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Solution of the Kardar-Parisi-Zhang equation on a quarter plane

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The KPZ equation is phenomenological model describing sputtering of a substance on a solid state surface:

Connection between 'physical' height of growing crystal and its reduced height:

$$h_{phys}(\vec{x}, t) = c\, t + h(\vec{x}, t)\,, \qquad \vec{x} = (x_1, x_2)\,. \tag{1}$$

The Kardar-Parisi-Zhang (KPZ) equation:

$$\frac{\partial h}{\partial t} = \frac{c}{2}\,(\nabla h)^2 + \nu\,\nabla^2 h\,, \qquad \nabla = (\partial/\partial x_1, \partial/\partial x_2)\,, \tag{2}$$

where c is a rate of this growth in the direction of of local normal to the surface and ν is coefficient of the surface diffusion with flux:

$$\vec{J} = -\nu\,\nabla h\,. \tag{3}$$

Kardar M., Parisi G., Zhang Y.C. Dynamical scaling of growing interfaces // Physical Review Letters. 1986. V. 56. P. 889 - 892.

As a rule the KPZ equation is solved on the whole two-dimensional plane with initial condition:

$$h(\vec{x},0)=h_0(\vec{x})\,,\qquad \vec{x}\in\mathbb{R}^2\,, \tag{4}$$

exact solution of the Cauchy problem (2) and (4) being equal to:

$$h(\vec{x},t)=\frac{2\,\nu}{c}\ln\left(\int_{\mathbb{R}^2}\exp\left[-\frac{(\vec{x}-\vec{\xi}\,)^2}{4\,\nu\,t}+\frac{c\,h_0(\vec{\xi}\,)}{2\,\nu}\right]\frac{d^{\,2}\xi}{4\,\pi\,\nu\,t}\right). \tag{5}$$

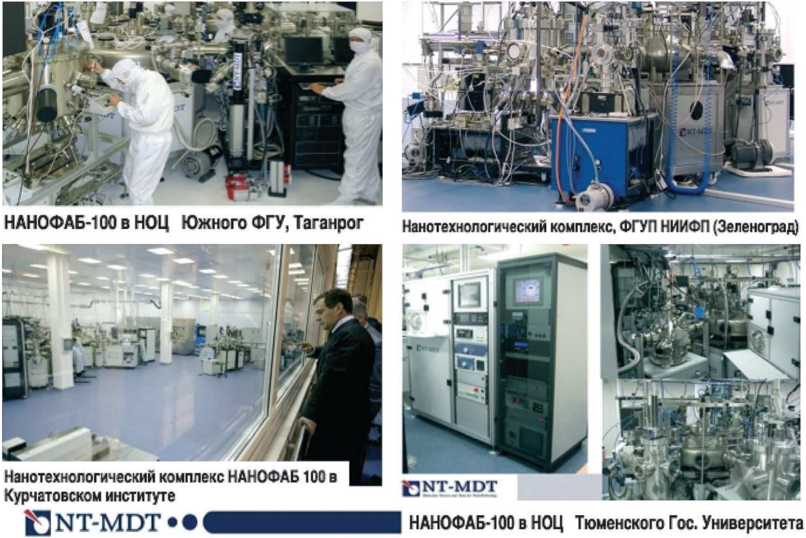
For further considerations it is convenient to rescale variables as follows (l is spatial scale of initial condition (4)):

$$x_1\to x/l\,,\quad x_2\to y/l\,,\quad \nu\,t/l^2\to t\,,\quad ch/\nu\to h\,, \tag{6}$$

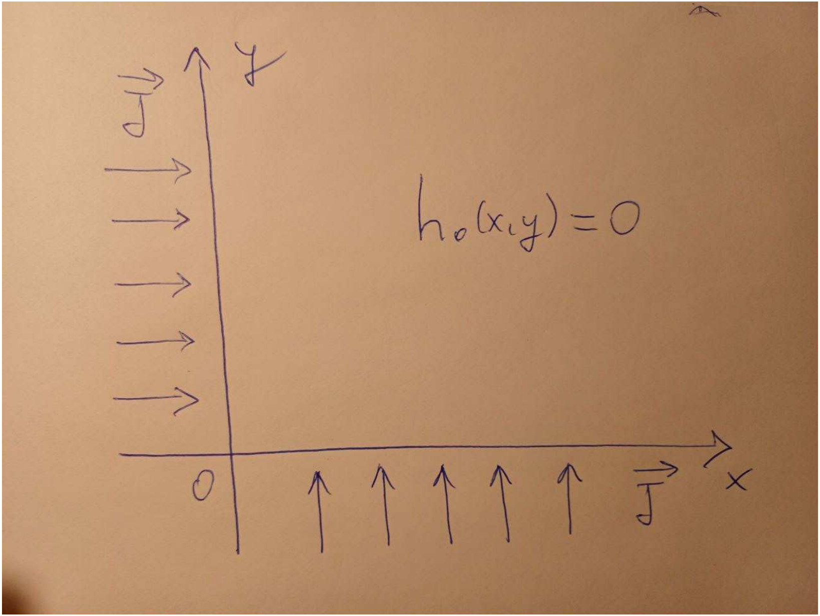
then the KPZ equation (2) is written in the next dimensionless form:

$$\frac{\partial h}{\partial t}=\frac{1}{2}\left(\frac{\partial h}{\partial x}\right)^2+\frac{1}{2}\left(\frac{\partial h}{\partial y}\right)^2+\frac{\partial^2 h}{\partial x^2}+\frac{\partial^2 h}{\partial y^2}\,. \tag{7}$$

Technologically, the process of crystal surface growth always occurs in a bounded domain:



Mixed problem for the KPZ equation on the quarter-plane:



Mixed problem for the KPZ equation on the quarter-plane:

$$\frac{\partial h}{\partial t} = \frac{1}{2}\left(\frac{\partial h}{\partial x}\right)^2 + \frac{1}{2}\left(\frac{\partial h}{\partial y}\right)^2 + \frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2}\,. \qquad (8)$$

Initial condition on the quarter-plane:

$$h(x,y,0) = 0\,, \quad (x,y) \in D = \left\{(x,y) \in \mathbb{R}^2 \,|\, x > 0\,, y > 0\,\right\} \qquad (9)$$

Boundary condition on the edge of quarter-plane (g is a constant):

$$\frac{\partial h}{\partial x}(0,y,t) = -2\,g\,, \qquad \frac{\partial h}{\partial y}(x,0,t) = -2\,g\,. \qquad (10)$$

Let one find solution of mixed problem (8)-(10) in the following form:

$$h(x,y,t) = H(x,t) + H(y,t)\,. \qquad (11)$$

SPLITTING OF THE PROBLEM!

Mixed problem for function $H(x,t)$ on half-line:

$$\frac{\partial H}{\partial t} = \frac{1}{2}\left(\frac{\partial H}{\partial x}\right)^2 + \frac{\partial^2 H}{\partial x^2}\,, \qquad x > 0\,. \tag{12}$$

$$H(x,0) = 0\,, \qquad \frac{\partial H}{\partial x}(0,t) = -2\,g\,. \tag{13}$$

Linearizing of mixed problem (12)-(13):

$$H(x,t) = 2\,\ln\varphi(x,t)\,. \tag{14}$$

Linear mixed problem on half-line:

$$\frac{\partial \varphi}{\partial t} = \frac{\partial^2 \varphi}{\partial x^2}\,, \quad \varphi(x,0) = 1\,, \quad \frac{\partial \varphi}{\partial x}(0,t) + g\,\varphi(0,t) = 0\,. \tag{15}$$

The Laplace transform:

$$\phi(x,p) = \int_0^{+\infty} \varphi(x,t)\,\exp(-p\,t)\,dt\,. \tag{16}$$

The Laplace transform of the linear mixed problem (15):

$$p\,\phi(x,p) - 1 = \frac{d^2\phi(x,p)}{d\,x^2}$$

$$\frac{d\phi(0,p)}{d\,x} + g\,\phi(0,p) = 0$$

$$\phi(x,p) = \frac{1}{p}\left[1 + \frac{g}{\sqrt{p} - g}\;\exp(-\sqrt{p}\,x)\right]. \qquad (17)$$

$$\varphi(x,t) = erf\,\frac{x}{2\sqrt{t}} + \exp(-g\,x + g^2\,t)\,erfc\,\frac{x - 2\,g\,t}{2\sqrt{t}}. \qquad (18)$$

$$erf\,(z) = \frac{2}{\sqrt{\pi}}\int_0^z \exp(-\zeta^2)\,d\zeta\,.$$

$$erfc(z) = 1 - erf\,(z)\,.$$

Exact solution for the auxiliary function:

$$H(x,t) = 2\ln\left[erf\,\frac{x}{2\sqrt{t}} + \exp(-g\,x + g^2\,t)\,erfc\,\frac{x - 2\,g\,t}{2\sqrt{t}}\right]. \quad (19)$$

Graph of the reduced height $h(x, y, t)$ of the surface:

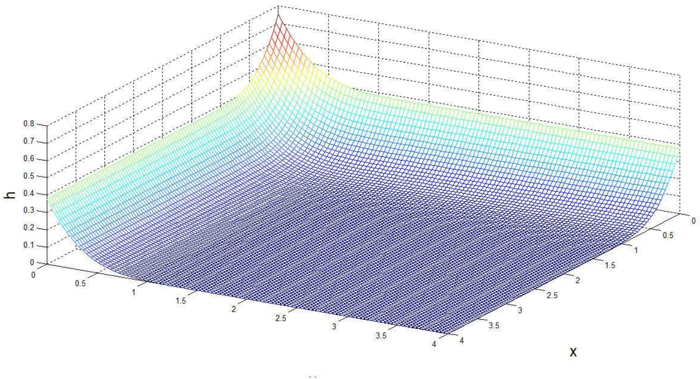


Рис.: $t = 0.1$, $g = 0.5$

Dimensionless flow of sputtering substance:

$$\vec{u} = -\nabla h\,, \qquad \nabla = (\partial/\partial x, \partial/\partial y)\,. \tag{20}$$

The Burgers vector equation:

$$\frac{\partial \vec{u}}{\partial t} + (\vec{u}\cdot\nabla)\,\vec{u} = \nabla^2\,\vec{u}\,, \quad \vec{u} \equiv (u_x(x,y,t), u_y(x,y,t))\,. \tag{21}$$

Mixed problem for equation (14) on the quarter-plane D :

$$\vec{u}\,|_{t=0} = 0\,, \qquad u_x(0,y,t) = u_y(x,0,t) = 2\,g\,. \tag{22}$$

Exact solution of mixed problem (14)-(15):

$$\vec{u}(x,y,t) = (U(x,t), U(y,t))\,, \qquad U(x,t) = -\frac{\partial H}{\partial x}(x,t)\,. \tag{23}$$

Graphs corresponding to the Burgers vector equation:

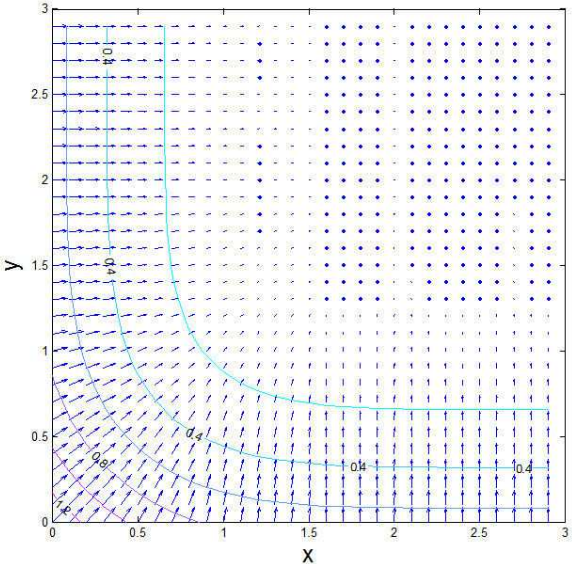


Рис.: $t = 0.3$, $g = 0.5$