Numerical study of chernoff approximations for parabolic heat-type equation with variable coefficients

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Introduction

The method of Chernoff approximation was discovered by Paul Chernoff in 1968 and now is a powerful and flexible tool of contemporary functional analysis. This method helps to solve numerically the Cauchy problem for evolution equations. The rate of convergence of Chenroff approximations were studied theoretically by O.E.Galkin and I.D.Remizov in a general setting for arbitrary C_0-semigroup.The present research is devoted to study of convergence rates of four families of Chernoff functions to the solution of cauchy problem with variable coefficient of thermal conductivity.

Definition of a C_0-semigroup

Let $\mathcal{F}$ be a Banach space, and $\mathscr{L}(\mathcal{F})$ be the space of all linear bounded operators on $\mathcal{F}$. Consider mapping $V\colon [0;+\infty) \to \mathscr{L}(\mathcal{F})$, which for every fixed $t \geq 0$ is a linear bounded operator $V(t)\colon \mathcal{F} \to \mathcal{F}$.
The family $(V(t))_{t\geq 0} \subset \mathscr{L}(\mathcal{F})$ is called C_0-**semigroup** iff the following holds:

1. $V(0) = I$, i.e. $V(0)f = f$ for all $f \in \mathcal{F}$;
2. $V(t+s) = V(t) \circ V(s)$ for any $t \geq 0, s \geq 0$;
3. V is continuous in strong operator topology, i.e. for any $f \in \mathcal{F}$ a mapping $t \longmapsto V(t)f$ is continuous.

For the C_0-semigroup, there is an analogue of the derivative at zero. This object is called its generator and is defined as follows.

Definition of a C_0-semigroup generator

By the **generator** of a C_0-semigroup of linear bounded operators in $\mathcal{F}$ we mean a linear operator L: $Dom(L) \to \mathcal{F}$ given by the formula

$$Lf = \lim_{t \to +0} \frac{V(t)f - f}{t},$$

defined on its domain $Dom(L)$, that is a dense subspace of $\mathcal{F}$ such that there exist a given limit where the limit is understood in the strong sense, i.e. it is defined in terms of the norm in space $\mathcal{F}$. The generator generates a C_0-semigroup, and one can use the notation $V(t) = e^{tL}$.

C_0-semigroup and linear evolution equations

let Q be some set. In the Cauchy problem for an evolution partial differential equation

$$\begin{cases} u_t'(t,x) = Lu(t,x) \ \text{ for } \ t>0, x\in Q, \\ u(0,x) = u_0(x) \ \text{ for } \ x \in Q. \end{cases}$$

we can assume $U(t) = u(t,\cdot) = [x \longmapsto u(t,x)]$ and get the Cauchy problem for an ordinary differential equation:

$$\begin{cases} \frac{d}{dt}U(t) = LU(t) \ \text{ for } \ t>0, \\ U(0) = u_0. \end{cases}$$

It is known that if $u(t,\cdot) \in \mathcal{F}$ and there exists a C_0-semigroup with generator L, that is, if there is an exponential form the operator tL, then both problems have a solution

$$U(t) = e^{tL}u_0, \quad u(t,x) = U(t)(x) = \left(e^{tL}u_0\right)(x).$$

Chernoff tangency

Chernoff tangency conditions the following:

1. Let $\mathcal{F}$ be a Banach space, and let $\mathscr{L}(\mathcal{F})$ be the space of all bounded linear operators on $\mathcal{F}$. Suppose a map $G\colon [0; +\infty) \to \mathscr{L}(\mathcal{F})$ is given;
2. The family G is strongly continuous in strong operator topology of the space $\mathscr{L}(\mathcal{F})$, i.e., the map $t \longmapsto G(t)f \in \mathcal{F}$ is continuous on $[0; +\infty)$ for each $f \in \mathcal{F}$;
3. $G(0) = I$;
4. There exists a linear subspace $D \subset \mathcal{F}$ dense in $\mathcal{F}$ such that for each $f \in D$ the limit

$$\lim_{t \to +0} \frac{G(t)f - f}{t}$$

exists. We denote its value by $G'(0)f$;

5. The closure of the operator $(G'(0), D)$ exists and is equal to $(L, Dom(L))$.

Chernoff theorem, summary

Chernoff's theorem is a theorem on the «second remarkable limit» for C_0-semigroup:

Let $\mathcal{F}$ — be a Banach space and L — be a closed linear operator in $\mathcal{F}$ with a dense domain. Let a family $(G(t))_{t\geq 0}$ of linear bounded operators in $\mathcal{F}$. Let the conditions also be true::
(E) C_0-semigroup $\left(e^{tL}\right)_{t\geq 0}$ exists
(N) There is such $\omega \in \mathbb{R}$ that $\|G(t)\| \leq e^{wt}$ for each $t \geq 0$
(CT) idea of the condition briefly: $G(t)f = f + tLf + o(t), t \to 0$
Then $e^{tL}f = \lim\limits_{n\to\infty} G(t/n)^n f$ for each $f \in \mathcal{F}$ and for each $t \geq 0$.
«second remarkable limit»

$$e^{tL} = \lim_{n\to\infty} G(t/n)^n = \lim_{n\to\infty} \left(I + \frac{tL}{n} + o(t/n)\right)^n$$

Model equation and problem statement

For the next Cauchy problem

$$\begin{cases} u'_t(t,x) = a(x)u''_{xx}(t,x) \\ u(0,x)=u_0(x) \end{cases}$$

We present the solution $u(t,x)$ in the form of a limit of fast converging Chernoff approximations under the conditions $\inf_{x\in\mathbb{R}} a(x) > 0$ and study the rate of convergence of Chernoff approximations.

Chernoff functions

We use the following Chernoff functions:

$$(G(t)f)(x) = \frac{1}{2}f(x) + \frac{1}{4}f\left(x + 2\sqrt{a(x)t}\right) + \frac{1}{4}f\left(x - 2\sqrt{a(x)t}\right)$$

$$(S(t)f)(x) = \frac{2}{3}f(x) + \frac{1}{6}f\left(x + \sqrt{6a(x)t}\right) + \frac{1}{6}f\left(x - \sqrt{6a(x)t}\right)$$

Chernoff functions

$$(H(t)f)(x) = \frac{2}{3}f(x) + \frac{1}{6}f\left(x + \sqrt{6a(x)t}\right) + \frac{1}{6}f\left(x - \sqrt{6a(x)t}\right) +$$
$$+ a(x)a'(x)t\Big(3f\left(x + \sqrt[3]{t}\right) - 3f\left(x + 2\sqrt[3]{t}\right) + f\left(x + 3\sqrt[3]{t}\right)\Big) +$$
$$+ \frac{1}{2}a(x)a''(x)t\Big(f\left(x + \sqrt{t}\right) + f\left(x - \sqrt{t}\right)\Big) - \Big(a'(x) + a''(x)\Big)a(x)tf(x)$$

$$(Q(t)f)(x) = \frac{2}{3}f(x) + \frac{1}{6}f\left(x + \sqrt{6a(x)t}\right) + \frac{1}{6}f\left(x - \sqrt{6a(x)t}\right) -$$
$$- a(x)a'(x)t\Big(\frac{7}{2}f\left(x + \sqrt[3]{t}\right) + \frac{1}{4}f\left(x - \sqrt[3]{t}\right) - \frac{7}{4}f\left(x + 2\sqrt[3]{t}\right) +$$
$$+ \frac{1}{4}f\left(x - 2\sqrt[3]{t}\right) + \frac{1}{4}f\left(x + 3\sqrt[3]{t}\right)\Big) +$$
$$+ \frac{1}{2}a(x)a''(x)t\Big(f\left(x + \sqrt{t}\right) + f\left(x - \sqrt{t}\right)\Big) +$$
$$+ \Big(\frac{5}{2}a'(x) - a''(x)\Big)a(x)tf(x)$$

Definition of norm in $UC_b(\mathbb{R})$

The standard norm in $UC_b(\mathbb{R})$ namely

$$\|u_n(t,\cdot) - u(t,\cdot)\| = \sup_{x\in\mathbb{R}} |u_n(t,x) - u(t,x)| = \sup_{x\in[a,b]} |u_n(t,x) - u(t,x)|,$$

where u is the solution of Cauchy problem and u_n is the Chernoff approximation,is reached at the interval $[a, b]$ corresponding to the period. So we have

$$d = \max_{k=1,\dots,100} \left| u_n\left(t, a + \frac{k}{100}(b-a)\right) - u\left(t, a + \frac{k}{100}(b-a)\right)\right|$$

$$\approx \sup_{x\in[a,b]} |u_n(t,x) - u(t,x)|$$

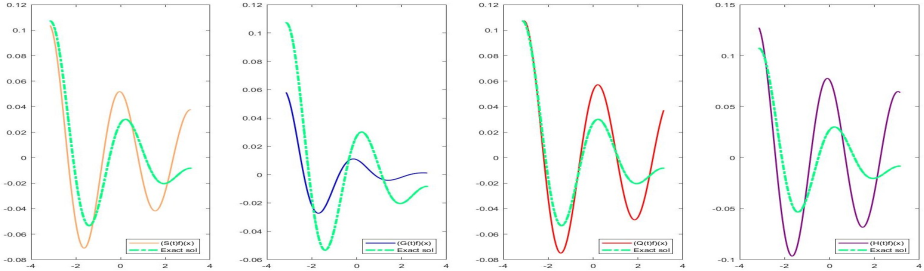
Comments on computational techniques

1. Cauchy problem was solved with numerical technique MATLAB PDE Solver pdepe method named MatlabPdepe
2. Calculations were performed in the Matlab environment.
3. Value of composition degree n varies from 1 to 4.

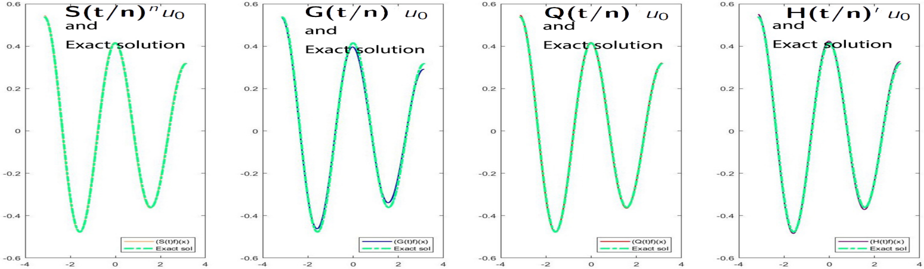
Numerical results

Numerical experiments for smooth initial conditions and smooth coefficient

$u_0(x) = \cos(2x) \qquad a(x) = 1.1 + \sin(\frac{x}{9}) \qquad n = 3$



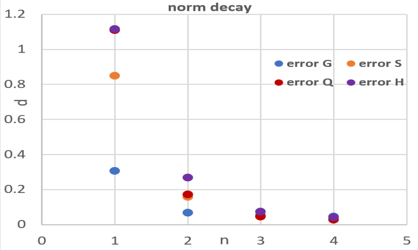
t=0.8

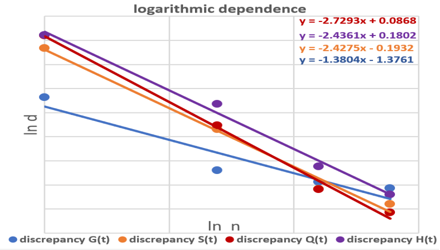


t=0.1

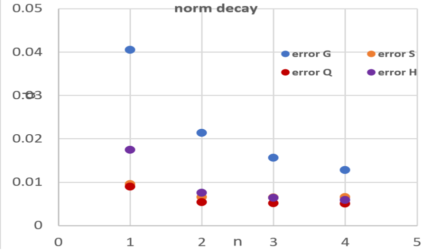
Examination of error and convergence rates

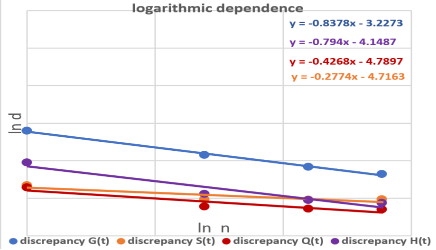
$u_0(x) = \cos(2x) \qquad a(x) = 1.1 + \sin\left(\frac{x}{9}\right)$





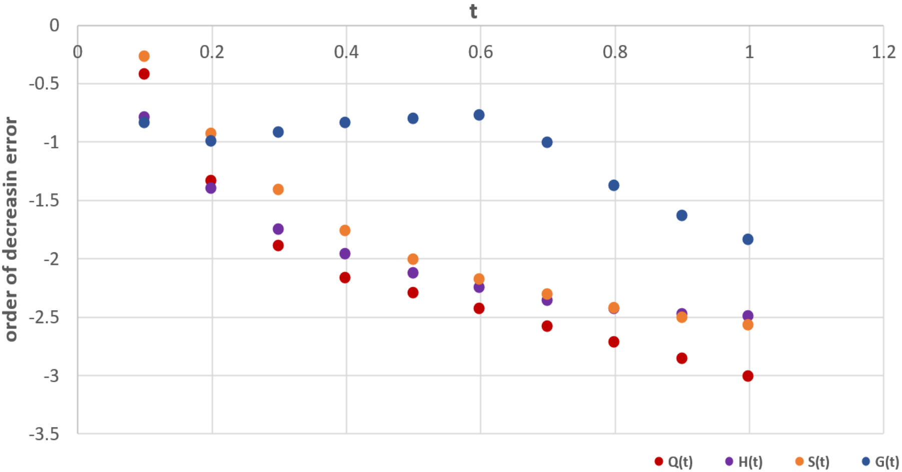
t=0.8





t=0.1

Estimation of convergence rates at different times

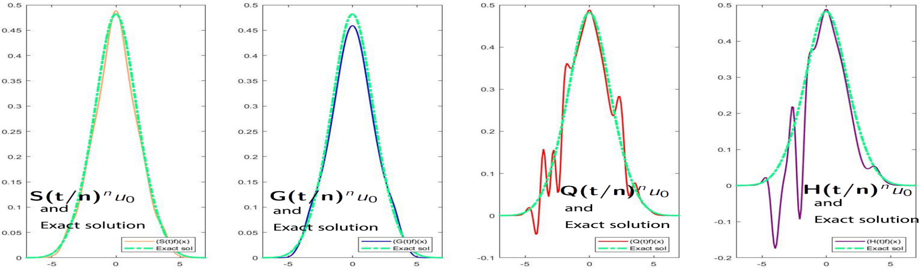


u_0 is the initial condition, t is time, we consider
$t \in \{0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1.0\}$

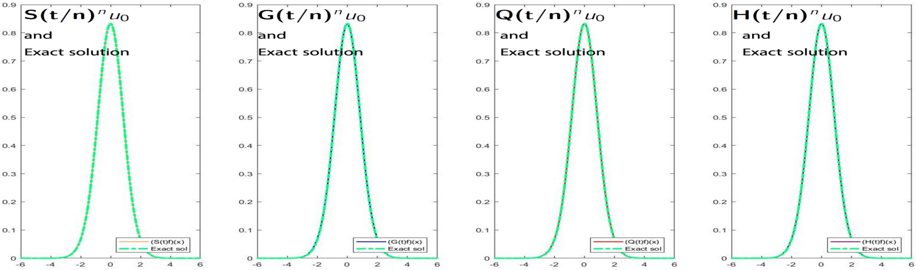
Numerical results

Numerical experiments for smooth initial conditions and non-smooth coefficient

$u_0(x) = \exp(-x^2) \qquad a(x) = 1.1 + |\sin(\frac{x}{2})|^{7/2} \qquad n = 3$



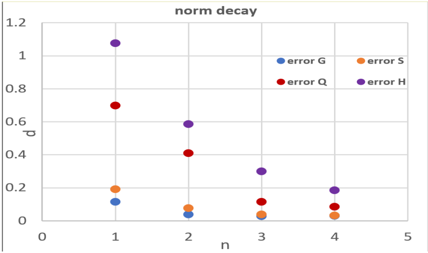
t=0.8

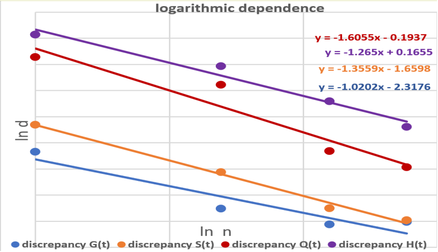


t=0.1

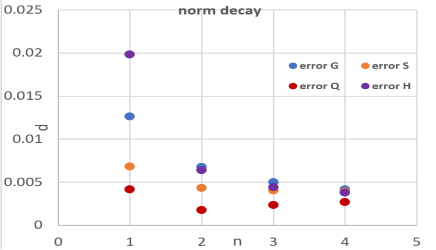
Examination of error and convergence rates

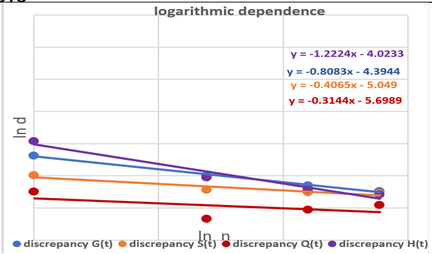
$u_0(x) = \exp(-x^2) \qquad a(x) = 1.1 + |\sin(\frac{x}{2})|^{7/2}$





t=0.8





t=0.1

Estimation of convergence rates at different times

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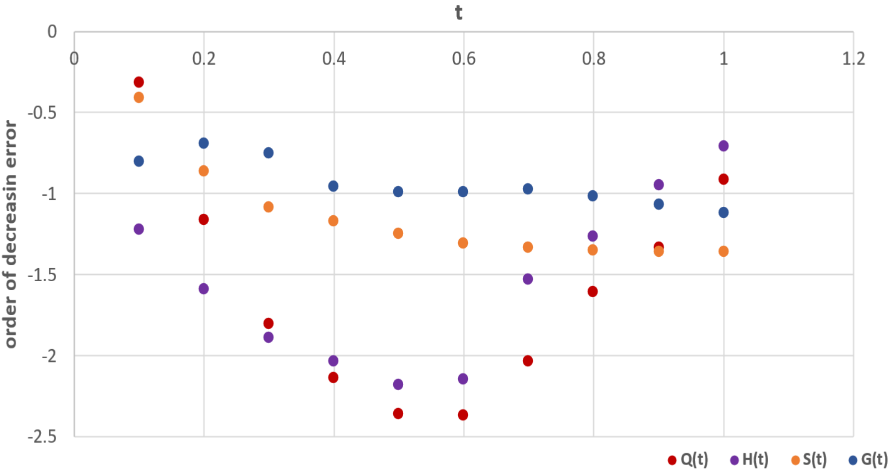


u_0 is the initial condition, t is time, we consider $t \in \{0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1.0\}$

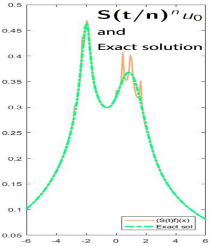
Numerical results

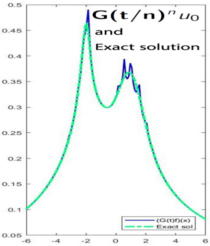
Numerical experiments for non-smooth initial conditions and smooth coefficient

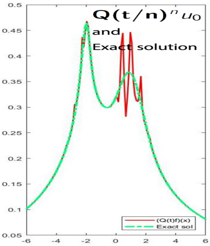
Examination of error and convergence rates

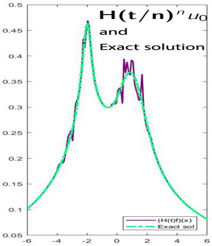
$u_0(x) = \exp(-|(x+2)(x-1)|^{1/4})$ $\quad a(x) = 1.1 + \sin(x)$

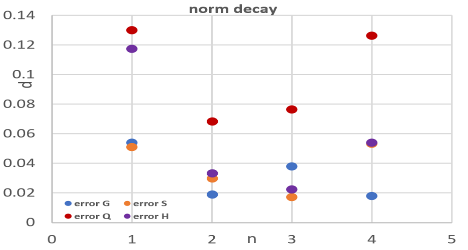
$t = 0.1$

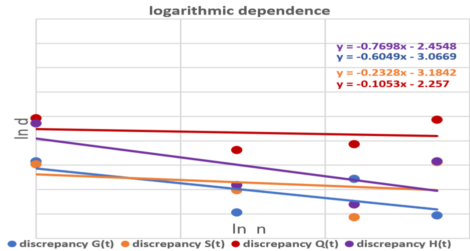








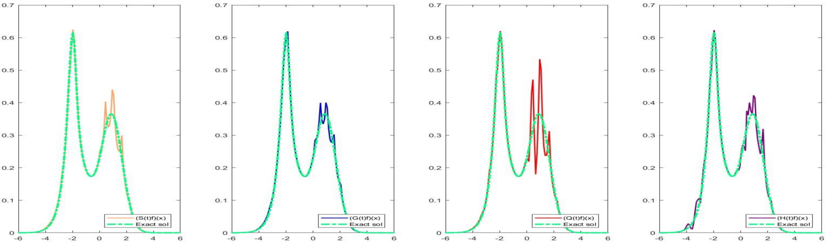


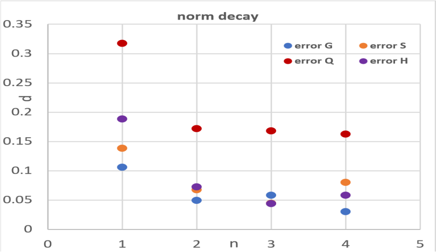


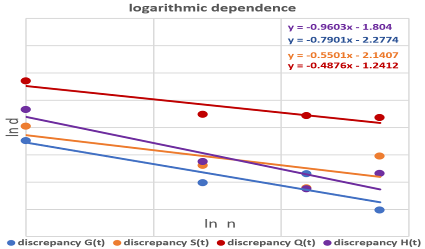
Examination of error and convergence rates

$u_0(x) = \exp(-|(x+2)(x-1)|^{3/4}) \qquad a(x) = 1.1 + \sin(x)$

$t = 0.1$



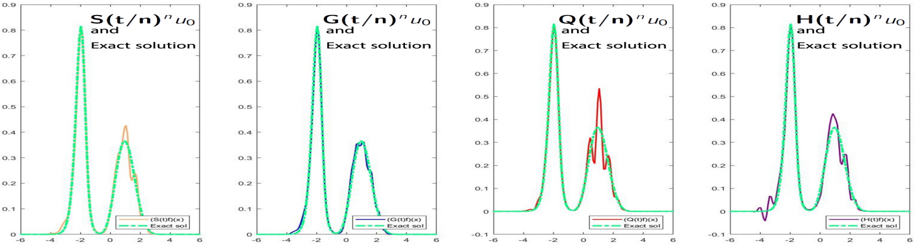


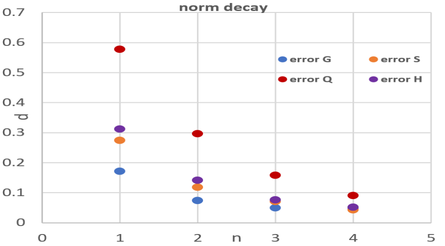


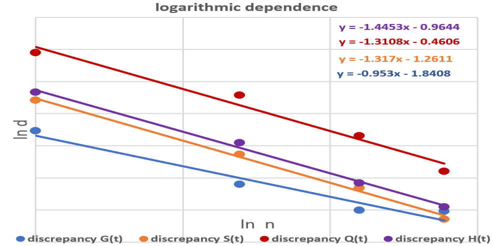
Examination of error and convergence rates

$u_0(x) = \exp(-|(x+2)(x-1)|^{5/2})$ $\quad a(x) = 1.1 + \sin(x)$

$t = 0.1$



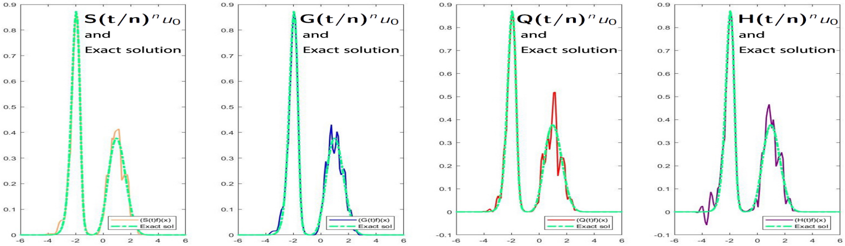


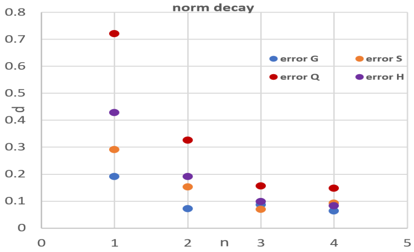


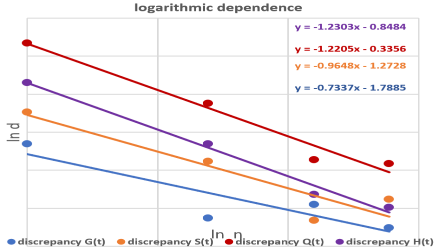
Examination of error and convergence rates

$u_0(x) = \exp(-|(x+2)(x-1)|^{9/2}) \qquad a(x) = 1.1 + \sin(x)$

$t = 0.1$







Estimation of convergence rates at different smoothness class of initial condition

$u_0(x) = \exp(-|(x+2)(x-1)|^q) \qquad a(x) = 1.1 + \sin(x)$

$t = 0.1$

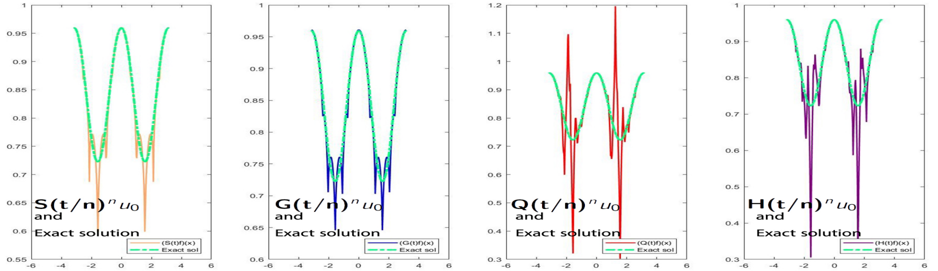
u_0 is the initial condition, q is the smoothness class of u_0, we consider $q \in \{1/4, 1/3, 1/2, 2/3, 3/4, 1, 3/2, 5/2, 7/2, 9/2, 11/2, 13/2\}$

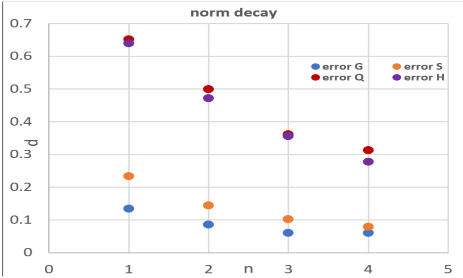
Numerical results

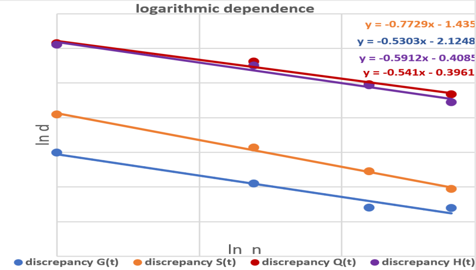
Numerical experiments for non-smooth initial conditions and non-smooth coefficient

Examination of error and convergence rates

$u_0(x) = |\cos(x)|^{1/3} \qquad a(x) = 1.1 + |\sin(x)|^{9/2} \qquad t = 0.1$

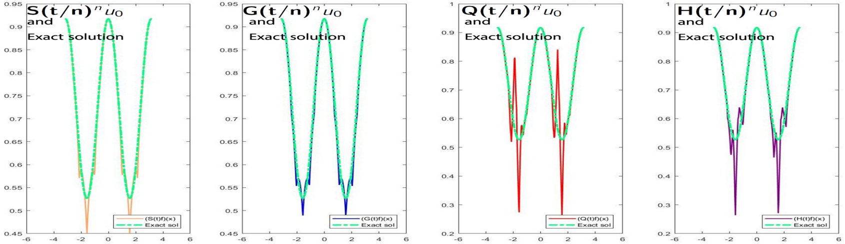


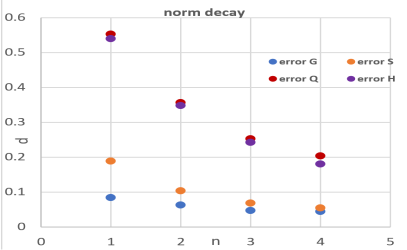


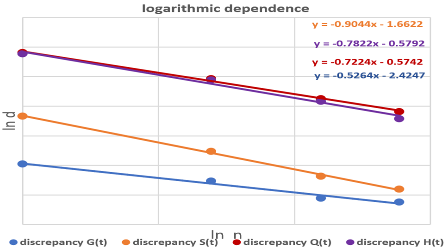


Examination of error and convergence rates

$u_0(x) = |\cos(x)|^{3/4} \qquad a(x) = 1.1 + |\sin(x)|^{9/2} \qquad t = 0.1$

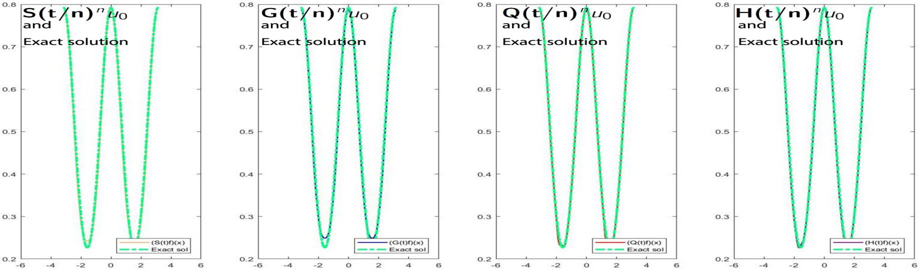


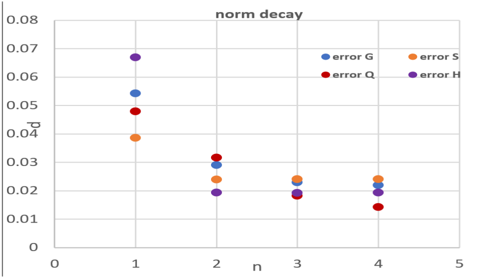


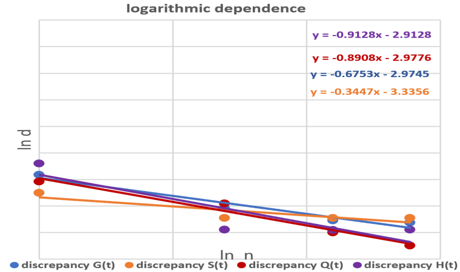


Examination of error and convergence rates

$u_0(x) = |\cos(x)|^{5/2} \qquad a(x) = 1.1 + |\sin(x)|^{9/2} \qquad t = 0.1$

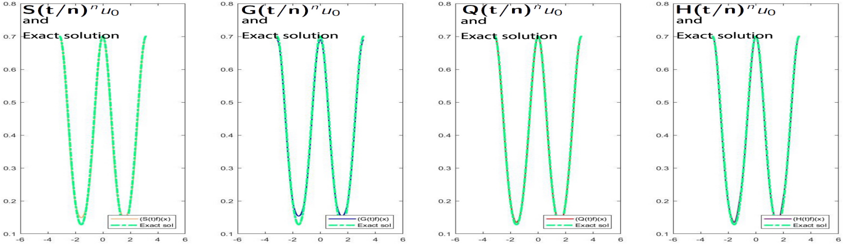


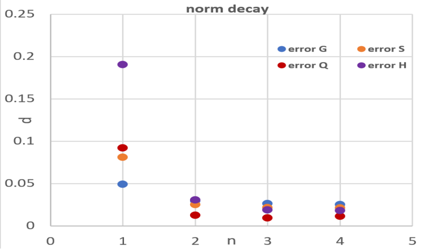


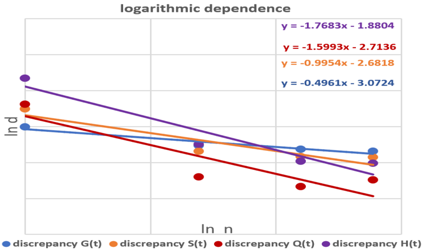


Examination of error and convergence rates

$u_0(x) = |\cos(x)|^{9/2} \qquad a(x) = 1.1 + |\sin(x)|^{9/2} \qquad t = 0.1$

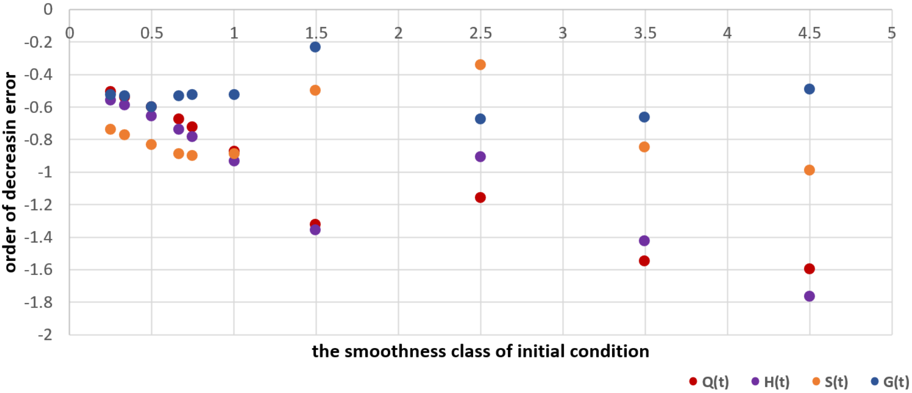






Estimation of convergence rates at different smoothness class of initial condition

$u_0(x) = |\cos(x)|^q \qquad a(x) = 1.1 + |\sin(x)|^{9/2} \qquad t = 0.1$



u_0 is the initial condition, q is the smoothness class of u_0, we consider $q \in \{1/4, 1/3, 1/2, 2/3, 3/4, 1, 3/2, 5/2, 7/2, 9/2\}$

Conclusion for smooth initial condition u_0

	Convergence rates	
$a(x)$	$t \in \{0.1, 0.2, 0.3, 0.4, 0.5\}$	$t \in \{0.6, 0.7, 0.8, 0.9, 1.0\}$
Smooth	As time increases, the convergence rate of all discussed functions rises. And Chernoff functions $Q(t)$ has the highest convergence rate compared to the $H(t)$, $S(t)$ and $G(t)$	As time increases, the convergence rate of all discussed functions increases. And Chernoff functions $Q(t)$ has the highest convergence rate compared to the $H(t)$, $S(t)$ and $G(t)$
Non-smooth	As time increases,the convergence rate of all discussed functions riseS. And Chernoff functions $H(t)$ has the highest convergence rate compared to the $Q(t)$, $S(t)$ and $G(t)$	Convergence rates of functions $S(t)$ and $G(t)$ have upward trend. In the case of functions $Q(t)$ and $H(t)$ the rates of convergence can increase or decrease

Conclusion for nonsmooth initial condition u_0

	Convergence rates	
a(x)	$q \in \{1/4, 1/3, 1/2, 2/3, 3/4, 1\}$	$q \in \{5/2, 7/2, 9/2, 11/2, 13/2\}$
Smooth	As the smoothness class q of initial condition u_0 increases ,the convergence rate of all discussed functions rises. And Chernoff functions $H(t)$ has the highest convergence rate compared to the $Q(t)$, $S(t)$ and $G(t)$	When $u_0(x) = \exp(-\|(x+2)(x-1)\|^q)$ the rates of convergence of all discussed functions have downward trend. conversely, convergence rates increase in the case of $u_0(x) = \|\cos(x)\|^q$
Non-smooth	As the smoothness class q of initial condition u_0 increases ,the convergence rate of all discussed functions rises	With the exception of Chernoff function $G(t)$, the rates of convergence of other Chernoff functions go up as q increases

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$$\mathbf{S(t/n)}^n u_0$$

$$\mathbf{G(t/n)}^n u_0$$

$$\mathbf{Q(t/n)}^n u_0$$

$$\mathbf{H(t/n)}^n u_0$$