Error of Chernoff approximations based on Chernoff function with a given t^2-coefficient

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Let $(X, \|\cdot\|)$ be any Banach space and $\mathscr{L}(X)$ denotes the set of all bounded linear operators on X. Next we will use the notions of *strongly continuous one-parameter semigroup* (or just C_0*–semigroup*), *contractive semigroup* and *generator of a strongly continuous semigroup*, definitions of which can be found, for example, in the book of Engel and Nagel [1]. In 1968 Paul Chernoff proved the following theorem:

Theorem 1 (Chernoff [2]). *Let X be a Banach space, $F(t)$ be a strongly continuous function from $[0,\infty)$ to the set of linear contraction operators on X, such that $F(0) = I$. Suppose that the closure A of the strong derivative $F'(0)$ is the generator of contractive C_0–semigroup $\{e^{tA}\}_{t\geq 0}$. Then $[F(t/n)]^n$ converges to e^{tA} in the strong operator topology.*

Let us note that this theorem does not contain an estimate of the rate of convergence. In 2022 was published the theorem that provides such estimate under certain conditions:

Theorem 2 (Galkin, Remizov [3]). *Suppose that:*

1) $T > 0$, $M_1 \geq 1$, $w \geq 0$. $(A, D(A))$ is generator of C_0-semigroup $(e^{tA})_{t\geq 0}$ in a Banach space X, such that $\|e^{tA}\| \leq M_1 e^{wt}$ for $t \in [0,T]$.

2) There are a mapping $F\colon (0,T] \to \mathscr{L}(X)$ and constant $M_2 \geq 1$ such that we have $\|(F(t))^k\| \leq M_2 e^{kwt}$ for all $t \in (0,T]$ and all $k \in \mathbb{N} = \{1,2,3,\ldots\}$.

3) $m \in \mathbb{N}\cup\{0\}$, $p \in \mathbb{N}$, subspace $\mathcal{D} \subset D(A^{m+p})$ is $(e^{tA})_{t\geq 0}$-invariant.

4) There exist such functions $K_j\colon (0,T] \to [0,+\infty)$, $j = 0,1,\ldots,m+p$ that for all $t \in (0,T]$ and all $x \in \mathcal{D}$ we have $\Big\|F(t)x - \sum_{k=0}^{m} \frac{t^k A^k x}{k!}\Big\| \leq t^{m+1} \sum_{j=0}^{m+p} K_j(t)\|A^j x\|$.

Then for all $t > 0$, all integer $n \geq t/T$ and all $x \in \mathcal{D}$ we have

$$\|(F(t/n))^n x - e^{tA}x\| \leq \frac{M_1 M_2 t^{m+1} e^{wt}}{n^m} \sum_{j=0}^{m+p} C_j(t/n)\|A^j x\|,$$

where $C_{m+1}(t) = K_{m+1}(t)e^{-wt} + M_1/(m+1)!$ and $C_j(t) = K_j(t)e^{-wt}$ for all $j \neq m+1$.

So the question arises: *what is the lower estimate of the error $\|(F(t/n))^n x - e^{tA}x\|$?*

In 2018, Ivan Remizov formulated the following conjecture:

Conjecture (Remizov [4]). Let $(e^{tA})_{t\geq 0}$ be a C_0-semigroup in a Banach space X, and F is a Chernoff function for operator A (recall that this implies $F(0)=I$ and $F'(0)=A$ but says nothing about $F''(0)$) and number $T>0$ is fixed. Suppose that vector x is from intersection of domains of operators $F'(t)$, $F''(t)$, $F'''(t)$, $F''''(t)$, $F'(t)F''(t)$, $(F'(t))^2F''(t)$, $(F''(t))^2$ for each $t\in[0,T]$, and suppose that if $Z(t)$ is any of these operators then function $t\to Z(t)x$ is continuous for each $t\in[0,T]$. Then there exists such a number $C>0$, that for each $t\in[0,T)$ and each $n\in\mathbb{N}$ the following inequality holds, where $B=F''(0)$:

$$\|(F(t/n))^n x - e^{tA}x - \frac{t^2}{2n}e^{tA}(B-A^2)x\| \leq \frac{C}{n^2}.$$

Although this hypothesis is not true in general, the following theorem holds:

Theorem 3. *Suppose that:*

1) C_0-semigroup $(e^{tA})_{t\geq 0}$ in a Banach space X has bounded generator $A\in\mathscr{L}(X)$.

2) $T>0$ and there are a mapping $F\colon[0,T]\to\mathscr{L}(X)$ and constants $M\geq 1$, $w\geq 0$ such that $\|(F(t))^k\|\leq Me^{kwt}$ for all $t\in[0,T]$, $k\in\mathbb{N}$.

3) There exist such bounded operator $B\in\mathscr{L}(X)$ and constant $K\geq 0$ that for all $t\in[0,T]$ we have $\|F(t)-I-tA-\frac{t^2}{2}B\|\leq Kt^3$.

Then there exists such a number $C>0$, that for each $t\in[0,T]$ and each $n\in\mathbb{N}$ the following inequality holds: $\|(F(t/n))^n - e^{tA} - \frac{t^2}{2n}\int_0^1 e^{tsA}(B-A^2)e^{t(1-s)A}ds\|\leq \frac{C}{n^2}$.

Acknowledgments. The authors are partially supported by Laboratory of Dynamical Systems and Applications NRU HSE, grant of the Ministry of science and higher education of the RF, ag. № 075-15-2022-1101

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