}{2}\right)\left\langle T; \mathbf{I}_{\frac{n}{2}-1}(z|x|)\right\rangle; \quad z \in \mathbb{C};$$

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where

$$\mathbf{I}\ (z) = \frac{J\ (z)}{z}, \qquad \in \mathbb{C}.$$

We note that $\widetilde{T}$ is an even entire function of exponential type and the Fourier transform $\widehat{T}$ is expressed in terms of $\widetilde{T}$ by

$$\widehat{T}(\ ) = \widetilde{T}\left(\sqrt{\ _1^2 + \ldots + \ _n^2}\right), \qquad \in \mathbb{C}^n.$$

The set of all zeros of the function $\widetilde{T}$ that lie in the half-plane $\operatorname{Re} z \geq 0$ and do not belong to the negative part of the imaginary axis will be denoted by $Z_+(\widetilde{T})$.

Using the well-known properties of the zeros of the Bessel functions one can obtain the corresponding information about the set $Z_+(\widetilde{\ }_r)$. In particular, all the zeros of $\widetilde{\ }_r$ are simple, belong to $\mathbb{R}\setminus\{0\}$ and

$$Z_+(\widetilde{\ }_r) = \left\{\frac{\ _1}{r}, \frac{\ _2}{r}, \ldots\right\}.$$

In addition, since the functions $J_{\frac{n}{2}-1}$ and $J_{\frac{n}{2}}$ do not have common zeros on $\mathbb{R}\setminus\{0\}$, the function

$$\ _r(x) = \frac{1}{r\ ^2}\frac{\mathbf{I}_{\frac{n}{2}-1}(\ |x|)}{\mathbf{I}_{\frac{n}{2}}(\ r)}\ _r(x), \qquad \in Z_+(\widetilde{\ }_r),$$

is well defined, where $\ _r$ is the indicator of the ball $B_r = \{x \in \mathbb{R}^n : |x| < r\}$.

Let

$$P_r(z) = \prod_{j=1}^{[\frac{n+5}{4}]}\left(z - \left(\frac{\ _j}{r}\right)^2\right), \qquad \ _r = P_r(\ )\ _r,$$

where $\ $ is the Laplace operator. Then, by virtue of the formula

$$p(\hat{\ })T(z) = p(-z^2)\widetilde{T}(z) \quad (p \text{ is an algebraic polynomial}),$$

we have

$$\widetilde{\ }_r(z) = P_r(-z^2)\widetilde{\ }_r(z),$$

$$Z_+\left(\widetilde{\ }_r\right) = \left\{\frac{\ _1}{r}, \frac{\ _2}{r}, \ldots\right\} \cup \left\{\frac{i\ _1}{r}, \frac{i\ _2}{r}, \ldots, \frac{i\ _m}{r}\right\},$$

and all zeros of $\widetilde{\ }_r$ are simple. In addition,

$$Z_+\left(\widetilde{\ }_{r_1}\right) \cap Z_+\left(\widetilde{\ }_{r_2}\right) = \varnothing \quad \Leftrightarrow \quad \frac{r_1}{r_2} \notin M_n.$$

For $\ \in Z_+\left(\widetilde{\ }_r\right)$, we set

$$\ _r = P_r(\ )\ _r \quad \text{if} \quad \ \in Z_+(\widetilde{\ }_r),$$

and

$$\ _r = Q_{r,\ }(\ )\ _r \quad \text{if} \quad P_r(-\ ^2) = 0,$$

where

$$Q_{r,\ }(z) = \frac{P_r(z)}{z + \ ^2}.$$

Theorem 2. Let $\frac{r_1}{r_2} \notin M_n$, $f \in D'(\mathbb{R}^n)$, $n \geq 2$. Then

$$f = \sum_{\ \in Z_+(\widetilde{\ }_{r_1})}\ \sum_{\ \in Z_+(\widetilde{\ }_{r_2})} \frac{4\ }{(\ ^2 - \ ^2)\widetilde{\ }'_{r_1}(\ )\widetilde{\ }'_{r_2}(\ )}\left(P_{r_2}(\ )((f * \ _{r_2}) * \ _{r_1}\right)$$

$$P_{r_1}(\quad)((f \qquad {}_{r_1}) \qquad {}_{r_2})\Big); \tag{2}$$

where the series in (2) converges unconditionally in the space $D^0(\mathrm{R}^n)$.

Equality (2) reconstruct an arbitrary distribution f from its known convolutions $f \quad {}_{r_1}$ and $f \quad {}_{r_2}$ (see formulas above). For other results related to the inversion of the spherical mean operator, see [6], [7].

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The range of C_0-semigroups

H. Zwart [29]

Keywords: right invertible; closed range; strongly continuous semigroup.

MSC2020 codes: 47D06, 47A05

Introduction. Let $(T(t))_{t\geq 0}$ be a strongly continuous semigroup on the Hilbert space Z. It is well-know that if the operator $T(t)$ is surjective for one $t > 0$, then it is surjective for all $t \geq 0$, see [1]. In this paper we study the question if other properties of the range of $T(t)$ are independent of t. By means of a counter example we show that the range of a semigroup can be change from non-closed to closed and back again. Thus properties of the range will be time-dependent, in general.

An illustrative example

In this section we construct an example showing that the range of a strongly continuous semigroup can change from closed to non-closed, and back again.

Let $H_0^1(0,3)$ denote the Sobolev space consisting of all functions in $L^2(0,3)$ whose first derivative exists in $L^2(0,3)$ and which are zero at $\zeta = 3$. It is a Hilbert space with the norm

$$\|f\|_{H^1}^2 = \|f\|^2 + \|\dot{f}\|^2,$$

where the later norms denote the standard L^2-norms of f and its derivative. It is well-known that $H_0^1(0,3)$ is a Hilbert space with this norm.

As Hilbert space Z we take $Z = H_0^1(0,3) \oplus L^2(-1,1)$, and as semigroup we define

$$\begin{aligned} T(t)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} &= \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \quad \text{with} \\ y_1(\zeta) &= x_1(t+\zeta)\mathbf{1}_{[0,3]}(t+\zeta), \qquad \zeta \in [0,3] \\ y_2(\zeta) &= x_1(t+\zeta)\mathbf{1}_{[-1,0]}(t+\zeta) + x_2(t+\zeta)\mathbf{1}_{[-1,1]}(t+\zeta), \qquad \zeta \in [-1,1], \end{aligned}$$

where $\mathbf{1}_{[a,b]}$ denotes the indicator function on the interval $[a,b]$, and we have extended x_1 and x_2 by zero "outside their own interval".

Next we study the range at different time instances.

- $\mathbf{t = 1}$. At $t = 1$, the second component equals zero for $\zeta \in (0,1)$, whereas in the interval $(-1,0)$ it consists of a function in H^1 plus an L^2-function. Since this L^2 function can be constructed freely by using a proper choice of x_2, the range of the second component is closed. It is easy to see that the range of the first component is closed, and thus the range of $T(1)$ is closed.

- $\mathbf{t = 2.5}$. For $t = 2.5$ we see that $x_2(t+\zeta)\mathbf{1}_{[-1,1]}(t+\zeta)$ equals zero for all $\zeta \in [-1,1]$, and the second component of the semigroup consists out of shifted H^1 functions. Since H^1 is not closed in L^2, the range cannot be closed.

- $\mathbf{t > 3}$. For time instances larger than 3, the semigroup equals zero, and thus its range is closed.

The above example shows that the range of a semigroup can change from closed to non-closed and back again. Using the above idea for the construction, it is not hard to see how examples can be constructed for which this change happens (infinitely) many times.

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An open problem

In [2] it is shown that if the semigroup is left invertible, then its left inverse can be chosen to be a strongly continuous semigroup as well. Until now this result is only known for Hilbert spaces, and although the proof uses Hilbert space techniques, the problem is well-formulated in a general Banach space. Hence the research question is to investigate whether this results extends to Banach spaces.

Note that when $T(t)$ is surjective, its adjoint is left invertible, and so there is a direct connection with the range of the semigroup.

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2. Nonlinear (semi)flows: ergodicity, chaos and other dynamical phenomena

Frequency-domain conditions for the exponential stability of compound cocycles generated by delay equations and effective dimension estimates of global attractors

M. M. Anikushin[30]

Keywords: compound cocycles; additive compound operators; dimension estimates; frequency theorem; delay equations

MSC2010 codes: 37L45, 37L30, 37L15, 34K08, 34K35

Introduction.

Usually, dimension estimates for global attractors are obtained via the Liouville trace formula (see S. Zelik [9]), possibly with the use of adapted metrics (see N.V. Kuznetsov and V. Reitmann [7]). In our paper [3], we showed that this approach provides rough bounds and does not allow to obtain effective estimates in the case of delay equations. This is caused not only by the non-self-adjointness of arising operators, but also by the specificity of delay equations where we deal with boundary perturbations.

It is also shown in [3] that a resolution to the problem can be given in the case of scalar equations with monotone feedback by combining results of J. Mallet-Paret and R.D. Nussbaum [8] with the ergodic variational principle and Poincaré-Bendixson trichotomy satisfied in these equations. However, there are many scalar equations, not to mention systems of equations, that exhibit chaotic behavior such as the Mackey-Glass equation or periodically forced Suarez-Schopf oscillator [2]. These models go beyond the described approach.

Main results. We are going to discuss our recent result [1] concerned with cocycles generated by the following class of nonautonomous delay equations in $\mathbb{R}^n$ over a semiflow π on a complete metric space $\mathcal{P}$:

$$\dot{x}(t) = \widetilde{A}x_t + \widetilde{B}F'(\pi^t(\wp))Cx_t, \tag{6}$$

where $\widetilde{A} \colon C([-\tau;0];\mathbb{R}^n) \to \mathbb{R}^n$, $C \colon C([-\tau;0];\mathbb{R}^n) \to \mathbb{M}$ are bounded linear operators; $\widetilde{B} \colon \mathbb{U} \to \mathbb{R}^n$ is a linear operator and $F' \colon \mathcal{P} \to \mathcal{L}(\mathbb{M};\mathbb{U})$ is a continuous mapping such that for some $\Lambda > 0$ we have

$$\|F'(\wp)\|_{\mathcal{L}(\mathbb{M};\mathbb{U})} \leq \Lambda \quad \text{for all } \wp \in \mathcal{P}. \tag{7}$$

Here $\mathbb{U}$ and $\mathbb{M}$ are finite-dimensional Euclidean spaces.

Equations such as (6) arise as linearizations of nonlinear delay equations.

It can be shown that (6) generates a cocycle Ξ over $(\mathcal{P},\pi)$ in the Hilbert space $\mathbb{H} = L_2([-\tau;0];\nu;\mathbb{R}^n)$, where ν is the sum of the Lebesgue measure on $[-\tau;0]$ and the δ-measure concentrated at 0 (see [3]). We study the m-fold compound cocycle Ξ_m given by the multiplicative extension of Ξ to the m-fold exterior power $\mathbb{H}^{\wedge m}$ of $\mathbb{H}$.

To the operator $\widetilde{A}$ from (6) there corresponds an operator A in $\mathbb{H}$ which generates an eventually compact C_0-semigroup G. It can be shown that the m-fold multiplicative extension $G^{\wedge m}$ of G onto $\mathbb{H}^{\wedge m}$ is an eventually compact C_0-semigroup in $\mathbb{H}^{\wedge m}$. Its generator $A^{[\wedge m]}$ is called the *additive (antisymmetric) compound* of A. Using the Spectral Mapping Theorem for Semigroups, one can describe the spectrum of $A^{[\wedge m]}$ through the spectrum of A.

We provide conditions for the uniform exponential stability or, more generally, for the existence of gaps in the Sacker-Sell spectrum of Ξ_m by considering it as a perturbation of $G^{\wedge m}$. On

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the infinitesimal level, we have that the generator of $_m$ is given by a nonautonomous boundary perturbation of $A^{[\wedge m]}$. Such perturbations can be described via unbounded in H quadratic constraints leading to the associated infinite-horizon quadratic regulator problem posed for a proper control system. The latter is resolved via the Frequency Theorem developed in our adjacent work [5] (see also [6]). As a consequence, we obtain frequency-domain conditions which guarantee the existence of a proper (indefinite) bounded quadratic Lyapunov-like functional. Such functionals can be used to obtain the desired dichotomy properties for $_m$ (see [4]).

Our frequency-domain conditions are given by strict frequency inequalities involving resolvents of additive compound operators $A^{[\wedge m]}$. Computing such operators requires solving a first order PDE with boundary conditions containing both partial derivatives and delays that makes it hard to deal with it analytically. However, verification of frequency inequalities reduces to the optimization of a Rayleigh quotient that is feasible to numerical investigation and reflects the computational complexity of the problem.

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Soliton interactions with an external forcing: The modified Korteweg-de Vries framework

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Keywords: solitons, trapped waves, modified Korteweg-de Vries equation, asymptotic theory

MSC2020 codes: 34A34

Abstract The aim of this work is to study asymptotically and numerically the interaction of solitons with an external forcing with a variable speed using the forced modified Korteweg-de Vries equation (mKdV). We show that the asymptotic predictions agree well with numerical solutions for forcings with constant speed and linear variable speed. Regarding forcing with linear variable speed, we find regimes in which the solitons are trapped at the external forcing and its amplitude increases or decreases in time depending on whether the forcing accelerates or decelerates.

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Application of the Kantorovich-Galerkin method for the analysis of resonant systems

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Keywords: emerging mechanical systems, di�erential operator, resonant amplitude, moving boundaries.

MSC2020 codes: 74H45, 74K05

The article considers the resonant characteristics of nonlinear oscillations of a rope with moving boundaries. The phenomena of resonance and passage through resonance are analyzed. An approximate method has been developed in relation to taking into account the influence of resistance forces and viscoelastic properties on the system. This method also allows considering a wider class of boundary conditions compared to other approximate methods for solving boundary value problems with moving boundaries. The resonance characteristics of viscoelastic rope with moving boundaries using the Kantorovich вЂ" Galerkin method are examined in the article. The phenomenon of resonance and steady passage through resonance are analyzed. One-dimensional systems whose boundaries move are widely used in engineering [1вЂ"5]. The presence of moving boundaries causes considerable difficulties in describing such systems. Exact methods for solving such problems are limited by the wave equation and relatively simple boundary conditions. Of the approximate methods, the Kantorovich-Galerkin method described in [5] is the most efficient. However, this method can also be used in more complex cases. This method makes it possible to take into account the effect of resistance forces on the system, the viscoelastic properties of an oscillating object, and also the weak nonstationarity of the boundary conditions. The paper considers the phenomena of steady-state resonance and passage through resonance for transverse oscillations of a rope of variable length, taking into account viscoelasticity and damping forces. Performing transformations similar to transformations [5], an expression is obtained for the amplitude of oscillations corresponding to the n-th dynamic mode. Expressions are also obtained that describe the phenomenon of steady state resonance and the phenomenon of passage through resonance. The expression that determines the maximum amplitude of oscillations when passing through the resonance was numerically investigated to the maximum. The dependence of the rope oscillation amplitude on the boundary velocity, viscoelasticity, and damping forces is analyzed. The results of numerical studies allow us to draw the following conclusions: - with a decrease in the velocity of the boundary, viscoelasticity and damping forces, the amplitude of oscillations increases; - as the boundary velocity, viscoelasticity and damping forces tend to zero, the oscillation amplitude tends to infinity; In conclusion, we note that the above results make it possible to carry out a quantitative analysis of the steady state resonance and the phenomenon of passage through the resonance for systems whose oscillations are described by the formulated problem.

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3. Interplay between linear infinite-dimensional systems and nonlinear finite-dimensional systems

4. Quantum physics, quantum information and quantum dynamical semigroups

On informational completeness of covariant positive operator-valued measures

G. G. Amosov [35]

Keywords: locally compact Abelian groups, Pontryagin duality, projective unitary representations, covariant positive operator-valued measures

MSC2010 codes: 81P18, 20C25

Let G be a locally compact Abelian group with the Haar measure . Due to the Pontraygin duality there exists the unique Haar measure ^ on the group of characters $\hat{G}$ such that the composition of forward and inverse Fourier transforms of functions from $H = L^2(G)$ gives the identity transformation. Following to [1-2] define a projective unitary representation of the group $\mathsf{G} = \hat{G} \quad G$ in H by the formula

$$(U_{\ ;g} f)(a) = \ (a) f(a+g); \quad 2\ \hat{G};\ g\ 2\ G;\ f\ 2\ H:$$

Then, the following statement holds true.

Theorem 1 [2]. Let f be a cyclic vector for the representation $(U_{\ ;g})$. Then,

$$\mathsf{M}(B) = \int_B j U_{\ ;g} f\, i\, h U_{\ ;g} f\, j\, d\hat{\ }(\ ) d\ (g);\ B \quad \mathsf{G};$$

is a positive operator-valued measure (POVM) on the space G.

Any POVM M determines an affine map from the set $\mathsf{S}(H)$ consisting of all states (positive unit trace operators) in H to the set (G) of all probability distributions on G defined by the formula

$$(\ )[B] = Tr(\ \mathsf{M}(B))$$

for measurable $B \quad \mathsf{G}$. The map is known as a measurement channel. We call a POVM M informationally complete if given a probability distribution 2 $(\mathsf{S}(H))$ there exists the unique state such that $(\ ) =$.

We introduced a family of contractions

$$T_{\ ;g} = \int_{\hat{G}\ G} {}^0(g)\ \overline{(g^0)} d\hat{\ }(\ ^0) d\ (g^0); \quad 2\ \hat{G};\ g\ 2\ G;$$

and proved that [3] there is a family of complex-valued functions $f(\ ;g);\ 0 < jf(\ ;g)j \quad 1$ such that

$$T_{\ ;g} = f(\ ;g) U_{\ ;g}; \quad 2\ \hat{G};\ g\ 2\ G;$$

and has shown that the following theorem takes palce.

Theorem 2 [3]. The POVM M is informationally complete. Moreover, the inverse formula for restoring a state is given by

$$= \int_{\mathsf{G}} d\hat{\ }(\ ) d\ (g) f(\ ;g)\ ^1 U_{\ ;g} \int_{\mathsf{G}} d\hat{\ }(\ ^0) d\ (g^0)\ (g^0)\ \overline{\ ^0(g)} p\ (\ ^0; g^0);$$

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where

$$p(\ ;g) = hU_{\ ;g}f;\ \ U_{\ ;g}fjU_{\ ;g}f;\ \ U_{\ ;g}fi$$

is the density of probability distribution $\ = \ (\ )$.

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On diagonal quantum channels

A. R. Arab [36]

Keywords: Cholesky decomposition; diagonal quantum channels; Kraus operators.

MSC2020 codes: 81P47, 81P45, 81P68

Introduction. In quantum information theory, the notions of the channel and its capacity, giving a measure of ultimate information-processing performance of the channel, play a central role. For a comprehensive introduction to quantum channels, see [1]. Diagonal quantum channels have significant applications in communication and physics. There are some studies on different types of diagonal channels, for instance depolarizing channels [2, 3] and diagonal channels with constant Frobenius norm [4].

Definition 1. Quantum channel $: \mathbb{C}^{n\ n} \to \mathbb{C}^{n\ n}$ is called *diagonal*, if its representation with respect to an orthonormal basis (constructed by the generalized Pauli matrices) is diagonal, i.e. $= \mathrm{diag}(1, a_1, a_2, \dots, a_{n^2\ 1})$.

Theorem 1. ([5]) *For every diagonal quantum channel , there is a collection of transition probabilities $\{P_{kj}\}_{j=1}^{n}$; i.e. $P_{kj}\ 0, \sum_{j=1}^{n} P_{kj} = 1$ such that*

$$(|k\rangle\langle k|) = \sum_{j=1}^{n} P_{kj}|j\rangle\langle j| \qquad (k = 1, 2, \dots, n).$$

Kraus representation for diagonal channel. Before we formulate the result of this section, we need to prove the following two lemmas [5].

Lemma 1. Let $= (x_1, x_2, \dots, x_n)$ where x_i's are rows of $n\ n$ matrix K, then $(K\ E_{ij}K)_{1\ i,j\ n} = (x_i x_j)_{1\ i,j\ n} =$.

Lemma 2. Let $: \mathbb{C}^{n\ n} \to \mathbb{C}^{n\ n}$ be a quantum channel, C be its Choi matrix, and $C = R\ R$ for some matrix R. If $_i$'s are rows of R, and K_i's are associated matrices to $_i$'s in lemma 1 $(1\ i\ n^2)$ then $\{K_i\}_{i=1}^{n^2}$ is a set of Kraus operators of .

Now we are in a position to assert the main result of this section:

Theorem 2. ([5]) *For hybrid depolarizing classical quantum channel*

$$= \mathrm{diag}(1, \underbrace{p, \dots, p}_{N}, \underbrace{p, \dots, p}_{N}, \underbrace{p, \dots, p}_{n\ 1});$$

Kraus operators can be determined in the following explicit form:

$$K_1 = \begin{pmatrix} \sqrt{a_0} & 0 & \dots & 0 \\ 0 & \sqrt{\frac{b_0}{a_0}} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \sqrt{\frac{b_0}{a_0}} \end{pmatrix}; \quad K_2 = \begin{pmatrix} 0 & \sqrt{\frac{1\ p}{n}} & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 \end{pmatrix};$$

$$\dots, K_n = \begin{pmatrix} 0 & 0 & \dots & \sqrt{\frac{1\ p}{n}} \\ 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 \end{pmatrix}; \quad K_{n+1} = \begin{pmatrix} 0 & 0 & \dots & 0 \\ \sqrt{\frac{1\ p}{n}} & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 \end{pmatrix};$$

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$$K_{n+2} = \begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & \sqrt{a_1} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \sqrt{\frac{b_1}{a_1}} \end{pmatrix}; \dots; K_{2n} = \begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & \sqrt{\frac{1-p}{n}} \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 \end{pmatrix};$$

$$\dots; K_{n^2-1} = \begin{pmatrix} 0 & \dots & 0 & 0 \\ 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots \\ 0 & \dots & \sqrt{\frac{1-p}{n}} & 0 \end{pmatrix}; \quad K_{n^2} = \begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \sqrt{a_{n-1}} \end{pmatrix};$$

where $a_m = (2p + \frac{1-p}{n})(1 + \frac{p}{pm+2p+\frac{1-p}{n}})$ *for* $m = 1, 2, \dots, n-1$; $b_m = (2p + \frac{1-p}{n})(\frac{p}{pm+2p+\frac{1-p}{n}})$ *for* $m = 1, 2, \dots, n-2$; $a_0 = p + \frac{1-p}{n}$, *and* $b_0 = -p$.

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Growth and divisor of complexified horocycle eigenfunctions

M. Dubashinskiy [37]

Furstenberg Theorem on unique ergodicity of horocycle flow over compact hyperbolic surfaces can be passed through a semiclassical quantization. We then arrive to a plenty of ***horocycle eigenfunctions* u** defined at the hyperbolic plane $\mathbb{C}^+$. They enjoy

$$\left(y^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)+2i\, y\frac{\partial}{\partial x}\right)u(x+iy)=s^2u(x+iy);\quad x+iy\in\mathbb{C}^+;$$

with $\to\infty$, $s=o(\;)$, $s;\ \in\mathbb{R}$, and possess Quantum Unique Ergodicity ($\hbar=1/\;$). At the left-hand side, we recognize ***magnetic*** Hamiltonian at hyperbolic plane.

Such functions can be analytically continued to a neighborhood of $\mathbb{C}^+$ in its complexification. The latter is just $\{(X;Y) : X;Y\in\mathbb{C}\}$. We establish asymptotic estimates for the growth of these continuations as $\to\infty$, and for de Rham currents given by their divisors.

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Superior resilience of non-Gaussian entanglement in local Gaussian semigroup quantum dynamics

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Keywords: quantum entanglement; Gaussian channel; semigroup dynamics; noisy attenuation; noisy amplification.

MSC2010 codes: 81P40, 81S22, 82C10, 81V80, 94A40

Problem setting and main result. Entanglement distribution task encounters a problem of how the initial entangled state should be prepared in order to remain entangled the longest possible time when subjected to local noises. In the realm of continuous-variable states and local Gaussian channels it is tempting to assume that the optimal initial state with the most robust entanglement is Gaussian too [1,2]; however, this is not the case [3,4]. In Ref. [5] we rigorously prove that specific non-Gaussian two-mode states remain entangled under the effect of deterministic local attenuation or amplification (Gaussian channels with the attenuation factor/power gain $_i$ and the noise parameter $_i$ for modes $i = 1;2$) whenever $_1\ ^2_2 + \ _2\ ^2_1 < \frac{1}{4}(\ _1 + \ _2)(1 + \ _1\ _2)$, which is a strictly larger area of parameters as compared to where Gaussian entanglement is able to tolerate noise. These results shift the "Gaussian world" paradigm in quantum information science (within which solutions to optimization problems involving Gaussian channels are supposed to be attained at Gaussian states).

Semigroup dynamics. A considered quantum channel $(\ ;\ )$ with fixed parameters and represents a snapshot of the dynamical map at a particular time moment t, which may correspond to a finite propagation time through a communication line. In a true time evolution the parameters and become functions of time t, (t) and (t). For instance, in the semigroup attenuation or amplification dynamics $= e^{Lt}$ with the generator L [6] we obtain the following dependencies:

$$(t) = e^{\quad t}; \qquad (t) = \quad (e^{\quad t} \quad 1)\left(n + \frac{1}{2}\right);$$

where the sign $+$ ( ) describes amplification (attenuation), $\quad 0$ is the process rate, and n is the average number of thermal photons in the environment. For such a semigroup dynamics the one-parameter family of maps $(\ (t);\ (t))$ is associated with a straight line in the parameter space $(\ ;\ )$.

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The Kossakowski Matrix and Strict Positivity of Markovian Quantum Dynamics

F. Franco [38]

We investigate the relationship between strict positivity of the Kossakowski matrix, irreducibility and positivity improvement properties of Markovian Quantum Dynamics. We show that for a Gaussian quantum dynamical semigroup strict positivity of the Kossakowski matrix implies irreducibility and, with an additional technical assumption, that the support of any initial state is the whole space for any positive time

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Quantum decoherence via Chernoff averages

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Keywords: Chernoff averages, Feynman-Chernoff iterations, quantum oscillator, decoherence.
MSC2020 codes: 47D06,28B10,60B12,47B80.

In this talk we study averages of Feynman-Chernoff iterations [1]

$$e^{-i\hat{H}_n\frac{t}{n}} \cdots e^{-i\hat{H}_1\frac{t}{n}}$$

for operator-valued quantum evolution functions.

The basic example is provided by Hamiltonians of the form

$$\hat{H}(t) = g(t)\hat{a}^{+}\hat{a} + f(t)\hat{a}^{+} + \overline{f(t)}\hat{a} + h(t), \tag{1}$$

where $g(t)$ and $h(t)$ are real-valued functions of time, $f(t)$ is a complex-valued function of time, $\hat{a}^{+}$ and $\hat{a}$ are the creation and annihilation operators.

In the works of Glauber [2], Meta and Sudarshan [3] it was shown that in the case of canonical commutation relations the formula (1) defines the general form of Hamiltonians, under which states that are initially coherent remain coherent during the time evolution. By coherent states we mean states corresponding to eigenvectors of the annihilation operator: $\hat{a}|z\rangle = z|z\rangle,\ z \in \mathbb{C}.$

Further note that if the Hamiltonian is of the form (1) than the action of the time evolution operator

$$\hat{S}(t) = T\left\{\exp\left(-i\int_0^t \hat{H}(\tau)\,d\tau\right)\right\},\ t > 0,$$

on any initial coherent state is described by the formulas:

$$|z(t)\rangle = \hat{S}(t)|z(0)\rangle,$$

$$z(t) = e^{-i\varphi(t)}z(0) - ie^{-i\varphi(t)}\int_0^t f(\tau)e^{i\varphi(\tau)}\,d\tau,$$

where $\varphi(t) = \int_0^t g(\tau)\,d\tau$. The symbol T denotes the operation of time-ordering.

In some problems [4,5] it becomes necessary to take into account the accumulated common phase factor. Substituting the vector $e^{i\alpha(t)}|z(t)\rangle$ into the Schrödinger equation leads to the equation

$$\left(i\frac{d\alpha(t)}{dt} + \frac{d}{dt}\right)|z(t)\rangle = -i\hat{H}(t)|z(t)\rangle,$$

from which it follows that

$$\alpha(t) - \alpha(0) = \int_0^t \langle z(\tau)|\hat{H}(\tau)|z(\tau)\rangle + i\int_0^t \langle z(\tau)|\frac{d}{d\tau}|z(\tau)\rangle,$$

where the first term on the right hand side is called a dynamic phase, and the second is called a geometric or Berry phase.

Thus the time evolution of the vectors of the Hilbert space H corresponding to coherent states under the action of the family of operators $\hat{S}(t)$ is completely described by the operator-valued function with values in the affine group of $\mathbb{R}^3 = \mathbb{C} \times \mathbb{R}$.

We consider the time evolution of quantum oscillator, which is given by compositions of random transformations described above, and the diffusion limit of such compositions in the sense of Feynman-Chernoff iterations. We provide the Fokker-Planck equation for the evolution

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of densities of mixed states, and numerically investigate the problem of decoherence of quantum states in interference experiments.

The talk is based on joint work with Y.N. Orlov and V.Z. Sakbaev.

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Dynamic law of large numbers for quantum stochastic filtering and related new nonlinear stochastic Schrodinger equations

V. N. Kolokoltsov [40]

As a far reaching extension of the quantum large number limit leading to the basic nonlinear Schrodinger equations, we derive the new nonlinear stochastic Schrödinger equations, as the limits of continuously observed and controlled systems of a large number of interacting quantum particles, evolving according to the Belavkin quantum filtering equation. This construction is a starting point for the quantum extension of the theory of quantum mean-field games. Our introduction of a new class of equations suggests many open problems concerning e.g. existence, uniqueness, regularity, etc. Ideas of the talk are taken from the papers

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On optimization of coherent and incoherent controls in one- and two-qubit open systems

O. V. Morzhin [41]

Keywords: quantum control, coherent control, incoherent control

MSC2020 codes: 81Q93, 34H05, 49Mxx

Control of quantum systems, e.g., individual atoms, molecules is an important direction in modern quantum technologies [1–5]. Often open quantum systems with Markovian dynamics are described via the Gorini–Kossakowski–Sudarshan–Lindblad (GKSL) master-equation, and controlling such a system is modelled in terms of coherent control entering in the system's Hamiltonian. However, there is known the approach (see the fundamental works [6, 7] and, e.g., [8–12]), where such a system's environment can be considered as a resource via introducing incoherent control in the superoperator of dissipation and also in the effective Hamiltonian.

The talk considers some one- and two-qubit open quantum systems whose dynamics is described via the GKSL master equation in the weak coupling limit (WCL) approach, and coherent u and incoherent n controls are used:

$$\frac{d\ (t)}{dt} = \quad i[H_0 + H_{\mathrm{e}\ ;n(t)} + H_{u(t)};\ (t)] + L_{n(t)}(\ (t)); \qquad (0) = \ _0; \tag{1}$$

where H_0, $H_{\mathrm{e}\ ;n(t)}$, and $H_{u(t)}$ are, correspondingly, some free, effective, and interaction Hamiltonians; density matrix of the system $(t)\ 2\ \mathbb{C}^{N \quad N}$ is a Hermitian positive semi-definite matrix, $(t) = \ ^{y}(t) \quad 0$, with unit trace, $\mathrm{Tr}\ (t) = 1$; $L_{n(t)}(\ (t))$ is the WCL type's superoperator of dissipation acting on (t); $[A; B]$ denote the commutator $[A; B] = AB + BA$ of operators $A; B$. Consider $N = 2$ and $N = 4$, i.e., correspondingly, for one- and two-qubit cases.

The talk discusses several directions related to some various control problems for the system (1). First, for the problem of generation of a given density matrix $_{\mathrm{target}}$ for the one-qubit system, we discuss, based on the article [13], a modification of the two-stage method [7] by using piecewise constant incoherent controls and the two-step gradient projection method at the first (incoherent) stage, where we obtain the possibility to decrease duration of this stage at the cost of complicating the first stage and losing the simplicity of the original method. Second, also for the one-qubit system, we consider such the steering control problem that initial and target density matrices have the same spectrum. We show when we can numerically obtain such a coherent control that, for some initial and target density matrices with the same spectrum, approximately solve the problem and can be used for such an another pair of initial and target density matrices that are related to the first pair due to the certain property. Also we show that increasing the dissipation rate breaks this possibility, and considering both coherent and incoherent controls can help here. This can be considered as a possible modification of the second stage of the two-stage method, and here we consider some special class of incoherent controls for avoiding large variations for each of them. Third, the talk considers the two-qubit system and the problems of minimizing the Hilbert-Schmidt distance between the final and target density matrices, maximizing the Hilbert-Schmidt overlap for them, steering the overlap to a given value [14, 15]. Here we outline the use of the Pontryagin maximum principle, gradient projection methods, stochastic optimization. For the problem of maximizing the overlap, we describe constructing some Krotov's type methods (in terms of density matrices) based on the special exact formulas for the increment of the objective functional [15].

Moreover, the talk notes, as an important direction, the problem of estimating the effectiveness of local and global methods for controlling one- and two-qubit systems.

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Environment as a resource for controlling quantum systems

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Keywords: quantum control, open quantum systems, incoherent control, engineered environment.

MSC2020 codes: 81Q93, 81S22

Introduction. Control of quantum systems is an important branch of modern quantum physics, whose development is motivated both by fundamental reasons and by existing and prospective applications to quantum technologies [1]. In many experimental circumstances, controlled quantum systems are interacting with the environment [2,3]. The environment is often considered as having deleterious effect on the ability to control the system. However, it can be also exploited as a useful resource. Various approaches for using the environment as a resource exist. In this talk we will discuss the *incoherent control* method proposed in [4]. This method, when combined with coherent control, was shown to provide approximate controllability of generic N–level open quantum systems in the set of all density matrices [5].

Incoherent control by the engineered environment. Density matrix ρ_t at time t of a quantum system under the action of coherent and incoherent controls evolves according to the master equation (we set Planck constant as $\hbar = 1$)

$$\frac{d\rho_t}{dt} = -i[H_0 + H_c(t), \rho_t] + \mathcal{L}_{n(t)}(\rho_t), \qquad \rho_{t=0} = \rho_0.$$

In this equation, H_0 is the free system Hamiltonian, $H_c(t)$ is the Hamiltonian describing interaction of the system with coherent control $u(t)$ [e.g., a laser field; a typical situation is when $H_c(t) = Vu(t)$], $n(t) \geq 0$ is generally time-dependent incoherent control (e.g., spectral density of incoherent photons). The key feature here is that the dissipator $\mathcal{L}_{n(t)}$ becomes dependent on incoherent control. Various physical forms of this dependence were considered in [4]. Equivalently, this master equation was written in [4] as a master equation with time dependent decoherence rates $\gamma_k(t)$,

$$\frac{d\rho_t}{dt} = -i[H_0 + H_c(t), \rho_t] + \sum_k \gamma_k(t) \mathcal{D}_k(\rho_t).$$

Here k denotes all possible different pairs of energy levels in the controlled system and $\mathcal{D}_k$ is a Gorini–Kossakowski–Sudarshan–Lindblad dissipator, for which two physical classes were exploited — incoherent photons and quantum gas, with two explicit forms of $\mathcal{D}_k$ derived in the weak coupling (describing atom interacting with photons) and low density (describing quantum system interacting with a quantum gas) limits, respectively [4]. Generally, coherent control can also enter in the dissipator, and in opposite, incoherent control also modifies the Hamiltonian via Lamb shift. Non-Markovian master equations can be considered for incoherent control as well.

Applications of incoherent control. The method of incoherent control was found to be successful when applied to various quantum systems. In [5], it was shown that for the explicit form of $\mathcal{D}_k$ derived in the weak coupling limit, generic N-level quantum systems subject to coherent and incoherent controls become approximately controllable in the set of density matrices. The original proposed scheme was significantly speed-up for a two-level case by minimizing time of the incoherent stage [6]. The set of reachable states for a single qubit was described analytically using methods of geometric control theory in [7]. Recently, incoherent

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control by the environment was combined with speed gradient approach to manipulate energy of a quantum oscillator interacting with the environment [8], where convergence of the differential form of the speed gradient approach to optimal solution was rigorously proved and moreover, the conditions which guarantee that the obtained incoherent control is physical (i.e., non-negative) were found. Various aspects of pure and mixed state preparation using coherent and incoherent controls were investigated also in two-qubit systems [9,10]. All of this show high capabilities of incoherent control [4] for controlling quantum systems.

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Optimization of state transfer and exact dynamics for two-level open quantum systems

V. N. Petruhanov [43], A. N. Pechen [44]

Keywords: quantum control; open quantum systems; qubit; incoherent control.

MSC2020 codes: 81Q93, 81S22, 49M05.

Quantum control which studies methods for manipulation of individual quantum systems is an important tool necessary for development of quantum technologies [1]. Often in experimental circumstances controlled systems can not be isolated from the environment, so that they are open quantum systems. Moreover, in some cases the environment can be used for actively controlling quantum systems, as for example in incoherent control [2,3]. While in some cases the solution for the optimal shape of the control can be obtained analytically, often it is not the case and various numerical optimization methods are needed. A large class of methods are gradient-based numerical optimization algorithms, one of which is GRadient Ascent Pulse Engineering (GRAPE) developed originally for design of NMR pulse sequences [4] and later applied to various problems, e.g. [5,6].

In this talk, we consider the state-to-state transfer control problem for an open two-level quantum system (qubit) whose evolution is governed by the GKSL master equation with coherent and incoherent controls [7,8]. General form of the GKSL master equation in the absence of controls was derived in particular in the weak coupling limit and in the stochastic limit of quantum theory. We consider the specific model of such master equation which includes coherent and incoherent controls. The state of the system is represented by a vector in the Bloch ball. We consider piecewise constant control as it commonly used in gradient optimization methods. Then we derive expressions for dynamics and objective functional gradient using matrix exponentials. Due to low dimension of the system, the corresponding $3 \quad 3$ matrix exponentials can be analytically diagonalized. For that we find eigenvalues and eigenvectors of the right-hand side matrix of the system evolution equation. Roots of the third order characteristic equation can be analytically found using the Cardano's formula. This enables obtaining exact form of matrix exponentials included in the dynamics and functional gradient expressions necessary for control landscape analysis.

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Feynman Integrals in Quantum 2D Gravity

E. T. Shavgulidze [45]

The enormous popularity of 2D gravity in the last several decades motivated by its role in string theory and studies of BH physics in the dimensional reduction approach has grown after realizing the Schwarzian nature of the JT dilaton gravity and the relation of this theory to SYK model.

The general form of the 2D gravity action up to the terms quadratic in curvature K is

$$\tilde{A} = c_0 \int \sqrt{G}\, d^2x + c_1 \int K \sqrt{G}\, d^2x + c_2 \int K^2 \sqrt{G}\, d^2x. \tag{8}$$

The first two terms do not determine the dynamics of 2D gravity. While the part of the action quadratic in the Gaussian curvature does.

Commonly it is transformed to the dilaton gravity action. An alternative way is to deal only with the geometric structures of the surface.

The action (8) is invariant under general coordinate transformations. Here, we reduce the set of coordinate transformations and consider the action restricted to the conformal gauge, where the metric of the 2D surface looks like

$$dl^2 = g(u, v)\, \left(du^2 + dv^2\right) = g(z, z)\, dz\, dz \qquad \sqrt{G} = g. \tag{9}$$

$$K = \frac{1}{2g} \quad \log g, \tag{10}$$

where stands for the Laplacian.

We consider the specific form of the action (8)

$$A = \frac{\ ^2}{2} \int_{d} (K + 4)^2\, g(z, z)\, dz\, dz = \frac{\ ^2}{2} \int_{d} (\quad)^2\, dz\, dz \tag{11}$$

where

$$= q \quad \log q + \frac{4}{q}, \qquad q = \frac{1}{\sqrt{g}}. \tag{12}$$

Now path integrals in the theory

$$\int \tilde{F}(g) \exp\{\ \tilde{A}(g)\} g\, dg \tag{13}$$

are path integrals

$$\int F(\ ) \exp\{\ A(\ )\} g\, d \tag{14}$$

over the Gaussian functional measure

$$(d\ ) = \frac{\exp\{\ A(\ )\} g\, d}{\int \exp\{\ A(\ )\} g\, d}. \tag{15}$$

We consider a model of 2D gravity with the action quadratic in curvature and represent path integrals as integrals over the $SL(2, \mathrm{R})$ invariant Gaussian functional measure. We reduce these path integrals to the products of Wiener path integrals and calculate the correlation function of the metric in the first perturbative order.

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On relation between exact and Markovian correlation functions for unitary dynamics with random Hamiltonian

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Keywords: Gorini-Kossakowski-Sudarshan-Lindblad equation; correlation functions; random Hamiltonian.

MSC2020 codes: 81S22, 82C10, 47B80

Introduction. In the theory of open quantum systems the Markovian approximation is very widespread. Usually it assumes the Gorini-Kossakowski-Sudarshan-Lindblad equation for density matrix dynamics and some specific formulae for correlation functions in terms of dynamical map for this equation [1]. These formulae are usually called regression formulae. We will call such a correlation functions the Markovian correlation functions, and we give more explicit definition of them below. Here we construct an explicit and simple example of a model, for which dynamics of the density matrix is defined by a dynamical semigroup with the Gorini-Kossakowski-Sudarshan-Lindblad generator, but exact correlation functions do not coincide with the Markovian correlation functions.

Model and definitions. We consider a unitary evolution with a random time-dependent Hamiltonian

$$\tilde{H}(t) = \frac{1}{2\sqrt{t}} H;$$

where H is a fixed n-dimensional Hermitian matrix, is a real random variable with the standard normal distribution.

Let $U(t)$ be a solution of the equation

$$\frac{d}{dt}U(t) = \quad i\tilde{H}(t)U(t)$$

for $t > 0$ such that $U(+0) = I$.

Definition 1. Suppose that A and B are $n \quad n$ matrices and $t > s > 0$. Then the exact correlation function $\langle A(t)B(s)\rangle$ is defined by the formula

$$\langle A(t)B(s)\rangle \quad \mathrm{E}\,\mathrm{Tr}\, A\mathcal{U}_t(\mathcal{U}_s)^{\;1}(B(\mathcal{U}_s \quad));$$

where $\mathcal{U}_t$ is a superoperator denfined as $\mathcal{U}_t(X) = U(t)X(U(t))^\dagger$ for an arbitrary $n \quad n$ matrix X.

Definition 2. Let us define

$$_t \quad \mathrm{E}\mathcal{U}_t.$$

Suppose that A, B_1, B_2 are $n \quad n$ matrices and $t > s > 0$. Then Markovian correlation function $\langle B_2(s)A(t)B_1(s)\rangle$ is defined by the formula

$$\langle B_2(s)A(t)B_1(s)\rangle_M \quad \mathrm{Tr}\, A \quad _t(\quad _s)^{\;1}(B_1(\quad _s \quad)B_2);$$

in particular for $B_2 = I$, $B_2 = B$

$$\langle A(t)B(s)\rangle_M \quad \mathrm{Tr}\, A \quad _t(\quad _s)^{\;1}(B(\quad _s \quad)).$$

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Theorem 1. The operator-valued function $\ _t$ has the explicit form

$$\ _t = e^{Lt};$$

where L has the Gorini-Kossakowski-Sudarshan-Lindblad form, namely,

$$L(\ ) = H\ \ H \quad \frac{1}{2}H^2 \quad \frac{1}{2}\ H^2:$$

The semigroup $\ _t$ defines evolution of the density matrix as $(t) = \ _t(\ (0))$.

Main result. The main result of the talk consists in the explicit formula, which expands the exact correlation functions in terms of Markovian correlation functions.

Theorem 2. Let A, B be $n \quad n$ matrices, $t > s > 0$, then

$$hA(t)B(s)i = hA(t)B(s)i_M + h(\ \ _{2(\sqrt{ts}\ \ s)}(A) \quad A)(t)B(s)i_M$$

$$\sum_{k=1}^{1} \frac{(\sqrt{ts} \quad t))^k}{k!} \sum_{m_1;m_2=0}^{k} \binom{k}{m_1}\binom{k}{m_2} h(H^{k \quad m_1})(s)(H^{k \quad m_2}\ \ _{2(\sqrt{ts}\ \ s)}(A)H^{m_2})(t)(H^{m_1}B)(s)i_M:$$

Thus, we have obtained non-trivial corrections to Markovian correlation functions for this simple model. These corrections are also repesented as sums of Markovian correlation functions, hence the exact correlation functions are still defined by Markovian correlation functions (but not only of the same operators), but there relation is much more complex than for the Markovian case.

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Higher order traps in quantum control landscapes

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Keywords: quantum control, control landscape, traps, quantum gate.

MSC2020 codes: 81P65, 81P16

Quantum control, that is control of single quantum systems such as atoms or molecules, attracts now high interest both due to fundamental reasons and applications in modern quantum technologies [1]. Controlled dynamics of an N-level closed quantum system (H_0, V) is described by the Schrödinger equation.

$$i\frac{dU_t^f}{dt} = (H_0 + f(t)V)U_t^f, \qquad U_{t=0}^f = \mathbb{I}. \tag{16}$$

Here H_0 and V ($[H_0, V] \neq 0$) are free and interaction Hamiltonians respectively (i.e., Hermitian $N \times N$-matrices), $f \in L_2([0, T]; \mathbb{R})$ is coherent control, and $T > 0$ is some target time. A typical quantum control problem can be formulated as the problem of maximizing the objective functional. In our talk, we consider the following Mayer type quantum control objective functionals:

1. Let O be a quantum observable (system's Hermitian operator) and ρ_0 an initial quantum density matrix (so that $\rho_0 \geq 0$ and $\mathrm{Tr}(\rho_0) = 1$). The objective functional for the expectation of the system observable O is:

$$J_O[f] = \mathrm{Tr}(OU_T^f \rho_0 U_T^{f\dagger}) \to \max. \tag{17}$$

2. The objective functional for generation of a quantum gate $W \in SU(N)$ is:

$$J_W[f] = \frac{1}{N^2}|\mathrm{Tr}(W^\dagger U_T^f)|^2 \to \max. \tag{18}$$

The goal of global optimization is to find a control which realizes global maximum of the objective. For global optimization, an important question for a controllable system is to establish whether or not the objective has trapping behaviour [2]. *Trap* for an objective functional is a point of local but not global optimum of this functional. The analysis for traps is important since traps, if they exist, determine the level of difficulty for the search for globally optimal controls. If $N = 2$ and $[H_0, V] \neq 0$ then the quantum system (H_0, V) is completely controllable. In this case, the absence of traps was proved in [3,4] for large times. In [5,6], some examples of third order traps were discovered for special N–level degenerate quantum systems with $N \geq 3$. Traps were discovered for some systems with $N \geq 4$ in [7].

In this talk, for the problem of controlled generation of single-qubit phase shift quantum gates we show that control landscape for small times is free of traps [8]. We also discuss the detailed structure of the quantum control landscape for this problem [9]. For the problem of maximizing or minimizing the expectation of a system observable O, we introduce the notion of *trap of n-th order*. We find the conditions under which the control landscape for a strongly degenerate controllable N-level system has trap of the order $2N - 3$ with $N \geq 3$ [10]. It is known that this special quantum system is completely controllable [11,12]. Properties of

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control landscapes for open quantum systems are related to optimization on complex Stiefel manifolds [13].

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5. Infinite-dimensional analysis, probability, stochastic processes and financial mathematics

A class of fractional Ornstein-Uhlenbeck processes mixed with a Gamma distribution

S. Bonaccorsi [49]

We aim to discuss the convergence of the empirical means of a sequence of realizations of the solution of a (fractional, in our case) stochastic differential equation (SDE) with random coefficients as a mean to construct new classes of stochastic processes. The talk is based on a recent paper by L. A. Bianchi, S. Bonaccorsi, L. Tubaro (Modern Stoch. Theory Appl. 10, no. 1, 37-57, 2023).

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New Applied Stochastic Models

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Keywords: Optimization, Limit behavior, Stability.

In order to study real-life processes or systems it is necessary to choose an appropriate mathematical model for their description, see, e.g. [1], [2]. The usual procedure includes the following steps: 1) Formulate a real problem. 2) Make assumptions. 3) Formulate a mathematical problem. 4) Solve the mathematical problem. 5) Interpret the solution. 6) Validate the model. If one establishes that the model describes correctly the real-life situation the solution can be used to explain, design, predict etc., that is, to make a necessary decision. Otherwise, one has to return to the second step (assumptions) and repeat the procedure once more. This explains, in particular, why there can exist a lot of models describing more or less precisely the same system. The applied probability models have input-output form, that is, they are specified by the following six-tuple $(T; Z; Y; U; \ ; L)$. Here T is a planning horizon, $Z = \{Z(t); t \in [0; T]\}$ is input process, whereas $Y = \{Y(t); t \in [0; T]\}$ and $U = \{U(t); t \in [0; T]\}$ are output and control processes, respectively. represents the system configuration and operation mode, hence, $X = \ (Z; Y; U)$ is the system state, while $L_T(U) = L(Z; Y; U; X; T)$ is objective function (target, valuation criterion, risk measure) evaluating the system performance quality. According to the choice of objective function there exist two main approaches, namely, reliability and cost ones, see, e.g., [3].

The aim of all investigations is to find the optimal control providing extremum (max or min) of the objective function. We are going to explain further procedures in terms of insurance, since it is the oldest applied probability research domain involving risk. So, the most frequently employed controls are reinsurance, investment, bank loans and dividends, see, e.g., [4]-[7]. We study the limit behavior of the insurance company capital under optimal control and carry out the sensitivity analysis of the models under consideration to small fluctuations of parameters and perturbations of underlying processes using the methods proposed in [8], [9].

It is interesting to mention that another interpretation of input and output processes enables us to use the obtained results for other application fields such as communications, inventory, finance, reliability or queueing theory.

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A mild Girsanov formula

G. Da Prato [51]

(Work in progress in collaboration with Enrico Priola (Pavia) and Luciano Tubaro (Trento))

ABSTRACT

We consider the SPDE

$$dZ = AZ + b(Z)dt + dW(t); \quad Z(0) = x$$

on a Hilbert space H, where $A : H \to H$ is self-adjoint, negative and such that $A^{\ 1}$ is of trace class, $b : H \to H$ is Lipschitz continuous and bounded, and W is a cylindrical process on H. Setting $P_T\varphi(x) = \mathrm{E}[\varphi(Z_x(T))]$ we prove, with the help of formula for nonlinear transformations of infinite dimensional Gaussian measures due to R. Ramer (*J. Functional Analysis*, **15**, 166–187, 1974), the identity

$$P_T\varphi(x) = \int_X \varphi(k(T) + e^{TA}x) \ (x;k)\, N_{Q_T}(dk); \tag{1}$$

where N_{Q_T} is the law of W in $L^2(0;T;H)$,

$$(x;k) = \exp\left\{ \ \tfrac{1}{2}|Q_T{}^{1/2} \ {}_x(k)|_X^2 + M \ (\ {}_x(k))\right\};$$

$$[\ {}_x(k)](t) = \int_0^t e^{(t \ s)A} b(k(s) + e^{sA}x)ds$$

and M is the adjoint of the Malliavin derivative in X. Finally, letting $T \to \infty$ in (1), we find an explicit formula for the invariant measure of P_T, which is ergodic, strongly mixing and absolutely continuous with respect the Gaussian measure $= N_{\ 1/2 A \ 1}$.

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Non-Equilibrium States in a Harmonic Crystal coupled to a Klein–Gordon Field

T. V. Dudnikova [52]

Keywords: harmonic crystal coupled to a Klein–Gordon field; Cauchy problem; random initial data; weak convergence of measures; Gibbs measures; energy current density; non-equilibrium states.

MSC2010 codes: 35Lxx, 60Fxx, 60G60, 82Cxx

In the talk, we discuss the long-time behavior of distributions of solutions for infinite-dimensional Hamiltonian systems and the existence of a nonzero heat flux in them. As a model, we consider a linear Hamiltonian system consisting of a real scalar Klein–Gordon field $\varphi(x)$ and its momentum $\pi(x)$, $x\in\mathbb{R}^d$, and a harmonic crystal described by the deviations $u(k)\in\mathbb{R}^n$ of the particles (atoms, ions, etc.) and their velocities $v(k)\in\mathbb{R}^n$, $k\in\mathbb{Z}^d$. The Hamiltonian functional of the coupled field–crystal system reads

$$H(\varphi,\pi,u,v)=\frac12\int\limits_{\mathbb{R}^d}\Big(|\nabla\varphi(x)|^2+m_0^2|\varphi(x)|^2+|\pi(x)|^2\Big)\,dx$$
$$+\frac12\sum_{k\in\mathbb{Z}^d}\Big(\sum_{k'\in\mathbb{Z}^d}u(k)\cdot V(k-k')u(k')+|v(k)|^2\Big)+\sum_{k\in\mathbb{Z}^d}\int\limits_{\mathbb{R}^d}R(x-k)\cdot u(k)\,\varphi(x)\,dx.$$

Here $m_0>0$, the coupled function $R(\cdot)\in\mathbb{R}^n$ is a smooth vector-valued function, exponentially decaying at infinity, "$\cdot$" denotes the inner product in $\mathbb{R}^n$, V is a real interaction matrix, $V(k)\in\mathbb{R}^n\times\mathbb{R}^n$, $d,n\ge 1$. This model can be considered as the description of the motion of electrons (so-called *Bloch electrons*) in the periodic medium which is generated by the ionic cores. Understanding of this motion is one of the central problem of solid state physics.

We study the Cauchy problem with the initial data $Y_0=(\varphi_0,\pi_0,u_0,v_0)$. We assume that Y_0 belong to the phase space $\mathcal{E}^s\equiv H^{s+1}\oplus H^s\oplus\ell^2\oplus\ell^2$, where $H^s\equiv H^s(\mathbb{R}^d)$ denotes the weighted Sobolev space, $\ell^2\equiv\ell^2(\mathbb{Z}^d)$ is the Hilbert space of vector-valued sequences $u(k)\in\mathbb{R}^n$, $k\in\mathbb{Z}^d$, with finite norm $\|\langle k\rangle^s u(k)\|_{\ell^2(\mathbb{Z}^d)}<\infty$, $\langle k\rangle:=\sqrt{k^2+1}$, $s<-d/2$.

We assume that Y_0 is a random function of the form $Y_0(p)=\sum_\pm\chi_\pm(p_1)Y_\pm(p)$, where $p=(p_1,\dots,p_d)\in\mathbb{P}^d\equiv\mathbb{R}^d\cup\mathbb{Z}^d$, $\chi_\pm$ are nonnegative cut-off functions equal to one for $\pm p_1>a$ and zero for $\pm p_1<-a$ with some $a>0$, the random functions $Y_\pm(p)$ have Gibbs distributions $g_\pm$, $\beta_\pm=T_\pm^{-1}$, with temperatures $T_\pm>0$. Given $t\in\mathbb{R}$, denote by μ_t the probability Borel measure in $\mathcal{E}^s$ that gives the distribution of the random solution $Y(t)\equiv(\varphi(\cdot,t),\pi(\cdot,t),u(\cdot,t),v(\cdot,t))$ with the random initial data Y_0. The main result is the following theorem.

Theorem. *The measures μ_t weakly converge to a Gaussian measure μ_∞ as $t\to\infty$ on the space $\mathcal{E}^s$, $s<-d/2$. The correlation matrix of μ_∞ is translation-invariant w.r.t. shifts in $\mathbb{Z}^d$. The explicit formulas for the limiting correlation functions are given.*

In non-equilibrium statistical mechanics, the heat flux is often calculated in models, which are an open system coupled to at least two reservoirs with different temperatures. These models differ in the description of the system, reservoirs and the type of interaction between them, see, e.g., [1]. Similar to these models, our system can be represented as a "system + two heat reservoirs", where "reservoirs" are described by the solutions $Y(p,t)$ with coordinates lying in two regions with $p_1\le -a$ and $p_1\ge a$, and an "open system" by the solutions with coordinates from the remaining part of the space. Initially, the reservoirs are assumed to be in thermal equilibrium with different temperatures T_- and T_+. The limiting energy current density is

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$J = \quad c(T_+ \quad T\ ;0;:::;0)$, $c > 0$, i.e., the heat flows (on average) from the "hot reservoir" to the "cold" one. Thus, we prove that there exist stationary non-equilibrium states, i.e., the probability limiting measures $\ _1$, in which there is a non-zero heat flux in the model under consideration.

For initial measures which are translation-invariant w.r.t. shifts in Z^d, the weak convergence of $\ _t$ was proved in [2]. The similar results were obtained in [3, 4] for the Klein–Gordon fields and in [5, 6] for the harmonic crystals.

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On a class of functionals Feynman integrable in the sense of analytic continuation

E. S. Kolpakov [53]

Keywords: Feynman integral; Wiener measure.

MSC2020 codes: 47D08, 35C15, 35J10

Introduction. This report is devoted to a method for calculating the Feynman integrals of some class of functional. In this case, the integration is carried out over some space E containing both continuous trajectories and trajectories with jumps. The integral over the space E is defined in the sense of analytic continuation. We also obtain a formula is that makes it possible to reduce the calculations of such an integral to calculations of some other integral with respect to the Wiener measure, and this integral is considered on the space of continuous trajectories.

The relationship between the integral over the space E and the integral over the space of continuous trajectories was first found by Belokurov and Shavgulidze [1], but the integral over the space E was considered not in the sense of analytic continuation.

We now recall the definition of the space E introduced in [2]:

Definition 1. $E = [_{n=0}^{1} X_n$, where X_n is the space of functions $x(t)$ of the form $x(t) = \sum_{j=1}^{n} \frac{1}{t\ \ t_j} + \ (t)$ where the function is Holder on $[0;1]$ with coefficient $\ 2\ (0;\frac{1}{2})$ and $\ (0) = \sum_{i=1}^{n} \frac{1}{t_i}$ and $\ (t_k) = \ \sum_{i \neq k} \frac{1}{t_k\ \ t_i}$ for $1\ \ k\ \ n$.

Definition 2. Consider the functional $f(x) = h(\int_0^1 (\int_0^{t_2} x(t_1)dt_1)\ '(t_2)dt_2)$ where $'$ - is a complex-valued continuously differentiable function and h analytic in the whole complex plane function is either of order at most 1 and of a finite type or is of order strictly less than 1. It was proved in [3] that this functional exists on functions of space E. Define G as the space of linear combinations of finite products of such functionals.

We now use the definition of the Feynman integral in terms of analytic continuation from the monographs by Smolyanov and Shavgulidze [4].

Theorem 1. Let $f\ 2$ G. The functional integral

$$I(\ ) = \frac{\int_E f(x) e^{\frac{1}{2}\ ^2 \int_0^1 (x'(t))^2 dt\ \ \int_0^1 x^4(t)dt + \frac{1}{3}x^3(1)} dx}{\int_E e^{\frac{1}{2}\ ^2 \int_0^1 (x'(t))^2 dt\ \ \int_0^1 x^4(t)dt + \frac{1}{3}x^3(1)} dx})$$

has an analytic continuation in the parameter to the domain

$$f\ \ j0\ \ \arg\ \ \ \frac{}{4};\frac{1}{2}\ \ j\ \ j\ \ 2g):$$

Acknowledgments. Author are thankful to E.T. Shavgulidze and O.G. Smolyanov.

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Averaging of random groups associated with random nonlinear differential equation

A. Malikov [54]

In this work, we study groups of transformations associated with nonlinear first-order partial differential equation

$$\frac{\partial u}{\partial t} = \quad \frac{\partial u}{\partial x} + u^2. \qquad (1)$$

Here   is random variable. Expected value of random group $U(t)$, $t \in \mathbb{R}$, associated with differential equation (1), is described. Low of large numbers for composition independent random group U is obtained.

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Semigroup methods in regularization of ill-posed stochastic problems

I. V. Melnikova[55]

Keywords: operator semigroup, generator, regularizing operator, stochastic equation, Wiener process

MSC2020 codes: 47A52 47D60 60J65 60H15

Introduction. The conference report is devoted to regularization of ill-posed stochastic Cauchy problems in Hilbert spaces:

$$du(t) = Au(t)dt + BdW(t); \; t \quad 0; \quad u(0) = \; ; \tag{19}$$

where the operator A, in general, is the generator of an R-semigroup in Hilbert space H, in particular, with $\;\; A$ generating a strongly continuous semigroup. The linear operator B acts from the space H, where the process $W = fW(t); t \quad 0g$ is defined in the form of series with respect to independent one dimensional Brownian motions, into the space H.

The need for regularization is connected with the fact that the operator A is not supposed to generate a strongly continuous semigroup and with the divergence of the series defining the infinite-dimensional Wiener process W.

We consider regularization of the problem (19) with the operator A, which is the generator of an R-semigroup in a Hilbert space H. The condition for A to be the generator of an R-semigroup in a Banach space means that the solution operators of the corresponding homogeneous Cauchy problem:

$$u^0(t) = Au(t); \; t > 0; \quad u(0) = \; ; \tag{20}$$

are generally unbounded in the space and defined only on some subset from the domain $D(A)$, but there is a family of bounded operators called a regularized semigroup or R-semigroup. This family gives a solution to some well-posed problem related to the homogeneous problem, but is not a semigroup in general.

The regularization of ill-posed stochastic Cauchy problems, as in the case of inhomogeneous deterministic problems, is closely related to the regularization of the corresponding homogeneous Cauchy problems.

The conference report consists of four sections.

Section 1 is devoted to the regularization of ill-posed homogeneous problems (20) with sectorial and half-strip operators A such that $\;\; A$ generates a strongly continuous semigroup. In continuation of earlier papers (see, e.g. [1], [2]) two types of regularizing operators $\mathrm{R}_{\;;t}$ are considered, that give fundamentally different error estimates of the exact solution to (20) from the solution to a regularized problem with initial data given with an error: $k \qquad k \quad$.

In section 2 a new approach to constructing regularizing operators $\mathrm{R}_{\;;t}$ in terms of $R\;$-semigroups is introduced. It is shown the connection between regularizing operators $\mathrm{R}_{\;;t}$ and $R\;$-semigroups depending on the regularization parameter $\;$. Nevertheless, the construction of such $R\;$-semigroups in the general case is not an easy problem.

In section 3 the Cauchy problem (20) with differential operators $A = A\left(i\frac{@}{@x}\right)$ is considered. Depending on whether A belongs to different classes in the Gelfand–Shilov classification, $R\;$-semigroups $fS\;(t); t \quad 0g$ with the generator A and matching regularizing operators $\mathrm{R}_{\;;t}$ are constructed.

Section 4 is devoted to correctness of infinite-dimensional stochastic problems and regularization of ill-posed stochastic problems.

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Representation of groups in infinite-dimensional Hilbert space equipped with an invariant measure

V. Zh. Sakbaev [56]

Keywords: finitely-additive measure; shift-invariant measure on an infinite-dimensional space; unitary representation of a group.

MSC2020 codes: 28C20, 28D05, 37A05

Finitely-additive measures invariant to the action of the group of shifts on a separable infinite-dimensional real Hilbert space are considered (see [1]). A considered invariant measure is locally finite, -finite, but it is not countably additive. The analog of ergodic decomposition of invariant finite-additive measure with respect to the group of shifts are obtained. The set of different invariant with respect to a group measures is parametrized by the obtained decomposition. A ring-ergodic component of this decomposition is used to obtain the irreducible representation of a group.

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Some Chebyshev type Inequalities for Riemann-Liouville type integral operator

A. Senouci [57]

Keywords: Chebyshev functional, integral inequalities, Riemann-Liouville fractional operator.

MSC2020 codes: 26D10, 26D15

Introduction. In this work, some weighted Chebyshev type inequalities are obtained by using a more general fractional integral operator, than the Riemann-Liouvile one.

Let $0 \quad a < b < 1$, f and g be two integrable functions on $[a; b]$ and

$$T(f; g) := \int_a^b f(t)g(t)dt \quad \frac{1}{(b \quad a)} \left(\int_a^b f(x)\,dx\right)\left(\int_a^b g(x)dx\right): \tag{1}$$

The Chebyshev functional (1) has many applications in numerical quadrature, transform theory, probability, study of existence of solutions of differential equations and in statistical problems.

In the following we give some basic definitions.

Definition 1. For $1 \quad p < 1$ we denote by $L_p := L_p(0; 1)$ the set of all Lebesgue measurable functions f such that

$$kfk_p = \left(\int_0^1 jf(x)j^p dx\right)^{\frac{1}{p}} < 1:$$

Definition 2. The Riemann-Liouville fractional integral operators of order $\quad 0$ of function $f(x)\ 2\ L_1[a; b]; \quad 1 < a < b < +1$ are defined by

$$J_{a+} f(x) = \frac{1}{(\ )} \int_a^x (x \quad t) \quad ^1 f(t)dt; \ x > a; \tag{2}$$

$$J_b \ f(x) = \frac{1}{(\ )} \int_x^b (t \quad x) \quad ^1 f(t)dt; \ x < b: \tag{3}$$

The following definition was introduced in [3].

Definition 3. Let $\quad > 0; \quad 1; 1 \quad p < 1$ and the integral operator $K_{u;v}^{\ ;}$ of the form

$$K_{u;v}^{\ ;} f(x) = \frac{v(x)}{(\ )} \int_0^x (x \quad t) \quad ^1 \left[\ln\left(\frac{x}{t}\right)\right] \quad ^1 f(t)\,u(t)dt; \tag{4}$$

defined from L_p to L_p space,with locally integrable non-negative weight functions u and v on $(0; 1)$.

Remark 1. If $v(x) = u(x) = 1; \quad = 1;$ the operator $K_{1;1}^{\ ;1}$ coincides with the classical Riemann-Liouville fractional integral operator.

The following theorem was proved in [2].

Theorem 1. Let f and g be two synchronous functions on $(0; 1):$ Then for all $t > 0; \quad > 0;$

$$J \ (fg)(t) \quad \frac{(\ +1)}{t} J \ f(t) J \ g(t): \tag{5}$$

The inequality (5) is reversed if the functions are asynchronous on $(0; 1):$

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The following theorem was proved in [1].

Theorem 2. Let $f f_i g_1 \;\; i \;\; n$ be n positive increasing functions on $(0; 1)$ then for all $x > 0; \quad > 0;$

$$J\left(\prod_{i=1}^{i=n} f_i\right)(x) \qquad (J \;\; (1)(x))^{(1 \quad n)} \prod_{i=1}^{i=n} J \;\; f_i(x): \tag{6}$$

To simplify we denote by $\mathrm{K} := \mathrm{K}_{u;v}^{\;;}$; and $k(x; t) := (x \quad t) \quad ^1 \ln \quad ^1 \left(\frac{x}{t}\right) \not\in 0;$ thus the integral operator in the inequality (4) takes the following form

$$\mathrm{K} f(x) = \frac{v(x)}{(\;)} \int_0^x k(x; t)\, f(t)\, u(t) dt; \qquad x > 0: \tag{7}$$

Theorem 3. Let $f; g$ be two synchronous functions on $(0; 1);$ u and v locally integrable non-negative weight functions.Then

$$\mathrm{K}(fg)(x) \qquad (\mathrm{K}(1)) \quad ^1 \; \mathrm{K} f(x)\, \mathrm{K} g(x); \tag{8}$$

where $\mathrm{K}(1)(x) = \frac{v(x)}{(\;)} \int_0^x k(x; t)\, u(t)\, dt:$

The inequality (8) is reversed if the functions are asynchronous on $(0; 1):$

Remark 2. By applying Theorem 3, for $v(x) = u(x) = 1; \quad = 1;$ we obtain Theorem 1.

Theorem 4. Let $f f_i g_1 \;\; i \;\; n$ be n positive increasing functions on $[0; 1[$ $\;u$ and v locally integrable non-negative weight functions, then for all $x > 0$

$$\mathrm{K}\left(\prod_{i=1}^{i=n} f_i\right)(x) \qquad (\mathrm{K}(1)(x))^{(1 \quad n)} \prod_{i=1}^{i=n} \mathrm{K} f_i(x): \tag{9}$$

Remark 3. By applying Theorem 4, for $v(x) = u(x) = 1; \quad = 1;$, we obtain Theorem 2.

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Numerical Solutions for Resolving Groups or Semigroups of Operators for Nonclassical Equations in the Space of Di□erential Forms

D. E. Shafranov [58], G. A. Sviridyuk [59]

Keywords: di□erential forms; Riemannian manifold; Laplace – Beltrami operator.

MSC2020 codes: MSC 35R60

Introduction.

Consider the following equations:

– the Barenblatt–Zheltov–Kochina equation [1] $(\quad)u_t = \quad u;$ which is a model of dynamics of a fluid filtering in a fractured-porous environment;

– the Dzektser equation [2] $(1 \quad)'_t = \quad ' \quad {}^2 ';$ which is a model of flow of a viscous-elastic incompressible zero-order Kelvin–Voigt fluid in the first approximation;

– the Ginzburg – Landau equation [3] $(\quad)u_t = \quad u + id \quad {}^2u$ from the phenomenological theory of superconductivity.

In the functional spaces U, F chosen by us, this equations are reduced to the linear equation of Sobolev type

$$L\underline{u} = Mu$$

with the irreversible operator L.

Earlier in school on Sobolev type equations the Cauchy problem and the Showalter–Sidorov problem

$$u(0) = u_0; P(u(0) \quad u_0) = 0$$

for abstract equations were considered.

We propose a transition of linear equation of Sobolev type to the stochastic Sobolev type equations

$$L \quad = M$$

with the condition

$$(0) = \quad {}_0; P(\quad (0) \quad {}_0) = 0$$

in spaces of Wiener stochastic processes in the case of an abstract $(L; p)$-bounded operator M, $(L; p)$-sectorial operator M and $(L; p)$-radial operator M, respectively. Since Wiener processes are continuous, but non-differentiable in the usual sense at each point, we use the Nelson–Gliklikh derivative. In this article, we study numerical solutions to all three equations (the Barenblatt – Zheltov – Kochina equation, the Dzektser equation and the Ginzburg – Landau equation in spaces of differential forms defined on a torus.

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On convergence rate bounds for linear and nonlinear Markov chains

A. Yu. Veretennikov [60]

Keywords: Markov chains, nonlinear Markov chains; small perturbations; uniform ergodicity; convergence rate; markovian coupling; spectral radius.

MSC2020 codes: 60J10, 60J05, 60J99, 37A30

A new approach for evaluating convergence rate for general linear and nonlinear Markov chains (MC) will be presented, on the base on the recently developed spectral radius technique for linear MC and on the idea of small nonlinear perturbations for nonlinear MC, [2]–[6].

For *linear MC* this approach uses a so called markovian coupling which provides naturally a certain sub-stochastic matrix or operator, say, V, and the rate of convergence under investigation is determined by its spectral radius $r(V)$. This value is in all cases no greater than the "supremum norm" $\|V\|$, which norm coincides with the so called Markov – Dobrushin ergodic coefficient. The latter coefficient was introduced by Markov for finite matrices in 1906, then it was used by Kolmogorov in 1938, and later by Dobrushin in 1956 in their famous papers; in the literature it is often called Dobrushin's ergodic coefficient, although, in the opinion of the author the name of Markov as the founder of this characteristic is a must. Since the spectral radius is always no greater and often is strictly less than the norm of the operator, the new bound is usually better than the one due to Markov and Kolmogorov: see examples in [5].

To evaluate the rate of convergence for *nonlinear MC,* a new important additional characteristic was proposed in [1]. It was later extended in [3], and eventually a way of using it for nonlinear MC in a combination with the spectral radius approach via small perturbations from linear ones was offered in [6]. The structure of the operator V will be explained in the talk.

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6. Related topics

Clark measures and composition operators in several variables

E. Doubtsov [64]

Keywords: Clark measures; compact composition operators; essential norms.

MSC2020 codes: 47B33, 32A35, 46E27

Introduction. Let B_n denote the open unit ball of $\mathbb{C}^n$, $n \ge 1$, and let ∂B_n denote the unit sphere. We also use symbols $\mathbb{D}$ and $\mathbb{T}$ for the unit disk B_1 and the unit circle ∂B_1, respectively.

Given $k \in \mathbb{N}$ and $n_j \in \mathbb{N}$, $j = 1, 2, \dots, k$, let

$$D = D[n_1, n_2, \dots, n_k] = B_{n_1} \times B_{n_2} \times \dots \times B_{n_k} \subset \mathbb{C}^{n_1+n_2+\dots+n_k}.$$

Model examples of D are B_n and the polydisk $\mathbb{D}^n$. Let $C(z,\zeta) = C_D(z,\zeta)$ denote the Cauchy kernel for D. Let ∂D denote the distinguished boundary $\partial B_{n_1} \times \partial B_{n_2} \times \dots \times \partial B_{n_k}$ of D. Then

$$C_D(z,\zeta) = \prod_{j=1}^{k} \frac{1}{(1-\langle z_j, \zeta_j\rangle)^{n_j}}, \quad z = (z_1, z_2, \dots, z_k) \in D, \quad \zeta = (\zeta_1, \zeta_2, \dots, \zeta_k) \in \partial D,$$

where $z_j = (z_{j,1}, z_{j,2}, \dots, z_{j,n_j}) \in B_{n_j}$ and $\zeta_j = (\zeta_{j,1}, \zeta_{j,2}, \dots, \zeta_{j,n_j}) \in \partial B_{n_j}$. The corresponding Poisson type kernel is given by the formula

$$P(z,\zeta) = \frac{C(z,\zeta)C(\zeta,z)}{C(z,z)}, \quad z \in D, \quad \zeta \in \partial D.$$

Clark measures. Let $M(\partial D)$ denote the space of complex Borel measures on ∂D. Given an $\alpha \in \mathbb{T}$ and a holomorphic function $\varphi : D \to \mathbb{D}$, the quotient

$$\frac{1-|\varphi(z)|^2}{|\alpha-\varphi(z)|^2} = \mathrm{Re}\left(\frac{\alpha+\varphi(z)}{\alpha-\varphi(z)}\right), \quad z \in D,$$

is positive and pluriharmonic. Therefore, there exists a unique positive measure $\sigma_\alpha = \sigma_\alpha[\varphi] \in M(\partial D)$ such that

$$P[\sigma_\alpha](z) = \mathrm{Re}\left(\frac{\alpha+\varphi(z)}{\alpha-\varphi(z)}\right), \quad z \in D.$$

After the seminal paper of Clark [3], various properties and applications of the measures on the unit circle $\mathbb{T}$ have been obtained; see [1] for further details and references in several variables.

Let σ denote the normalized Lebesgue measure on ∂D. Specific properties of Clark measures are illustrated by the following theorem on disintegration of Lebesgue measure.

Theorem 1. Let $\varphi : D \to \mathbb{D}$ be a holomorphic function and let $\sigma_\alpha = \sigma_\alpha[\varphi]$, $\alpha \in \mathbb{T}$. Then

$$\int_{\mathbb{T}} \int_{\partial D} f\, d\sigma_\alpha \, dm_1(\alpha) = \int_{\partial D} f\, d\sigma$$

for all $f \in C(\partial D)$.

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Essential norms of composition operators. Let $Hol(D)$ denote the space of holomorphic functions in D. For $0 < p < \infty$, the classical Hardy space $H^p = H^p(D)$ consists of those $f \in Hol(D)$ for which

$$\|f\|^p_{H^p} = \sup_{0<r<1} \int_{\partial D} |f(r\zeta)|^p \, d\sigma(\zeta) < \infty.$$

Each holomorphic function $\varphi : D \to \mathbb{D}$ generates the composition operator $C_\varphi : Hol(\mathbb{D}) \to Hol(D)$ by the following formula:

$$(C_\varphi f)(z) = f(\varphi(z)), \quad z \in D.$$

It is well known that C_φ maps $H^2(\mathbb{D})$ into $H^2(D)$. So, a natural problem is to characterize the compact operators $C_\varphi : H^2(\mathbb{D}) \to H^2(D)$. A more general problem is to compute or estimate the essential norm of the composition operator under consideration. For the unit disk $\mathbb{D}$, a solution to this problem in terms of the Nevanlinna counting function was given in the seminal paper of Shapiro [4]. A solution in terms of the family $\sigma_\alpha[\varphi]$, $\alpha \in \mathbb{T}$, was later obtained by Cima and Matheson [2]. Extending the theorem of Cima and Matheson to several variables, we prove the following result:

Theorem 2. Let $\varphi : D \to \mathbb{D}$ be a holomorphic function. Then the essential norm of the composition operator $C_\varphi : H^2(\mathbb{D}) \to H^2(D)$ is equal to the following quantity:

$$\sqrt{\sup\{\|\sigma_\alpha^s\| : \alpha \in \mathbb{T}\}},$$

where σ_α^s denotes the singular part of the Clark measure $\sigma_\alpha = \sigma_\alpha[\varphi]$.

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On entropy correct spatial discretizations for 1D regularized systems of equations for gas mixture dynamics

A. S. Fedchenko [65]

Keywords: regularized equations for gas mixture dynamics; multi-velocity and multi-temperature gas mixture; one-velocity and one-temperature gas mixture; nonstandard symmetric three-point spatial discretization; semi-discrete entropy balance equation.

MSC2020 codes: 65M06, 76M20, 76T99

Introduction. Multicomponent gas mixtures are widespread in nature and industry. Therefore, dynamics of these mixtures is of great theoretical and applied interest. The compressible multicomponent gas mixture dynamics is described by various complicated systems of equations and under different assumptions, for example, in [1]. For considered models, the fulfillment of the entropy balance equation with non-negative entropy production plays the key role in both their physical and mathematical aspects. This property confirms physical correctness of the derived equations and allows one to prove basic a priori estimates of solutions.

Numerical simulation of the gas dynamics is performed by research teams around the world, and a large number of numerical methods was proposed during last decades, for example, in [2]. Methods based on preliminary regularizations of these equations include quasi-gasdynamic (QGD) and quasi-hydrodynamic (QHD) regularizations which are presented, in particular, in [3]. The QGD and QHD equations for the general (multi-velocity and multi-temperature) as well as one-velocity and one temperature gas mixture dynamics were developed and thoroughly studied, and the validity of the entropy balance equations with non-negative entropy production for these systems was proved, see [3-6].

In this report, one-dimensional regularized systems of equations for the general and one-velocity and one-temperature compressible multicomponent inert gas mixture dynamics are considered. Two types of the regularization are studied, and the entropy balance equation is obtained in both cases. The discretization from [7] is generalized, and new nonstandard symmetric three-point spatial discretizations are performed. The suggested discretizations are conservative in mass, momentum, and total energy. Semi-discrete balance equations for the mass, kinetic and internal energies of the mixture are derived as well. The discretizations also satisfy semi-discrete counterparts of the entropy balance equations, and the property of non-negativity of the entropy-production is also proven. The basic discretization in the one-velocity and one-temperature case is constructed by aggregation of the discretization in the case of general mixture, which is a new approach. In addition, an adequate discretization is performed for the terms describing the diffusion fluxes between the mixture components to ensure the non-negative entropy production. The results are obtained together with A.A. Zlotnik and are published in [8].

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On the uniqueness class and the correctness class of one fourth-order partial differential equation from the theory of heat transfer

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Keywords: uniqueness class; correctness class; heat conduction; Cauchy problem; parabolic equation; biparabolic equation.

MSC2010 codes: 35A01, 35A02, 35E20, 35K30, 80A99

Introduction. In the article [1], in order to describe heat and diffusion processes, a new fourth-order partial differential equation was introduced

$$\lambda_1(\partial_t - a^2\Delta)u(t,\mathrm{x}) + \lambda_2(\partial_t - a^2\Delta)^2u(t,\mathrm{x}) = f(t,\mathrm{x}), \tag{1}$$

where $\mathrm{x} = (x_1, x_2, \dots, x_n)$, λ_1 and λ_2 are some real parameters, $a > 0$ is a physical constant characteristic of the medium, and Δ is the Laplace operator. Also, in the paper [1], the solution of the Cauchy problem for Eq. (1) was formally constructed in the one-dimensional case. However, in the article [1], the most important thing about the Cauchy problem for equation (1) is not presented: the uniqueness class and the correctness class.

Main result. The uniqueness class for the Cauchy problem for Eq. (1) consists of functions g which satisfy the inequality

$$g(\mathrm{x}) \leqslant C\exp\left(b|\mathrm{x}|^2\right). \tag{2}$$

The correctness class for the Cauchy problem for Eq. (1) is the is the class of locally integrable functions g, which satisfy the inequality (2).

Thus, Eq. (1) does not improve the uniqueness class and correctness class of the heat equation [3].

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Resonance in oscillators with bounded nonlinearities

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Keywords: Nonlinear resonance, self-consistent source, oscillatory systems with bounded nonlinearity

MSC2020 codes: 34A34

Abstract. The present paper discusses a method for finding self-consistent external influences on a nonlinear oscillator that lead to the phenomenon of resonance as in the linear case. It is shown that for bounded nonlinear systems it is possible to find such a self-consistent external force. To illustrate the search for self-consistent external influences, the simplest nonlinear systems are selected.

Introduction. Resonant phenomena in linear oscillatory systems are well studied and described in all books on general physics and oscillation theory. If the external force frequency coincides with one of the partial natural frequencies of the linear system, the oscillation amplitude in the absence of attenuation increases according to the linear law and can reach significant values, thus, leading to the structure destruction. In the case of nonlinear systems, the monochromatic effect does not lead to a significant increase in the oscillation amplitude, since their frequency depends on the amplitude and, consequently, the equality of the frequencies of the external force and natural oscillations is violated. This problem was encountered during the construction of the first cyclotrons [1], [2]. As a result, the resonance curve becomes limited and asymmetric with respect to the linear oscillation frequency. This process in weakly nonlinear systems is also well described in literature; see, for example, [3], [4], [5], [6].

One of the ways to overcome the movement of the natural frequency of a nonlinear system from resonance is to control the external force by adjusting its frequency to the local natural frequency. Such a mechanism is called autoresonance, it has become widespread in discrete [7] and distributed systems [8], [9].

The purpose of this study is to search for self-consistent external influences that make it possible to swing oscillations in nonlinear systems. Here we will limit ourselves to the simplest nonlinear oscillator models and show that it is possible to select limited external influences that lead to the resonant phenomena similar to those existing in the linear system.

Self-consistent source in a bounded nonlinear oscillator. Let us consider the following bounded nonlinear system which are described by the equation:

$$\frac{d^2u}{dt^2} + u + F(u) = 0 \tag{1}$$

where $F(u)$ is some continuous nonlinear function. We will consider the nonlinearity bounded by $|F(u)| < F_0$, $F_0 \in R$, $F(0) = 0$ and $F(u) \to \alpha u^2$ for $u \to 0$, $\alpha \in R$, for the nonlinearity to be infinitely small of a higher order than u, in order for a linear resonance to be obtained in this neighborhood with a sinusoidal effect with a unit frequency.

Then, instead of solving this equation, we can assume that we will be able to obtain a resonant solution, as in the linear case, due to an some external force $f(t)$.

$$\frac{d^2u}{dt^2} + u + F(u) = 2\cos t + f(t) \tag{2}$$

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Assuming that the solution of this equation is $u(t) = t\sin t$, we can find out expression for the function $f(t)$:

$$f(t) = F(t\sin t)$$

The resulting external force is continuous and limited, but not monochromatic, as in the linear case, but with a wide spectrum.

Examples of finding a self-consistent source. As an illustration of this approach, we chose a system with sinusoidal nonlinearity (3), as well as with saturation-type nonlinearity (4).

$$\frac{d^2u}{dt^2} + u + a\sin^2 bu = 0 \tag{3}$$

The phase portrait of system (3) consists of a finite number of alternating centers and saddles, the number of which depends on the parameters of this system. In this case, the external effect will look like this

$$f(t) = a\sin^2(bt\sin t) \tag{5}$$

It is shown in [10] that the amount of energy to maintain this system in a state of resonance increases linearly over time, and the external force becomes more and more high-frequency.

Along with sinusoidal nonlinearity, we also analyzed an example of saturation type nonlinearity. The saturation nonlinearity systems are very common in technical applications [11], [12].

$$\frac{d^2u}{dt^2} + u + \frac{au^2}{1+b^2u^2} = 0 \tag{4}$$

The phase portrait in this case has only 2 possible positions - it is only the center or 2 centers and the saddle. And in this case external force will be

$$f(t) = a\frac{t^2\sin^2 t}{1+b^2t^2\sin^2 t} \tag{6}$$

We have numerically shown that with a small deviation of the parameters of the external force from the parameters of the original system, it will still cause resonance. It is shown that as the amplitude of the nonlinear saturation function increases, the system becomes more sensitive to changes in the amplitude of a self-consistent external force.

In our opinion, the search and study of resonance in nonlinear isochronous systems, which are not rare or exceptional examples of nonlinear systems, is an interesting subject for subsequent research.

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Phase volume invariants of dynamical systems with dissipation

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Keywords: dynamical system; dissipation; transcendental first integral; tensor invariant.

MSC2010 codes: 37C, 37F

Tensor invariants (differential forms) for homogeneous dynamical systems on tangent bundles of smooth finite-dimensional manifolds are presented. The connection between the presence of these invariants and the full set of first integrals necessary for the integration of geodesic, potential, and dissipative systems is shown. The force fields introduced into the considered systems make them dissipative with dissipation of different signs and generalize previously considered force fields.

It is well known [1, 2, 3] that a system of differential equations can be completely integrated when it has a sufficient number of not only first integrals (scalar invariants) but also tensor invariants. For example, the order of the considered system can be reduced if there is an invariant form of the phase volume. For conservative systems, this fact is natural. However, for systems having attracting or repelling limit sets, not only some of the first integrals, but also the coefficients of the invariant differential forms involved have to consist of, generally speaking, transcendental (in the sense of complex analysis) functions [4, 5, 6].

For example, the problem of a n-dimensional pendulum on a generalized spherical hinge placed in nonconservative force field leads to a system on the tangent bundle of the $(n-1)$-dimensional sphere with a special metric on it induced by an additional symmetry group. Dynamical systems describing the motion of such a pendulum have the various dissipation, and the complete list of first integrals consists of transcendental functions expressed in terms of a finite combination of elementary functions. There are also problems concerning the motion of a point over n-dimensional surfaces of revolution, the Lobachevsky spaces, etc. The results obtained are especially important in the context of a nonconservative force field present in the system.

In this activity, we present tensor invariants for homogeneous dynamical systems on tangent bundles of smooth finite-dimensional manifolds. The relation between the existence of these invariants and the existence of a complete set of first integrals necessary for the integration of geodesic, potential, and dissipative systems is shown. The force fields introduced into the considered systems make them dissipative with dissipation of different signs and generalize previously considered force fields.

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Statistical properties of synchronous soliton collisions

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Keywords: soliton collisions, Korteweg-de Vries equation, statistical moments.

MSC2020 codes: 00A79, 35C08, 35Q53

The concept of soliton gas, soliton or integrable turbulence appeared shortly after the discovery of outstanding properties of solitons – isolated waves, maintained by a stable balance between the counteracting effects of nonlinearity and dispersion. They exhibit an exceptional stability which becomes apparent through elastic interactions with other solitons and quasilinear waves. The existence of solitons originates from the fact of integrability of a series of nonlinear equations, such as the Korteweg-de Vries equation or the Nonlinear Schrödinger equation, by the Inverse Scattering Transform. From this perspective, solitons correspond to waves of the discrete spectrum of the associated scattering problem, which do not disperse and represent the large-time asymptotic solution of the Cauchy problem for localized initial conditions. Solitons exist in various fields of physics: hydrodynamics, plasma and optics, and play a particular role in the dynamics of nonlinear waves. They are deeply intertwined with the problem of the emergence of rogue waves, represented by extreme deviations from the average wave amplitude.

For the description of the soliton gas dynamics kinetic equations were derived [1,2]. They characterize the transport of the soliton spectral density, but due to the violation of the wave linear superposition principle, do not provide information about the wave solution itself (which can be water surface displacement, intensity of electromagnetic fields, etc.). In particular, the questions about the probability distribution for wave amplitudes or about the values of the wave field statistical moments remain unanswered. Multisoliton solutions, which can be formally written in a closed form using the Inverse Scattering Transform or related methods for integrable equations, are very cumbersome, what makes their analytical and even numerical analysis difficult. The direct numerical simulation of evolution equations is commonly used to study the soliton gas evolution, which also becomes complicated in the case of a dense gas (i.e. when many solitons interact simultaneously) [3].

The focus of this study is made on the dynamics of soliton interactions governed by the classic Korteweg – de Vries (KdV) equation, which has the standard dimensionless form

$$u_t + 6uu_x + u_{xxx} = 0, \tag{21}$$

where the real functions $u(x;t)$ describes the wave field, the variable $x \in (-\infty; +\infty)$ serves as a space coordinate, $t \in (-\infty; +\infty)$ is the time. Its exact N-soliton solution $u_N(x;t)$ can be obtained via consecutive Darboux transformations, which allow a compact representation

$$u_N(x;t) = 2\frac{\partial^2}{\partial x^2} \ln W_N(\psi_1, \psi_2, \ldots, \psi_N). \tag{22}$$

Here $W(\psi)$ denotes the Wronskian for N "seed" functions $\psi_{2s-1} = \cosh \varphi_{2s-1}$, $\psi_{2s} = \sinh \varphi_{2s}$ for integer $s \geq 1$, where the phases are $\varphi_j = k_j(x - V_j t - x_j)$, $j = 1, 2, \ldots, N$. The parameters k_j specify the soliton amplitudes $A_j = 2k_j^2$ and velocities $V_j = 4k_j^2$, while the constants x_j are responsible for the respective positions of solitons at a given time. The solution (22) is always positive, $u_N(x;t) > 0$. The use of an ultra-high-precision procedure made it possible to

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compute the exact N-soliton solutions (22) when N is large [4] and to calculate their statistical moments $\mu_n(t) = \int_{-\infty}^{+\infty} u_N^n(x;t)dx$, $n \in \mathbb{N}$, with high accuracy.

In this work we present a general idea that dense ensembles of KdV-type solitons of the same sign can be considered as strongly-nonlinear / small-dispersion wave states, what allows to express the statistical moments in terms of the spectral parameters of the associated scattering problem. A particular case when dense soliton states can occur is synchronous multisoliton collisions (see Fig. 1), for which the reference locations of all the solitons at $t = 0$ coincide with the coordinate origin, $x_j = 0$, $j = 1, \ldots, N$. This property can be formalized through the following symmetry condition, $u_N(-x; -t) = u_N(x; t)$. The soliton amplitudes are set decaying exponentially, so that they form a geometric series with the ratio $d > 1$, $A_j = 1/d^{j-1}$, $j = 1, \ldots, N$.

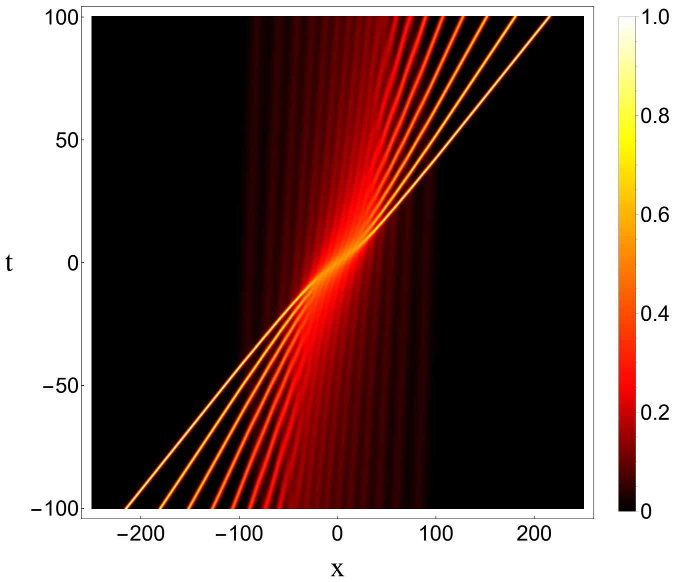

Figure 1: Interaction of $N = 20$ KdV solitons with $d = 1.2$

Time dependences of statistical moments are investigated for many-soliton solutions. It is shown that during the interaction of solitons of the same sign the wave field is effectively smoothed out. When d is sufficiently close to 1, and N is large, the statistical moments remain approximately constant within long time spans, when the solitons are located most densely. This quasi-stationary state is characterized by greatly reduced statistical moments and by the density of solitons close to some critical value. This state may be treated as the small-dispersion limit, what makes it possible to analytically estimate all high-order statistical moments. While the focus of the study is made on the Korteweg–de Vries equation and its modified version, a much broader applicability of the results to equations that support soliton-type solutions is discussed.

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Structure of essential spectra and discrete spectrum of the energy operator of six-electron systems in the Hubbard model. third triplet state

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Keywords: Hubbard Model, Six-electron systems, Spectra, Essential Spectra, Discrete Spectrum, Triplet State, Octet State, Quintet State

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Introduction. In the 1963, three papers [1]-[3], where a simple model of a metal was proposed that has become a fundamental model in the theory of strongly correlated electron systems, appeared almost simultaneously and independently. In that model, a single nondegenerate electron band with a local Coulomb interaction is considered. The model Hamiltonian contains only two parameters: the parameter B of electron hopping from a lattice site to a neighboring site and the parameter U of the on-site Coulomb repulsion of two electrons. In the secondary quantization representation, the Hamiltonian can be written as $H = B\sum_{m,\tau,\sigma} a^+_{m,\sigma} a_{m+\tau,\sigma} + U\sum_m a^+_{m,\uparrow}a_{m,\uparrow}a^+_{m,\downarrow}a_{m,\downarrow},$ where $a^+_{m,\sigma}$ and $a_{m,\sigma}$ denote Fermi operators of creation and annihilation of an electron with spin σ on a site m and the summation over τ means summation over the nearest neighbors on the lattice. The model proposed in [1]-[3] was called the Hubbard model after John Hubbard, who made a fundamental contribution to studying the statistical mechanics of that system. The Hubbard model is currently one of the most extensively studied multielectron models of metals [4]. The spectrum and wave functions of the system of two electrons in a crystal described by the Hubbard Hamiltonian were studied in [4]. The spectrum and wave functions of the system of three electrons in a crystal described by the Hubbard Hamiltonian were studied in [5]. In the three-electron systems are exists quartet state, and two type doublet states.

Hamiltonian of the system. We consider the energy operator of six-electron systems in the Hubbard model and describe the structure of the essential spectra and discrete spectrum of the system for third triplet state in the lattice. The Hamiltonian of the chosen model has the form $H = A\sum_{m,\gamma} a^+_{m,\gamma} a_{m,\gamma} + B\sum_{m,\tau,\gamma} a^+_{m,\gamma} a_{m+\tau,\gamma} + U\sum_m a^+_{m,\uparrow}a_{m,\uparrow}a^+_{m,\downarrow}a_{m,\downarrow}.$ Here A is the electron energy at a lattice site, B is the transfer integral between neighboring sites (we assume that $B > 0$ for convenience), $\tau = \pm e_j,\ j = 1,2,\ldots,\nu,$ where e_j are unit mutually orthogonal vectors, which means that summation is taken over the nearest neighbors, U is the parameter of the on-site Coulomb interaction of two electrons, γ is the spin index, $\gamma = \uparrow$ or $\gamma = \downarrow$, $\uparrow$ and $\downarrow$ denote the spin values $\frac{1}{2}$ and $-\frac{1}{2},$ and $a^+_{m,\gamma}$ and $a_{m,\gamma}$ are the respective electron creation and annihilation operators at a site $m \in Z^\nu.$

In the six electron systems has a octet state, and quintet states, and triplet states, and singlet states. The energy of the system depends on its total spin $S.$ Hamiltonian H commutes with all components of the total spin operator $S = (S^+, S^-, S^z)$, and the structure of eigenfunctions and eigenvalues of the system therefore depends on $S.$ The Hamiltonian H acts in the antisymmetric Fo'ck space $\mathcal{H}_{as}$.

Six-electron third triplet state in the Hubbard model.

Let φ_0 be the vacuum vector in the space $\mathcal{H}_{as}$. The third triplet state corresponds to the free motion of six electrons over the lattice and their interactions with the basic functions ${}^3t^1_{p,q,r,k,l,n\in Z^\nu} = a^+_{p,\uparrow}a^+_{q,\uparrow}a^+_{r,\uparrow}a^+_{k,\downarrow}a^+_{l,\downarrow}a^+_{n,\uparrow}\varphi_0.$ The subspace ${}^3\mathcal{H}^1_t,$ corresponding to the third triplet state is the set of all vectors of the form ${}^3\psi^1_t = \sum_{p,q,r,k,l,n\in Z^\nu} f(p,q,r,k,l,n)\,{}^3t^1_{p,q,r,k,l,n\in Z^\nu},\ f \in l_2^{as},$ where l_2^{as} is the subspace of antisymmetric functions in the space $l_2((Z^\nu)^6).$ We denote by ${}^3H^1_t$ the restriction of operator H to the subspace ${}^3\mathcal{H}^1_t.$

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Theorem 1. (coordinate representation of the actions of operator ${}^3H_t^1$) The subspace ${}^3H_t^1$ is invariant under the operator $H;$ and the restriction ${}^3H_t^1$ of operator H to the subspace ${}^3H_t^1$ is a bounded self-adjoint operator. It generates a bounded self-adjoint operator ${}^3\overline{H}_t^{\,1}$ acting in the space l_2^{as} as

$${}^3\overline{H}_t^1{}^3\ \ {}_t^1 = 6Af(p;q;r;k;l;n)+B\sum[f(p+\ \ ;q;r;k;l;n)+f(p;q+\ \ ;r;k;l;n)+f(p;q;r+\ \ ;k;l;n)+$$

$$+f(p;q;r;k+\ \ ;l;n)+f(p;q;r;k;l+\ \ ;n)+f(p;q;r;k;l;n+\ \ )]+U[\ {}_{p;k}+\ {}_{q;k}+\ {}_{r;k}+\ {}_{k;n}+\ {}_{p;l}+\ {}_{q;l}+$$

$+\ {}_{r;l}+\ {}_{l;n}]f(p;q;r;k;l;n):$ The operator ${}^3H_t^1$ acts on a vector ${}^3\ \ {}_t^1\ 2^3\ H_t^1$ as

$${}^3H_t^1{}^3\ \ {}_t^1 = \sum_{p;q;r;k;l;n2Z}({}^3\overline{H}_t^1 f)(p;q;r;k;l;n){}^3t^1_{p;q;r;k;l;n2Z}\ : \qquad (4)$$

Lemma 1. The spectra of operator's ${}^3H_t^1$ and ${}^3\overline{H}_t^1$ coincide.

We call the operator ${}^3H_t^1$ the six-electron third triplet state operator in the Hubbard model.

Let $F\ :\ l_2((Z\ )^6)\ \ !\ \ L_2((T\ )^6)\qquad {}^3\widetilde{H}_t^1$ be the Fourier transform, where T is the dimensional torus endowed with the normalized Lebesgue measure $d\ \ ;$ i.e. $\ (T\ \ ) = 1:$

We set ${}^3\widetilde{H}_t^1 = F\ {}^3\overline{H}_t^1F\ \ {}^1:$ In the quasimomentum representation, the operator ${}^3\overline{H}_t^1$ acts in the Hilbert space $L_2^{as}((T\ \ )^6);$ where L_2^{as} is the subspace of antisymmetric functions in $L_2((T\ \ )^6):$

Theorem 2. (quasimomentum representation of the actions of operator ${}^3H_t^1$) The Fourier transform of operator ${}^3\overline{H}_t^1$ is an operator ${}^3\widetilde{H}_t^1 = F\ {}^3\overline{H}_t^1F\ \ {}^1$ acting in the space $L_2^{as}((T\ \ )^6)$ be the formula

$${}^3\widetilde{H}_t^1{}^3\ \ {}_t^1 = h(\ ;\ ;\ ;\ ;\ ;\ )f(\ ;\ ;\ ;\ ;\ ;\ ) + U[\int_T f(s;\ ;\ ;\ \ +\ \ \ s;\ ;\ )ds+$$

$$+\int_T f(\ ;s;\ ;\ +\ \ \ s;\ ;\ )ds+\int_T f(\ ;\ ;s;\ +\ \ \ s;\ ;\ )ds+\int_T f(\ ;\ ;\ ;s;\ ;\ +\ \ \ s)ds+$$

$$+\int_T f(s;\ ;\ ;\ ;\ +\ \ \ s;\ )ds+\int_T f(\ ;s;\ ;\ ;\ +\ \ \ s;\ )ds+\int_T f(\ ;\ ;s;\ ;\ +\ \ \ s;\ )ds+$$

$+\int_T f(\ ;\ ;\ ;\ ;s;\ +\ \ \ t)ds];$ where $h(\ ;\ ;\ ;\ ;\ ;\ ) = 6A+2B\sum_{i=1}[\cos\ {}_i+\cos\ {}_i+\cos\ {}_i+\cos\ {}_i+\cos\ {}_i+\cos\ {}_i];$ and L_2^{as} is the subspace of antisymmetric functions in $L_2((T\ \ )^6):$

Structure of the essential spectrum and discrete spectrum of operator ${}^3\widetilde{H}_t^1:$

Using tensor products of Hilbert spaces and tensor products of operators in Hilbert spaces, and taking into account that the function $f(\ ;\ ;\ ;\ ;\ ;\ )$ is an antisymmetric function, we can describe the structure of essential spectra and discrete spectrum the operator ${}^3H_t^1$:

Theorem 3. Let $\ \ = 1$ and $U<0:$ Then the essential spectrum of the operator ${}^3H_t^1$ is consists of the union of seven segments: ${}_{ess}({}^3H_t^1) = [a+c+e;b+d+f]\,[\,[a+c+z_3;b+d+z_3]\,[\,[a+e+\widetilde{z}_2;b+f+\widetilde{z}_2]\,[\,[a+\widetilde{z}_2+z_3;b+\widetilde{z}_2+z_3]\,[\,[c+e+z_1;d+f+z_1]\,[\,[c+z_1+z_3;d+z_1+z_3]\,[\,[e+z_1+\widetilde{z}_2;d+z_1+\widetilde{z}_2];$ and discrete spectrum of the operator ${}^3H_t^1$ is consists of no more then one eigenvalue ${}_{disc}({}^3H_t^1) = fz_1+\widetilde{z}_2+z_3g;$ or ${}_{disc}({}^3H_t^1) = ;;$ here and hereafter $a = 2A\ \ \ 4B\cos\frac{1}{2};\ b = 2A+4B\cos\frac{1}{2};$ $c = \ \ 2A\ \ \ 4B\cos\frac{2}{2};\ d = \ \ 2A+4B\cos\frac{2}{2};\ e = 2A\ \ \ 4B\cos\frac{3}{2};\ f = 2A+4B\cos\frac{3}{2};\ z_1 =$ $2A\ \ \ 2\sqrt{U^2+4B^2\cos^2\frac{1}{2}};\ \widetilde{z}_2 = \ \ 2A+\sqrt{9U^2+16B^2\cos^2\frac{2}{2}};\ z_3 = 2A\ \ \ \sqrt{U^2+16B^2\cos^2\frac{3}{2}};$ and ${}_1 = \ \ +\ \ ;\ {}_2 = \ \ +\ \ ;\ {}_3 = \ \ +\ \ :$

Theorem 4. Let $\ \ = 1$ and $U>0:$ Then the essential spectrum of the operator ${}^3H_t^1$ is consists of the union of seven segments: ${}_{ess}({}^3H_t^1) = [a+c+e;b+d+f]\,[\,[a+c+\widetilde{z}_3;b+d+\widetilde{z}_3]\,[\,[a+e+z_2;b+f+z_2]\,[\,[a+z_2+\widetilde{z}_3;b+z_2+\widetilde{z}_3]\,[\,[c+e+\widetilde{z}_1;d+f+\widetilde{z}_1]\,[\,[c+\widetilde{z}_1+\widetilde{z}_3;d+\widetilde{z}_1+\widetilde{z}_3]\,[$

$[e + \widetilde{z}_1 + z_2; d + \widetilde{z}_1 + z_2]$; and discrete spectrum of the operator ${}^3H_t^1$ is consists of more then one eigenvalue $_{disc}({}^3H_t^1) = f\widetilde{z}_1 + z_2 + \widetilde{z}_3 g$; or $_{disc}({}^3H_t^1) = ;$; where $\widetilde{z}_1 = 2A + 2\sqrt{U^2 + 4B^2 cos^2 \frac{{}_1}{2}}$; $z_2 = \quad 2A \quad \sqrt{9U^2 + 16B^2 cos^2 \frac{{}_2}{2}}$; $\widetilde{z}_3 = 2A + \sqrt{U^2 + 16B^2 cos^2 \frac{{}_3}{2}}$:

Theorem 5. Let $\quad = 3$ and $U > 0$; and $_1 = \quad + \quad ;$ $_2 = \quad + \quad ;$ $_3 = \quad + \quad ;$ and $_1 = ({}^0_1; {}^0_1; {}^0_1)$; $_2 = ({}^0_2; {}^0_2; {}^0_2)$; and $_3 = ({}^0_3; {}^0_3; {}^0_3)$:

a). If $U > \frac{12B \cos \frac{{}^0_3}{2}}{W}$; $\cos\frac{{}^0_1}{2} > \frac{2}{3}\cos\frac{{}^0_2}{2}$; and $\cos\frac{{}^0_1}{2} < 2\cos\frac{{}^0_3}{2}$; or $U > \frac{12B \cos \frac{{}^0_3}{2}}{W}$; $\cos\frac{{}^0_1}{2} < \frac{2}{3}\cos\frac{{}^0_2}{2}$; and $\cos\frac{{}^0_1}{2} < 2\cos\frac{{}^0_3}{2}$; or $U > \frac{6B \cos \frac{{}^0_1}{2}}{W}$; $\cos\frac{{}^0_3}{2} < \frac{1}{2}\cos\frac{{}^0_1}{2}$; and $\cos\frac{{}^0_3}{2} < \frac{1}{3}\cos\frac{{}^0_2}{2}$; or $\cos\frac{{}^0_3}{2} > \frac{1}{3}\cos\frac{{}^0_2}{2}$; or $U > \frac{4B \cos \frac{{}^0_2}{2}}{W}$; $\cos\frac{{}^0_3}{2} < \frac{1}{3}\cos\frac{{}^0_2}{2}$; and $\cos\frac{{}^0_3}{2} > \frac{1}{2}\cos\frac{{}^0_1}{2}$; or $\cos\frac{{}^0_3}{2} < \frac{1}{2}\cos\frac{{}^0_1}{2}$; then the essential spectrum of the operator ${}^3H_t^1$ is consists of the union of seven segments: $_{ess}({}^3H_t^1) = [a + c + e; b + d + f]\ [\ [a + c + \widetilde{z}_3; b + d + \widetilde{z}_3]\ [\ [a + e + z_2; b + f + z_2]\ [\ [a + z_2 + \widetilde{z}_3; b + z_2 + \widetilde{z}_3]\ [\ [c + e + \widetilde{z}_1; d + f + \widetilde{z}_1]\ [\ [c + \widetilde{z}_1 + \widetilde{z}_3; d + \widetilde{z}_1 + \widetilde{z}_3]\ [\ [e + \widetilde{z}_1 + z_2; d + \widetilde{z}_1 + z_2]$; and discrete spectrum of the operator ${}^3H_t^1$ is consists of no more one eigenvalue $_{disc}({}^3H_t^1) = f\widetilde{z}_1 + z_2 + \widetilde{z}_3 g$; or $_{disc}({}^3H_t^1) = ;$; where $a = 2A \quad 12B\cos\frac{{}^0_1}{2}$; $b = 2A + 12B\cos\frac{{}^0_1}{2}$; $c = \quad 2A \quad 12B\cos\frac{{}^0_2}{2}$; $d = \quad 2A + 12B\cos\frac{{}^0_2}{2}$; $e = 2A \quad 12B\cos\frac{{}^0_3}{2}$; $f = 2A + 12B\cos\frac{{}^0_3}{2}$; $\widetilde{z}_1$; z_2; $\widetilde{z}_3$ are the same concrete numbers and W is the Watson integral.

b). If $U > 0$; $\frac{6B \cos \frac{{}^0_1}{2}}{W} < U \quad \frac{12B \cos \frac{{}^0_3}{2}}{W}$; $\cos\frac{{}^0_1}{2} > \frac{2}{3}cos\frac{{}^0_2}{2}$; and $cos\frac{{}^0_3}{2} > \frac{1}{2}cos\frac{{}^0_1}{2}$; or $U > 0$; $\frac{4B \cos \frac{{}^0_2}{2}}{W} < U \quad \frac{12B \cos \frac{{}^0_3}{2}}{W}$; $\cos\frac{{}^0_2}{2} < cos\frac{{}^0_1}{2}$; and $cos\frac{{}^0_1}{2} < cos\frac{{}^0_3}{2}$; or $U > 0$; $\frac{4B \cos \frac{{}^0_2}{2}}{W} < U \quad \frac{6B \cos \frac{{}^0_1}{2}}{W}$; $\cos\frac{{}^0_3}{2} < \frac{1}{3}cos\frac{{}^0_2}{2}$; and $cos\frac{{}^0_1}{2} > \frac{2}{3}cos\frac{{}^0_2}{2}$; or $U > 0$; $\frac{12B \cos \frac{{}^0_3}{2}}{W} < U \quad \frac{6B \cos \frac{{}^0_1}{2}}{W}$; $\cos\frac{{}^0_3}{2} > \frac{1}{3}cos\frac{{}^0_2}{2}$; and $cos\frac{{}^0_3}{2} > \frac{1}{2}cos\frac{{}^0_1}{2}$; or $U > 0$; $\frac{12B \cos \frac{{}^0_3}{2}}{W} < U \quad \frac{4B \cos \frac{{}^0_2}{2}}{W}$; $\cos\frac{{}^0_3}{2} > \frac{1}{2}cos\frac{{}^0_1}{2}$; and $cos\frac{{}^0_3}{2} < \frac{1}{3}cos\frac{{}^0_2}{2}$; or $U > 0$; $\frac{6B \cos \frac{{}^0_1}{2}}{W} < U \quad \frac{4B \cos \frac{{}^0_2}{2}}{W}$; $\cos\frac{{}^0_3}{2} < \frac{1}{2}cos\frac{{}^0_1}{2}$; and $cos\frac{{}^0_1}{2} < \frac{2}{3}cos\frac{{}^0_2}{2}$; then the essential spectrum of the operator ${}^3H_t^1$ is consists of the union of four segments: $_{ess}({}^3H_t^1) = [a + c + e; b + d + f]\ [\ [a + e + z_2; b + f + z_2]\ [\ [c + e + \widetilde{z}_1; d + f + \widetilde{z}_1]\ [\ [e + \widetilde{z}_1 + z_2; d + \widetilde{z}_1 + z_2]$; or $_{ess}({}^3H_t^1) = [a+c+e; b+d+f]\ [\ [a+c+\widetilde{z}_3; b+d+\widetilde{z}_3]\ [\ [c+e+\widetilde{z}_1; d+f+\widetilde{z}_1]\ [\ [c+\widetilde{z}_1+\widetilde{z}_3; d+\widetilde{z}_1+\widetilde{z}_3]$; or $_{ess}({}^3H_t^1) = [a+c+e; b+d+f]\ [\ [a+c+\widetilde{z}_3; b+d+\widetilde{z}_3]\ [\ [a+e+z_2; b+f+z_2]\ [\ [a+z_2+\widetilde{z}_3; b+z_2+\widetilde{z}_3]$; and discrete spectrum of the operator ${}^3H_t^1$ is empty set: $_{disc}({}^3H_t^1) = ;$:

There is also the case when the essential spectrum of the operator ${}^3H_t^1$ is consists of the unions of two segments, and the discrete spectrum of the operator ${}^3H_t^1$ is empty set: $_{disc}({}^3H_t^1) = ;$, and the case when the essential spectrum of the operator ${}^3H_t^1$ is consists of a single segment, and the discrete spectrum of the operator ${}^3H_t^1$ is empty set: $_{disc}({}^3H_t^1) = ;$.

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Analogues of Carleman's tangent approximation theorem

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Carleman's famous tangent approximation theorem derived in 1927 states that for every function $f \in C(\mathbb{R})$ and every error function ε, i.e. any positive function $\varepsilon \in C(\mathbb{R})$, there exists an entire function $g : \mathbb{C} \to \mathbb{C}$ such that

$$|f(t) - g(t)| < \varepsilon(t)$$

for all $t \in \mathbb{R}$ (see, for example, [1], Chap. 4, Sect. 3). Carleman's theorem has been further developed and refined in many papers (see bibliography in [1] and [2]). Carleman himself had already generalized his result by replacing $\mathbb{R}$ by more general curves and systems of curves in the complex plane. Many authors have studied, in connection with Carleman's theorem, approximation in combination with interpolation, as well as tangent approximation of smooth functions together with their derivatives. In addition, approximation with a certain rate of decrease of the error function was considered. Questions related to tangent and uniform approximation under restrictions on the growth of the approximating function were also studied. We also note the multidimensional analog of Carleman's theorem obtained by S. Sheinberg (see references in [1]). Carleman's theorem and its generalizations play an important role in the study of boundary properties of analytic functions and in the study of the distribution of their values (see [1], Chap. 4, Sect. 5).

The class of entire functions $g : \mathbb{C} \to \mathbb{C}$ coincides with the set of solutions of the differential equation

$$\left(\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2}\right) g = 0, \quad (x_1, x_2) \in \mathbb{R}^2.$$

In this regard, it is of interest to obtain analogues of Carleman's theorem in which the approximation is made by solutions of other linear partial differential equations in $\mathbb{R}^n$, $n \geq 2$, with constant coefficients. For the solutions of most of these equations, many important and useful properties of the class of entire functions are not fulfilled (for example, they as a rule do not form an algebra), which prevents them from obtaining analogues of Carleman's theorem by known methods. The simplest example is the class of eigenfunctions of the Laplace operator in $\mathbb{R}^2$, that is, the set of solutions of the equation

$$\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\right) g + \lambda g = 0, \quad (x_1, x_2) \in \mathbb{R}^2,$$

for $\lambda \neq 0$.

Here we study the approximation of continuous functions on rays in $\mathbb{R}^n$ by solutions of a multidimensional convolution equation of the form

$$g * T = 0, \tag{1}$$

where T is a given radial distribution with compact support in $\mathbb{R}^n$, $n \geq 2$. The theory of equations (1) originates in the work of the famous Romanian mathematician D. Pompeiu who considered the case when T is the indicator of a ball in $\mathbb{R}^n$ (see, e.g., [3], [4]). Equation (1)

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as well as its various analogues and generalizations have been intensively studied over the past fifty years by F. John, J. Delsarte, J.D. Smith, L. Zalcman, C.A. Berenstein, and others (see the overviews in [3], [4] and monographs [5]–[7] which provide extensive bibliographies). We note that with an appropriate choice of T they characterize such important classes of functions as functions with zero spherical (or ball) means, functions with the property of mean values from the theory of harmonic functions, and also solutions of elliptic differential equations of the form

$$p(\Delta)g = 0,$$

where Δ is the Laplace operator in $\mathbb{R}^n$, and p is an arbitrary algebraic polynomial other than the identical constant.

Everywhere in what follows, $\mathbb{R}^n$ is a Euclidean space of dimension $n \geq 2$. Denote by $\mathcal{D}'(\mathbb{R}^n)$ (respectively, $\mathcal{E}'(\mathbb{R}^n)$) the space of distributions (respectively, distributions with compact supports) in $\mathbb{R}^n$, $\mathcal{D}(\mathbb{R}^n)$ is the space of finite infinitely differentiable functions in $\mathbb{R}^n$, $\mathcal{E}(\mathbb{R}^n) = C^\infty(\mathbb{R}^n)$.

Let $T \in \mathcal{E}'(\mathbb{R}^n)$, $T \neq 0$. For every $f \in \mathcal{D}'(\mathbb{R}^n)$, the convolution $f * T$ is defined by the equality

$$\langle f * T, \varphi \rangle = \langle f_y, \langle T_x, \varphi(x+y)\rangle\rangle, \quad \varphi \in \mathcal{D}(\mathbb{R}^n),$$

as a distribution in $\mathcal{D}'(\mathbb{R}^n)$ (the index at the bottom of the distributions f and T means the action on the specified variable). A distribution of the class

$$\mathcal{D}'_T(\mathbb{R}^n) = \{f \in \mathcal{D}'(\mathbb{R}^n) : f * T = 0\}$$

is called mean periodic with respect to T.

Let $SO(n)$ be the rotation group of $\mathbb{R}^n$. A distribution $T \in \mathcal{E}'(\mathbb{R}^n)$ is called radial if it is invariant under the group $SO(n)$, i.e.

$$\langle T, \varphi(\tau x)\rangle = \langle T, \varphi(x)\rangle \quad \text{for all} \quad \varphi \in \mathcal{E}(\mathbb{R}^n), \quad \tau \in SO(n).$$

Denote by $\mathcal{E}'_\natural(\mathbb{R}^n)$ the set of all radial distributions $T \in \mathcal{E}'(\mathbb{R}^n)$. The simplest example of distribution in the class $\mathcal{E}'_\natural(\mathbb{R}^n)$ is the Dirac delta function δ_0 with support at zero, i.e.

$$\langle \delta_0, \varphi \rangle = \varphi(0), \quad \varphi \in \mathcal{E}(\mathbb{R}^n).$$

Let $\mathbb{S}^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$, $l \in \mathbb{S}^{n-1}$, and assume that $a \in \mathbb{R}^n$. As usual, the ray in $\mathbb{R}^n$ with vertex a in direction l is the set

$$L_{a,l} = \{x = (x_1, \dots, x_n) \in \mathbb{R}^n : x_j = a_j + tl_j,\ t \geq 0,\ j = 1, \dots, n\}.$$

Theorem 1. Let $T \in \mathcal{E}'_\natural(\mathbb{R}^n)$ and

$$T \neq c\,\delta_0, \quad c \in \mathbb{C}\setminus\{0\}. \tag{2}$$

Suppose also that $a \in \mathbb{R}^n$, $l \in \mathbb{S}^{n-1}$, and $g \in C(L_{a,l})$. Then for every positive function $\varepsilon \in C(L_{a,l})$ there exists a function $f \in (\mathcal{E} \cap \mathcal{D}'_T)(\mathbb{R}^n)$ satisfying the conditions

(i) for every $x \in L_{a,l}$

$$|g(x) - f(x)| < \varepsilon(x); \tag{3}$$

(ii) there exists a function $w \in C^\infty(\mathbb{R}^2)$ such that

$$f(x) = w\left((x, l), \sqrt{|x|^2 - (x, l)^2}\right) \tag{4}$$

for all $x \in \mathbb{R}^n$.

By the arbitrariness of $\varphi \in C(L_{a,l})$, inequality (3) guarantees the tangent approximation of g on $L_{a,l}$ by smooth solutions to (1). Note that (4) means that the approximating function f is radial in any hyperplane orthogonal to the ray $L_{a,l}$.

Observe that (2) is necessary in Theorem 1. Indeed, if $T = c\delta_0$ for some $c \in \mathbb{C}\setminus\{0\}$ then the zero function is the only solution to (1); therefore, the claim of Theorem 1 fails.

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