Feynman formula that (maybe) provides a solution to a generalized Black-Scholes equation

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Introduction

- This is a talk on the research that is only in the very beginning of it.
- There will be no proved theorems.
- There will be ideas and problem settings.
- There will be a formula that we wish to prove to be a solution of a generalized Black-Scholes equation.
- This is my second in life talk in English, your kind support is appreciated :)

Chernoff theorem

Theorem. Suppose that the following three conditions are met:

1. C_0-semigroup $(e^{tL})_{t\geq 0}$ with generator $(L, D(L))$ in Banach space $\mathcal{F}$ is given.
2. There exists a strongly continuous mapping $S\colon [0,+\infty) \to \mathscr{L}(\mathcal{F})$ such that $S(0) = I$ and the inequality $\|S(t)\| \leq e^{wt}$ holds for all $t \geq 0$.
3. There exists a dense linear subspace $D \subset \mathcal{F}$ such that for all $f \in D$ there exists a limit $S'(0)f := \lim_{t\to+0}(S(t)f - f)/t$. Moreover, $S'(0)$ on D has a closure that coincides with the generator $(L, D(L))$.

Then the following statement holds:

(C) For every $f \in \mathcal{F}$, as $n \to \infty$ we have $S(t/n)^n f \to e^{tL}f$ locally uniformly with respect to $t \geq 0$, i.e. for each $T > 0$ and each $f \in \mathcal{F}$ we have

$$\lim_{n\to\infty} \sup_{t\in[0,T]} \|S(t/n)^n f - e^{tL}f\| = 0.$$

Above $S(t/n)^n = \underbrace{S(t/n) \circ \cdots \circ S(t/n)}_{n}$ is the composition of n copies of linear bounded operator $S(t/n)$ defined everywhere on $\mathcal{F}$.

Definition. Let C_0-semigroup $(e^{tL})_{t\geq 0}$ with generator L in Banach space $\mathcal{F}$ be given. The mapping $S\colon [0,+\infty) \to \mathscr{L}(\mathcal{F})$ is called a *Chernoff function for operator* L iff it satisfies the condition (C) of Chernoff theorem above.
In this case expressions $S(t/n)^n$ are called *Chernoff approximations to the semigroup* e^{tL}.

Heat equation and heat semigroup: known facts

Cauchy problem for the heat equation with constant coefficient $a > 0$ is

$$\begin{cases} u_t(t,x) = au_{xx}(t,x), x \in \mathbb{R}, t > 0, \\ u(0,x) = u_0(x), x \in \mathbb{R}. \end{cases}$$

Let us define operator H by the rule $(Hf)(x) = af''(x)$ for all $x \in \mathbb{R}$ and all f from some dense subspace of appropriate Banach space of functions $f : \mathbb{R} \to \mathbb{R}$. Let us introduce function-valued function U by the rule $U(t) = u(t, \cdot) = [x \longmapsto u(t,x)]$. Then the above Cauchy problem can be rewritten as

$$\begin{cases} U'(t) = HU(t), t > 0, \\ U(u) = u_0. \end{cases}$$

If H is the generator of C_0-semigroup in the space of functions that we work in, then the solution of both Cauchy problems are given by the so-called heat semigroup $(e^{tH})_{t \geq 0}$:

$$u(t,x) = (U(t))(x) = (e^{tH}u_0)(x) = \frac{1}{\sqrt{2at}} \int_{\mathbb{R}} \exp\left(\frac{-(x-y)^2}{2at}\right) u_0(y)dy.$$

Heat equation with variable coefficient

Consider Cauchy problem for the heat equation with variable coefficient $a(x) > 0$

$$\begin{cases} u_t(t,x) = a(x)u_{xx}(t,x), x \in \mathbb{R}, t > 0, \\ u(0,x) = u_0(x), x \in \mathbb{R}. \end{cases}$$

Let us define operator H_v by the rule $(H_v f)(x) = a(x)f''(x)$ for all $x \in \mathbb{R}$ and all f from some dense subspace of appropriate Banach space of functions $f \colon \mathbb{R} \to \mathbb{R}$. Let us introduce function-valued function U by the rule $U(t) = u(t,\cdot) = [x \longmapsto u(t,x)]$. Then the above Cauchy problem can be rewritten as

$$\begin{cases} U'(t) = H_v U(t), t > 0, \\ U(u) = u_0. \end{cases}$$

As before, the solution is given by the semigroup $(e^{tH_v})_{t\geq 0}$, but the analogue of previous formula does not give the semigroup anymore:

$$u(t,x) = (U(t))(x) = (e^{tH_v}u_0)(x) \neq \frac{1}{\sqrt{2a(x)t}} \int_{\mathbb{R}} \exp\left(\frac{-(x-y)^2}{2a(x)t}\right) u_0(y)$$

Heat equation with variable coefficient

Operator-valued function S

$$(S(t)u_0)(x) = \frac{1}{\sqrt{2a(x)t}} \int_{\mathbb{R}} \exp\left(\frac{-(x-y)^2}{2a(x)t}\right) u_0(y)dy$$

does not posess the semigroup property, i.e. we should not expect that $S(t_1 + t_2) = S(t_1)S(t_2)$ and $e^{tH_v} \neq S(t)$, but this function S is still useful for the following reason. Under sertain conditions it is known that the semigroup e^{tH_v} is given in the form

$$(e^{tH_v}u_0)(x_0) = \left(\lim_{n\to\infty} S(t/n)^n u_0\right)(x_0) = \lim_{n\to\infty} \underbrace{\int_{\mathbb{R}} \cdots \int_{\mathbb{R}}}_{n} \prod_{k=0}^{n-1} \frac{1}{(2a(x_k)t)^{n/2}} \times$$

$$\times \exp\left(\sum_{k=0}^{n-1} \frac{-(x_k - x_{k+1})}{2(t/n)a(x_k)}\right) u_0(x_n)dx_1 \ldots dx_n$$

This is a very particular case of what is written in the paper:
Ya.A. Butko, M. Grothaus, O.G. Smolyanov. Lagrangian Feynman formulas for second-order parabolic equations in bounded and unbounded domains.// IDAQP vol. 13, No. 3 (2010), 377-392.

Now we are coming to the main topic of the talk!

Our plan is to consider the Black-Scholes equation and do with it exactly what was done with the heat equation above.

Consider linear differential operator A given by

$$(Af)(x) = a(x)f''(x) + b(x)f'(x) + c(x)f(x) \qquad (1)$$

If operator A generates a C_0-semigroup $(e^{tA})_{t\geq 0}$ then Cauchy problem for parabolic equation

$$\begin{cases} u_t(t,x) = a(x)u_{xx}(t,x) + b(x)u_x(t,x) + c(x)u(t,x), \\ u(0,x) = u_0(x) \end{cases} \qquad (2)$$

has solution $u(t,x) = (e^{tA}u_0)(x)$. Moreover for each $T > 0$ Cauchy problem for parabolic equation

$$\begin{cases} -v_t(t,x) = a(x)v_{xx}(t,x) + b(x)v_x(t,x) + c(x)v(t,x), \\ v(T,x) = v_T(x) \end{cases} \qquad (3)$$

has solution $v(t,x) = (e^{(T-t)A}v_T)(x)$. Note that (3) becomes the (useful in mathematical finance) Black-Scholes equation for option pricing if we use the following notation: v is the price of the option as a function of stock price x and time t, $a(x) = \frac{1}{2}\sigma^2 x^2$ where $\sigma > 0$ is the volatility of the stock, $b(x) = rx$ and $c(x) = -r$ where $r > 0$ is the risk-free interest rate.

Black-Scholes equation and Black-Scholes semigroup

The operator L defined as

$$(Lf)(x) = \frac{1}{2}\sigma^2 x^2 f''(x) + rxf'(x) - rf(x)\cdot$$

is an unbounded operator in Banach space

$$Y^{s,\tau} = \left\{ u \in C(0,\infty) : \lim_{x\to\infty} \frac{u(x)}{1+x^s} = 0,\ \lim_{x\to 0} \frac{u(x)}{1+x^{-\tau}} = 0 \right\}$$

with respect to the norm

$$\|u\|_{Y^{s,\tau}} = \sup_{x>0} \left| \frac{u(x)}{(1+x^s)(1+x^{-\tau})} \right|.$$

In the case of constant parameters $\sigma > 0$ and $r > 0$ the solution to the Cauchy problem for the Black-Scholes equation

$$\begin{cases} u_t(t,x) = Lu(t,x), x > 0 \\ u(0,x) = u_0(x), x > 0 \end{cases}$$

with $u(t,x) > 0$ is given (see e.g. Goldstain-Goldstain papers) as

$$u(t,x) = (e^{tL}u_0)(x) = (4\pi t)^{-1/2} e^{-rt} \int_{\mathbb{R}} e^{-y^2/(4t)} u_0\left(x e^{(r-\sigma^2/2)t - \sigma y/\sqrt{2}} \right) dy$$

Generalized Black-Scholes equation: possible Chernoff function and Feynman formula

What if the coefficients σ and r are not constants, but are bounded, continuous and positive functions?

$$(S(t)u_0)(x) = (4\pi t)^{-1/2} e^{-r(x)t} \int_{\mathbb{R}} e^{-y^2/(4t)} u_0 \left(x e^{(r(x)-\sigma^2(x)/2)t - \sigma(x)y/\sqrt{2}} \right) dy$$

If $S(t)$ is a Chernoff function, then Chernoff approximations will be

$$\begin{aligned}
&(S(t/n)\, u_0)^n (x) = \\
&= (-1)^n \left(\frac{n}{2\pi t}\right)^{n/2} \int_0^\infty \cdots \int_0^\infty \exp\left(- \left(r(x) + \sum_{k=1}^{n-1} r(y_k) \right) \frac{t}{n} \right) \times \\
&\times \exp\left[-\frac{n}{2t} \left(\frac{[t(r(x) - \sigma^2(x)/2)/n - \ln(y_1/x)]^2}{\sigma^2(x)} + \right.\right. \\
&\left.\left. + \sum_{k=1}^{n-1} \frac{[t(r(y_k) - \sigma^2(y_k)/2)/n - \ln(y_{k+1}/y_k)]^2}{\sigma^2(y_k)} \right) \right] \times \\
&\times \frac{u_0(y_n)\, dy_1 \ldots dy_n}{\sigma(x) y_n \prod_{k=1}^{n-1} \sigma(y_k) y_k}
\end{aligned}$$

Work in progress

Our plan is to prove, that the function $S(t)$ given by

$$(S(t)u_0)(x) = (4\pi t)^{-1/2} e^{-r(x)t} \int\limits_{\mathbb{R}} e^{-y^2/(4t)} u_0 \left(x e^{(r(x)-\sigma^2(x)/2)t - \sigma(x)y/\sqrt{2}} \right) dy$$

is a Chernoff function. It means we have to show that all the conditions of Chernoff theorem hold.

Thank you for your attention!