On embedding of two-dimensional separatrices of saddle equilibria in four-dimensional manifolds

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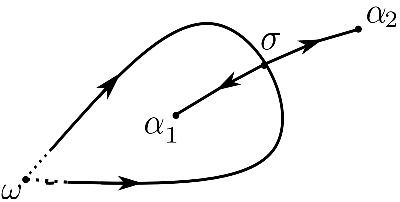
Gradient-like flows

Let M^n be a closed connected smooth manifold of dimension $n \geq 1$. A smooth map $F : M^n \times \mathbb{R} \to M^n$ such that:

1. $F(x, 0) = x$ for any $x \in M^n$;
2. $F(F(x, s), t) = F(x, s + t)$ for any $x \in M^n$, s, $t \in \mathbb{R}$

is called a smooth flow. It is usually denoted $F(x, t) = f^t(x)$. A smooth flow $f^t : M^n \to M^n$ is called gradient-like if its non-wandering set Ω_{f^t} consists of a finite number of hyperbolic equilibria and invariant manifolds of equilibria transversally intersect each other. We will say that a hyperbolic equilibrium p is of type $(i, n - i)$ if $\dim W^u_p = i$. Let $G(M^n)$ be a class of gradient-like flows on M^n such that for any flow f^t invariant manifolds of different saddles do not intersect each other.

History of the issue



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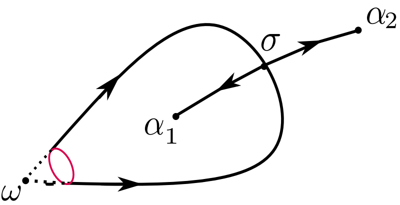

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Fundamental impossibility of combinatorial classification for the class $G(M^n)$, $n \geq 4$

Proposition

There are non-equivalent flows f^t, $f'^t \in G(M^n)$ with three states of equilibria, $n \in \{8, 16\}$

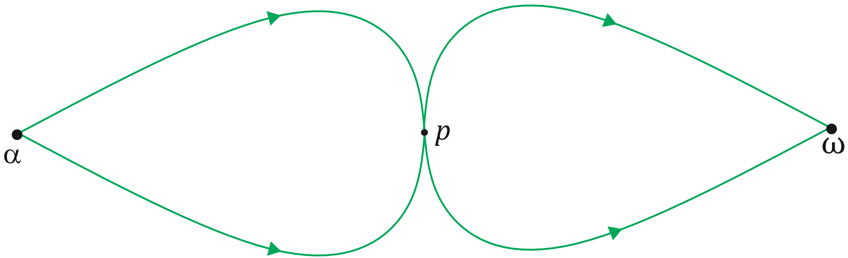

Figure: A gradient-like flow with wildly embedded two-dimensional separatrix

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Fundamental impossibility of combinatorial classification for the class $G(M^n)$, $n \geq 4$

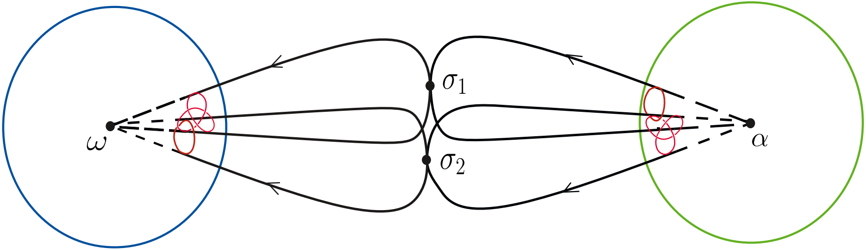

Figure: A gradient-like flow with wildly embedded two-dimensional separatrix

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Flows with only one saddle of type $(2,2)$

We consider a subclass $G_1(M^4)$ of $G(M^4)$ such that for any flow $f^t \in G_1(M^4)$ there is only one saddle of type $(2,2)$.

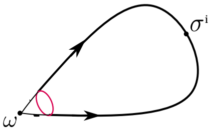
Theorem

An ambient manifold M^4 of any flow $f^t \in G_1(M^4)$ is homeomorphic to a connected sum of a finite number of copies of $\mathbb{S}^1 \times \mathbb{S}^3$ and $\mathbb{C}\mathrm{P}^2$ if and only if f^t carries only one saddle of type $(2,2)$.

Grines V.Z., Zhuzhoma E.V., Medvedev V.S., *On the structure of the ambient manifold for Morse-Smale systems without heteroclinic intersections*, Trudy Matematicheskogo instituta im. V. A. Steklova RAN, 2017, V. 297, p. 201-210.

Topology of embeddings of separatrices of flows from $G_1(M^4)$

Let $f^t \in G_1(M^4)$, $\sigma^i \in \Omega_{f^t}$ be a saddle of type $(i, n-i)$, $i \in \{2,\ 3\}$. Then there is a unique sink equilibrium ω such that $\mathrm{cl}W^u_{\sigma^i} = W^u_{\sigma^i} \cup \{\omega\}$.



Lemma

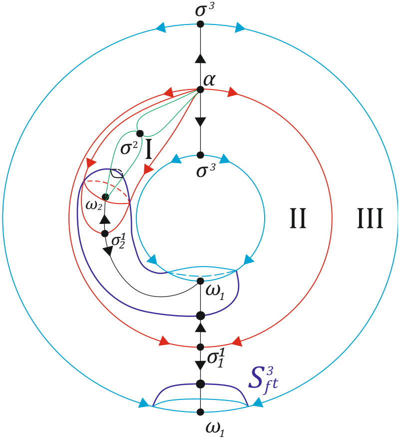
Let $f^t \in G_1(M^4)$. Then:

- *for $i = 3$ the set $\mathrm{cl}W^u_{\sigma^3}$ is a locally flat three-dimensional sphere;*
- *for $i = 2$ the set $\mathrm{cl}W^u_{\sigma^2}$ is a locally flat two-dimensional sphere.*

Combinatorial classification for the class $G_1(M^4)$

Theorem

Classes of topological equivalence of flows from $G_1(M^4)$ can be described in combinatorial terms.



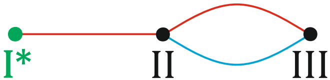


Figure: Bicolor graph of the flow f^t