# Chernoff approximations for resolvents of generators of $C_0$-semigroups

**I. D. Remizov** [1]

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*Summary of the talk.* The method of Chernoff approximation [1] is an extremely effective tool for expressing $e^{tL}$ in terms of variable coefficients of operator $L$. The talk shows that this method can be also be used for expressing $(\lambda I - L)^{-1}$ in terms of variable coefficients of operator $L$, and for finding the solution of the corresponding differential equation $\lambda f - Lf = g$. We demonstrate this on the second order differential operator $L$. As a corollary, we obtain two new representations of the solution of an inhomogeneous second order linear ordinary differential equation in terms of functions that are the coefficients of this equation playing the role of parameters for the problem. This reasoning also works in the multi-dimensional situation, where we have an elliptic PDE instead of ODE. Full proofs are available in the preprint [2].

Let us recall the Chernoff theorem.

*Chernoff theorem, one of the wordings.* Suppose that the following three conditions are met:
1. $C_0$-semigroup $(e^{tL})_{t\geq 0}$ with generator $(L, D(L))$ in Banach space $\mathcal{F}$ is given, such that for some $w \geq 0$ the inequality $\|e^{tL}\| \leq e^{wt}$ holds for all $t \geq 0$.

2. There exists a strongly continuous mapping $S\colon [0, +\infty) \to \mathscr{L}(\mathcal{F})$ such that $S(0) = I$ and the inequality $\|S(t)\| \leq e^{wt}$ holds for all $t \geq 0$.

3. There exists a dense linear subspace $D \subset \mathcal{F}$ such that for all $f \in D$ there exists a limit $S'(0)f := \lim_{t\to+0}(S(t)f - f)/t$. Moreover, $S'(0)$ on $D$ has a closure that coincides with the generator $(L, D(L))$. Then the following statement holds:

(C) For every $f \in \mathcal{F}$, as $n \to \infty$ we have $S(t/n)^n f \to e^{tL} f$ locally uniformly with respect to $t \geq 0$, i.e. for each $T > 0$ and each $f \in \mathcal{F}$ we have $\lim_{n\to\infty} \sup_{t\in[0,T]} \|S(t/n)^n f - e^{tL} f\| = 0$.

*Remark 1.* Above $S(t/n)^n = \underbrace{S(t/n) \circ \cdots \circ S(t/n)}_{n}$ is the composition of $n$ copies of linear bounded operator $S(t/n)$ defined everywhere on $\mathcal{F}$.

*Definition 1.* Let $C_0$-semigroup $(e^{tL})_{t\geq 0}$ with generator $L$ in Banach space $\mathcal{F}$ be given. The mapping $S\colon [0, +\infty) \to \mathscr{L}(\mathcal{F})$ is called a *Chernoff function for operator* $L$ iff it satisfies the condition (C) of Chernoff theorem above. In this case expressions $S(t/n)^n$ are called *Chernoff approximations to the semigroup* $e^{tL}$.

**Main idea of the talk.** Thanks to Chernoff theorem we have $e^{tL} f = \lim_{n\to\infty} S(t/n)^n f$ for all vectors $f$ and for properly selected operator-valued function $S$. Also, there is a standard fact that for $\lambda$ with $Re\lambda$ large enough for the resolvent of $L$ we have the followng representation: $(\lambda I - L)^{-1} f = \int_0^\infty e^{-\lambda t} e^{tL} f dt$, so we can substitute $e^{tL}$ by $S(t/n)^n$ and get approximations for the resolvent:

$$(\lambda I - L)^{-1} f = \int_0^\infty e^{-\lambda t} e^{tL} f dt = \int_0^\infty e^{-\lambda t} \lim_{n\to\infty} S(t/n)^n f dt = \lim_{n\to\infty} \int_0^\infty e^{-\lambda t} S(t/n)^n f dt.$$

Above the first (left) equality is a classical fact, the second inequality is due to Chernoff theorem, and the last (the right) equality is the main idea of all results that follow below.

*Theorem 1.* Let $\mathcal{F}$ be real or complex Banach space, and let $\mathscr{L}(\mathcal{F})$ be the set of all linear bounded operators in $\mathcal{F}$. Suppose that linear operator $L\colon \mathcal{F} \supset D(L) \to \mathcal{F}$ generates $C_0$-semigroup $(e^{tL})_{t\geq 0}$ satisfying for some constants $M \geq 1$ and $\omega \geq 0$ inequality $\|e^{tL}\| \leq Me^{\omega t}$ for

[1] HSE University, International Laboratory of Dynamical Systems and Applications, Russia, Nizhny Novgorod. Email: ivremizov@yandex.ru

all $t \geq 0$. Suppose that function $S\colon [0,+\infty) \to \mathscr{L}(\mathcal{F})$ is given and $\|S(t)^k\| \leq Me^{\omega t k}$ for all $t \geq 0$ and all $k = 1, 2, 3, \dots$ Let us denote the resolvent of $(L, D(L))$ by the symbol $R_\lambda = (\lambda I - L)^{-1}$ for all $\lambda \in \rho(L)$. Suppose that the number $\lambda \in \mathbb{C}$ is given and $Re\lambda > \omega$. Then $\lambda \in \rho(L)$ and:

1. If for all $T > 0$ we have $\lim_{n\to\infty} \sup_{t\in[0,T]} \left\|e^{tL}f - (S(t/n))^n f\right\| = 0$ for all $f \in \mathcal{F}$, then for all $f \in \mathcal{F}$ we have

$$\lim_{n\to\infty} \left\| R_\lambda f - \int_0^\infty e^{-\lambda t} (S(t/n))^n f dt \right\| = 0.$$

2. If for all $T > 0$ we have $\lim_{n\to\infty} \sup_{t\in[0,T]} \left\|e^{tL} - (S(t/n))^n\right\| = 0$, then we have

$$\lim_{n\to\infty} \left\| R_\lambda - \int_0^\infty e^{-\lambda t} (S(t/n))^n dt \right\| = 0.$$

*Theorem 2.* Consider second order ordinary differential equation for function $f\colon \mathbb{R} \to \mathbb{R}$

$$a(x)f''(x) + b(x)f'(x) + (c(x) - \lambda)f(x) = -g(x) \text{ for all } x \in \mathbb{R}, \tag{1}$$

where functions $a, b, c, g\colon \mathbb{R} \to \mathbb{R}$ are known parameters and number $\lambda \in \mathbb{C}$ is also a known parameter. Assume that there exists constant $a_0 > 0$ such that $a(x) > a_0$ for all $x \in \mathbb{R}$. Assume that there exists $\beta \in (0, 1]$ such that function $c$ is bounded and Hölder continuous with Hölder exponent $\beta$, and functions $a$, $x \mapsto 1/a(x)$, $b$ are bounded and Hölder continuous with Hölder exponent $\beta$ with derivatives of order one and two. Assume that function $g$ is continuous and vanishes at infinity. Assume that $\mathbb{R} \ni \lambda > \max(0, \sup_{x\in\mathbb{R}} c(x))$.

Then for equation (1) there exists a unique continuous and vanishing at infinity solution $f$ given for all $x_0 \in \mathbb{R}$ by the formula

$$f(x_0) = \lim_{n\to\infty} \int_0^\infty e^{-\lambda t} \Bigg[ \underbrace{\int_{\mathbb{R}} \cdots \int_{\mathbb{R}}}_{n} \exp\left( \frac{t}{n} \sum_{j=1}^n \left( c(x_{j-1}) - \frac{b(x_{j-1})^2}{2a(x_{j-1})} \right) \right) \times$$

$$\times \exp\left( \sum_{j=1}^n \frac{b(x_{j-1})(x_j - x_{j-1})}{a(x_{j-1})} \right) \times p_a(t/n, x_0, x_1) \dots p_a(t/n, x_{n-1}, x_n) g(x_n) dx_1 \dots dx_n \Bigg] dt,$$

where the limit $\lim_{n\to\infty}$ exists uniformly in $x_0 \in \mathbb{R}$, and we denoted

$$p_a(t, x, y) = \frac{1}{\sqrt{2\pi t a(x)}} \exp\left( \frac{-(x-y)^2}{2ta(x)} \right) \text{ for all } x, y \in \mathbb{R}, t > 0.$$

*Some notation.* Let us use symbol $UC_b(\mathbb{R})$ to denote Banach space of all bounded and uniformly continuous functions $f\colon \mathbb{R} \to \mathbb{R}$ with the uniform norm $\|f\| = \sup_{x\in\mathbb{R}} |f(x)|$. Let us use symbol $C_b^\infty(\mathbb{R})$ for the subspace of $UC_b(\mathbb{R})$ consisting of all infinitely differentible functions that are bounded and have bounded derivatives of all orders.

*Theorem 3.* Suppose that functions $a, b, c \in UC_b(\mathbb{R})$ are bounded with their derivatives up to order 3, and there exists such a constant $a_0 > 0$ that estimate $\inf_{x\in\mathbb{R}} a(x) \geq a_0 > 0$ is satisfied for all $x \in \mathbb{R}$. For each function $\phi \in C_b^\infty(\mathbb{R}) = D(A)$ define $A\phi = a\phi'' + b\phi' + c\phi$. For each $t \geq 0$, each $x \in \mathbb{R}$ and each $f \in UC_b(\mathbb{R})$ define

$$(S(t)f)(x) = \frac{1}{4} f\left(x + 2\sqrt{a(x)t}\right) + \frac{1}{4} f\left(x - 2\sqrt{a(x)t}\right) + \frac{1}{2} f\left(x + 2b(x)t\right) + tc(x)f(x). \tag{2}$$

Assume also that $\mathbb{R} \ni \lambda > \sup_{x\in\mathbb{R}} |c(x)| = \|c\|$. Then:

1. Closure $\overline{A}$ of operator $A$ generates a $C_0$-semigroup in $UC_b(\mathbb{R})$.

2. For each $g \in UC_b(\mathbb{R})$ the solution $f\colon \mathbb{R} \to \mathbb{R}$ of the equation

$$a(x)f''(x) + b(x)f'(x) + (c(x) - \lambda)f(x) = -g(x) \text{ for all } x \in \mathbb{R},$$

exists, is unique in $UC_b(\mathbb{R})$ and is given for all $x \in \mathbb{R}$ by the formula

$$f(x) = \int_0^\infty e^{-\lambda t} \left(e^{\overline{A}}g\right)(x)dt = \lim_{n\to\infty} \int_0^\infty e^{-\lambda t} \left((S(t/n))^n g\right)(x)dt, \tag{3}$$

where $S(t/n)$ is obtained by substitution of $t$ with $t/n$ in (2), and $(S(t/n))^n$ is the composition of $n$ copies of linear bounded operator $S(t/n)$.

Suppose additionally that function $g$ is bounded with derivatives up to order 5. Then:

3. There exist nonnegative constants $C_0, C_1, \ldots, C_4$ such that for all $t > 0$ and all $n \in \mathbb{N}$ the following inequality holds:

$$\|S(t/n)^n g - e^{t\overline{A}}g\| \le \frac{t^2 e^{\|c\|t}}{n}\big(C_0\|g\| + C_1\|g'\| + C_2\|g''\| + C_3\|g'''\| + C_4\|g^{(IV)}\|\big).$$

4. Error bound in (3) for all $n \in \mathbb{N}$ is given by inequality

$$\sup_{x\in\mathbb{R}} \left| f(x) - \int_0^\infty e^{-\lambda t} \left((S(t/n))^n g\right)(x)dt \right| \le \frac{2C_g}{n\cdot(\lambda - \|c\|)^3},$$

where $C_g = C_0\|g\| + C_1\|g'\| + C_2\|g''\| + C_3\|g'''\| + C_4\|g^{(IV)}\|$.

5. Integral in item 2 can be calculated over $[0, T]$ instead of $[0, \infty)$ with controlled level of error. This means that for each $\varepsilon > 0$ there exists $T = \max\left(0, \frac{1}{\lambda - \|c\|}\ln\frac{2}{(\lambda-\|c\|)\varepsilon}\right)$ such that for all $n \in \mathbb{N}$ we have

$$\sup_{x\in\mathbb{R}} \left| f(x) - \int_0^T e^{-\lambda t} \left((S(t/n))^n g\right)(x)dt \right| \le \frac{2C_g}{n\cdot(\lambda - \|c\|)^3} + \varepsilon.$$

*Remark 2.* Independently of Chernoff function used (is it based on integral operators as in theorem 2 or on translation operators as in theorem 3), Chernoff approximations are allowing to calculate value of the solution in only one point of the domain of solution (in one point $x \in \mathbb{R}$ in our examples). Meanwhile methods based on a computational grid calculate values of the solution in all points of the computational grid. Moreover, values of Chernoff approximations at different points of the domain can be calculated in parallel, using multi-core processors and GPU which is an advantage of this approach.

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