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Contents

About optimal harvesting of renewable resource at a finite period of time

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The problem of rational use of renewable resources is one of the important tasks in mathematical biology. The optimal harvesting of the excess portion of individuals from the population contributes to more intensive reproduction of resources. In recent years a large number of works devoted to analytical and numerical study of dynamics regimes of two-age population have been published, for example [1, 2]. In more detail, the current state of research in the field of optimal resource extraction for different models of exploited populations is described in [3].

This paper considers the structured population at which individuals are divided into age or typical groups, given a normal autonomous system of difference equations. For such a population, the task of optimal collection of a renewable resource at a finite period of time.

Define $x_i(k)$, $i = 1, \dots, n$ the number of resources of each of the $n \geq 2$ of the species or classes at the moment $k = 0, 1, 2, \dots$. We will consider the model of the exploited population in the form

$$x(j+1) = F\big((1-u(j))x(j)\big), \quad j = 0, 1, 2, \dots, k-1,$$

where $x(j) = \big(x_1(j), \dots, x_n(j)\big) \in R_+^n$, $R_+^n \doteq \big\{x \in R^n : x_1 \geq 0, \dots, x_n \geq 0\big\}$,
$u(j) = \big(u_1(j), \dots, u_n(j)\big) \in [0,1]^n$ — control that can be varied to achieve the best collection result, $(1 - u_i(j))x_i(j)$ — number of remaining resource of the i-th species at the moment k after harvesting, $F(x) = \big(f_1(x), \dots, f_n(x)\big)$, $f_i(x)$ — real non-negative functions defined for all $x \in R_+^n$ of them are $f_i(0) = 0$, $f_i \in C^2(R_+^n)$, and Jacobi matrix $\left(\frac{\partial f_i}{\partial x_j}\right)_{i,j=1,\dots,n}$ is nondegenerate for all $x \in R_+^n$.

Let $C_i \geq 0$, $i = 1, \dots, n$ — the cost of the conditional unit of each class, then the cost of all all extracted products at the moment k is equal $z(k) = \sum\limits_{i=1}^{n} C_i x_i(k) u_i(k)$.

Define $\overline{u}(k) \doteq (u(0), \dots, u(k-1))$, where $u(j) = (u_1(j), \dots, u_n(j)) \in [0,1]^n$, $j = 0, 1, \dots, k-1$. For any $k = 1, 2, \dots$ consider the function

$$h\big(\overline{u}(k), x(0)\big) \doteq \sum_{j=0}^{k-1} z(j) = \sum_{j=0}^{k-1} \sum_{i=1}^{n} C_i x_i(j) u_i(j),$$

this is equal to the value of a resource extracted for k seizures.

Theorem 1. *Let the function* $D(x) \doteq \sum\limits_{i=1}^{n} C_i\big(f_i(x) - x_i\big)$ *reaches the maximum value in the only one point* $x^* \in R_+^n$ *и* $x_i^* \leq f_i(x^*) \neq 0$ *for any* $i = 1, \dots, n$. *Then for any* $x(0) \in R_+^n$ *such that* $x_i(0) \geq x_i^*$, $i = 1, \dots, n$, *function* $h\big(\overline{u}(k), x(0)\big)$ *reached the highest value*

$$h\big(\overline{u}^*(k), x(0)\big) = (k-1) \cdot D(x^*) + \sum_{i=1}^{n} C_i x_i(0)$$

on multiple $[0,1]^{kn}$ *at the following exploitation mode:* (1) *if* $k = 1$, *then* $u^*(0) = (1, \dots, 1)$;

(2) *if* $k = 2$, *then* $\overline{u}^*(2) = \big(u^*(0), u^*(1)\big)$, $u^*(0) = \Big(1 - \frac{x_1^*}{x_1(0)}, \dots, 1 - \frac{x_n^*}{x_n(0)}\Big)$, $u^*(1) = (1, \dots, 1)$;

(3) *if* $k \geq 3$, *then* $\overline{u}^*(k) = \big(u^*(0), \dots, u^*(k-1)\big)$, *where* $u^*(0) = \Big(1 - \frac{x_1^*}{x_1(0)}, \dots, 1 - \frac{x_n^*}{x_n(0)}\Big)$; $u^*(j) = \Big(1 - \frac{x_1^*}{f_1(x^*)}, \dots, 1 - \frac{x_n^*}{f_n(x^*)}\Big)$ *при* $j = 1, \dots, k-2$; $u^*(k-1) = (1, \dots, 1)$.

The work was carried out under the guidance of Professor of the Department of Functional Analysis and its Applications of VlSU L.I. Rodina.

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About the bifurcations of the Logistic Equation with Diffusion and Non-linear Multiplier of Delay

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We consider bifurcations in the problem

$$\frac{\partial u}{\partial t} = d\frac{\partial^2 u}{\partial x^2} + ru(1 - a(x)u(t-1,x)) \tag{1}$$

with boundary conditions

$$\left.\frac{\partial u}{\partial x}\right|_{x=0} = \left.\frac{\partial u}{\partial x}\right|_{x=1} = 0. \tag{2}$$

For consideration, the function $a(x)$ is chosen

$$a(x) = Cx^{-\alpha}, \quad 0 < \alpha < 1, C > 0. \tag{3}$$

The problem (1), (2) under the condition (3) has a clear biological meaning and models the dynamics of the development of a population of animals living in mountainous areas. For convenience, the function $a(x)$ is normalized so that

$$\int_0^1 a(x)dx = 1 \tag{4}$$

and the parameter C was chosen

$$C = 1 - \alpha. \tag{5}$$

In the problem (1), (2) for $r > 4.05265$, a stable cycle is born. At the left border of the habitat, a small change in the number of individuals is observed, whereas, when approaching the right border, these indicators increase markedly. It is shown that for $x \to 0$ the solution is $u(t,x) \to 0$. A decrease in the degree of α in the term with delay increases the amplitude of the oscillations.

Absorbing domain and Smale horseshoe in multidimensional Henon map

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In this talk we consider a multidimensional Henon map of a general form [1, 2]

$$\begin{cases} \bar{x} = f(x) + \sum\limits_{i=1}^{n} a_i v_i, \\ \bar{v_1} = x, \\ \bar{v_i} = v_{i-1}, i = \overline{2, n}, \end{cases} \tag{1}$$

where $a_i \in \mathbb{R}^1, f(x)$ is the quadratic function of the form

$$f(x) = \mu - x^2, \mu \in \mathbb{R}^1. \tag{2}$$

This map written in reverse numbering with the help of the variables change $v_j = u_j + x, j = \overline{1, n}, u_0 \equiv 0$ takes the next form

$$\begin{cases} \bar{x} = x + \sum\limits_{j=1}^{n} a_j u_j + F(x), \\ \bar{u_j} = -a_1 u_1 - a_2 u_2 - ... - (a_{j-1} - 1) u_{j-1} - ... - a_n u_n - F(x), \end{cases} \tag{3}$$

where $u_0 \equiv 0, F(x) = \big(\sum\limits_{i=1}^{n} a_i - 1\big) + f(x)$.

This map is the Lurie-type map with one nonlinearity and admits the comparison principle [3]. Using this approach for some region of parameters we prove the existence of absorbing domain G containing an attractor of the Henon map (**??**). We find another region of parameters for which G is no longer absorbing domain. We prove that this domain G and its image form the multidimensional Smale horseshoe. Therefore in this region of parameters the chaotic component of non-wandering set exists.

This work was supported by the Russian Foundation for Basic Research under Grant No. 18-01-00556 and Russian Scientific Foundation (numerics) under Grant No. 19-12-00367.

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Algebraic Constructions Generated By Causal Structure Of Space-times

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General relativity and quantum field theory both require systematic approach for working with families of causal subsets of space-times (suffice it to mention constructing the causal hierarchy of space-times, singularity theorems [1] and causal nets of operator algebras [2]). Here we shall confine ourselves to organizing subsets of Minkowski space-times M_1^D into algebraic systems (lattices) and describing their interrelations.

Let's consider the families of upper and lower cones on Minkowski space-time

$$\overset{+}{Con}\left(M_1^D\right) \equiv \left\{ \emptyset,\ \underset{x}{\overset{+}{con}} \middle| x \in M_1^D, x \cdot x \geq 0, x^0 \geq 0 \right\},$$
$$\overset{-}{Con}\left(M_1^D\right) \equiv \left\{ \emptyset,\ \underset{x}{\overset{-}{con}} \middle| x \in M_1^D, x \cdot x \geq 0, x^0 \leq 0 \right\},$$

where $x \cdot x \equiv \eta_{\alpha\beta} x^\alpha x^\beta, \quad \alpha, \beta = \overline{0, D-1}, \quad (\eta_{\alpha\beta}) \equiv diag\,(+, -, -, ..., -)$.

Now we define binary operations on $\overset{+}{Con}$ ("addition" $\overset{+}{\vee}$ and "multiplication" $\overset{+}{\wedge}$) as follows

$$\overset{+}{\vee}:\ \overset{+}{Con}\left(M_1^D\right) \times \overset{+}{Con}\left(M_1^D\right) \to \overset{+}{Con}\left(M_1^D\right)$$
$$\left(\underset{x}{\overset{+}{con}},\ \underset{y}{\overset{+}{con}}\right) \mapsto \underset{x}{\overset{+}{con}} \overset{+}{\vee} \underset{y}{\overset{+}{con}} \equiv \begin{array}{l} \textit{smallest upper cone,} \\ \textit{containing } \underset{x}{\overset{+}{con}} \text{ and } \underset{y}{\overset{+}{con}} \end{array}$$
$$\overset{+}{\wedge}:\ \overset{+}{Con}\left(M_1^D\right) \times \overset{+}{Con}\left(M_1^D\right) \to \overset{+}{Con}\left(M_1^D\right)$$
$$\left(\underset{x}{\overset{+}{con}},\ \underset{y}{\overset{+}{con}}\right) \mapsto \underset{x}{\overset{+}{con}} \overset{+}{\wedge} \underset{y}{\overset{+}{con}} \equiv \underset{x}{\overset{+}{con}} \bigcap \underset{y}{\overset{+}{con}}$$

It's easy to identify the properties of idempotency, commutativity, associativity of both operations, and also the validity of absorption identities

$$\underset{x}{\overset{+}{con}} \overset{+}{\wedge} \left(\underset{x}{\overset{+}{con}} \overset{+}{\vee} \underset{y}{\overset{+}{con}}\right) = \underset{x}{\overset{+}{con}} = \underset{x}{\overset{+}{con}} \overset{+}{\vee} \left(\underset{x}{\overset{+}{con}} \overset{+}{\wedge} \underset{y}{\overset{+}{con}}\right) \quad \forall\, x, y \in M_1^D$$

and conclude that $\left(\overset{+}{Con}\left(M_1^D\right), \overset{+}{\wedge}, \overset{+}{\vee}\right)$ is a lattice for which the property of distributivity is also true. In the same way we establish the distributivity of the lattice $\left(\overset{-}{Con}\left(M_1^D\right), \overset{-}{\wedge}, \overset{-}{\vee}\right)$, where "multiplication" $\overset{-}{\wedge}$ and "addition" $\overset{-}{\vee}$ of the lower cones are defined as in the previous case. A bijection between these lattices transforms one cone into another without changing the vertex

$$T:\ \left(\overset{+}{Con}\left(M_1^D\right), \overset{+}{\wedge}, \overset{+}{\vee}\right) \to \left(\overset{-}{Con}\left(M_1^D\right), \overset{-}{\vee}, \overset{-}{\wedge}\right)$$
$$\underset{x}{\overset{+}{con}} \mapsto T\left(\underset{x}{\overset{+}{con}}\right) \equiv \underset{x}{\overset{-}{con}}$$

Let's consider the family of diamonds on M_1^D

$$Dmd\left(M_1^D\right) \equiv \left\{ \begin{array}{l} \underset{x}{\overset{y}{dmd}} = \underset{x}{\overset{+}{con}} \bigcap \underset{y}{\overset{-}{con}} = \underset{x}{\overset{+}{con}} \bigcap T\left(\underset{y}{\overset{+}{con}}\right) = \\ = T^{-1}\left(\underset{x}{\overset{-}{con}}\right) \bigcap \underset{y}{\overset{-}{con}} \end{array} \middle|\, x, y \in M_1^D \right\},$$

and define "addition" $\vee$ and "multiplication" $\wedge$ as follows:

$$\vee: \; Dmd\left(M_1^D\right) \times Dmd\left(M_1^D\right) \to Dmd\left(M_1^D\right)$$
$$\left(\underset{'x}{\overset{'y}{dmd}}, \underset{''x}{\overset{''y}{dmd}}\right) \mapsto \underset{'x}{\overset{'y}{dmd}} \;\vee\; \underset{''x}{\overset{''y}{dmd}} \equiv \left(\underset{'x}{\overset{+}{con}} \overset{+}{\vee} \underset{''x}{\overset{+}{con}}\right) \bigcap \left(\underset{'y}{\overset{-}{con}} \overset{-}{\vee} \underset{''y}{\overset{-}{con}}\right)$$

$$\wedge: \; Dmd\left(M_1^D\right) \times Dmd\left(M_1^D\right) \to Dmd\left(M_1^D\right)$$
$$\left(\underset{'x}{\overset{'y}{dmd}}, \underset{''x}{\overset{''y}{dmd}}\right) \mapsto \underset{'x}{\overset{'y}{dmd}} \;\wedge\; \underset{''x}{\overset{''y}{dmd}} \equiv \left(\underset{'x}{\overset{+}{con}} \overset{+}{\wedge} \underset{''x}{\overset{+}{con}}\right) \bigcap \left(\underset{'y}{\overset{-}{con}} \overset{-}{\wedge} \underset{''y}{\overset{-}{con}}\right)$$

The properties of "addition" and "multiplication" on cones allow extracting the properties of operations on double cones. These causal subsets of the Minkowski spacetime, partially ordered by inclusion $\supseteq$, are directed sets, what makes them potentially interesting as a start when constructing the nets of C^*-algebras.

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Algorithms for computing Intersection Numbers and Bases of Cohomology Groups for Triangulated Closed Three-Dimensional Manifolds

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We solve some computational problems for triangulated closed three-dimensional manifolds P using groups of simplicial homology and cohomology modulo 2. One of the interesting and important tasks of the computational topology is the development of algorithms for calculating the intersection numbers of cycles of a closed manifold. For two-dimensional manifolds, the first versions of its solution were proposed in [1] and [2]. In [3], this problem is solved for a simple $(n-1)$-dimensional cycle x and one-dimensional cycle y of a manifold of arbitrary dimension n. But the proposed algorithm was not applicable for the case of a non-simple cycle x.

In this work two efficient algorithms for computing intersection numbers of 1- and 2-dimensional cycles are developed. We show that computational complexity of described algorithms are $O(|P|)$ and $O(|P|\ log\ |P|)$ respectively, where $|P|$ is the size of the polyhedron, i.e. the number of simplices in the model. Thus, the described algorithms have similar or better efficiency than the previously developed algorithms, but can be applied to the cases not covered by existing algorithms.

Using these algorithms it is possible to obtain a basis of the cohomology group $H^1(P)$ $(H^2(P))$ from a given basis of the homology group $H_2(P)$ $(H_1(P))$ of complementary dimension. This cohomology group basis can be used for constructing covering polyhedron $\hat{P}$ of a specific form. This covering polyhedron, in turn, can be used to reduce the problem on finding conditional minimum in original polyhedron P, like finding shortest path or cycle inside particular homology class, to finding absolute minimum in covering polyhedron $\hat{P}$.

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Analysis of Some Non-Smooth Bifurcations with Applications to Ship Maneuvering

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The super/subcriticality of a Hopf bifurcation in a generic smooth 2D system can be readily determined by the sign of the first Lyapunov coefficient. However, for a system of continuous but non-smooth equations this cannot be applied in general. We show new results for autonomous systems of arbitrary finite dimension with focus on non-smooth nonlinearities of the form $|u_i|u_j$. This is motivated mainly by models for ship maneuvering and its control. We present the unfolding of Hopf-type bifurcations for such systems and discuss generalizations to bifurcations at switching points for continuous piecewise smooth systems.

An unguided tour started from chirality

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We will survey an unguided mathematical tour of research by topologists at Peking University and their collaborators over many years. The tour starts with work on chirality and, drawn by questions related to attractors, goes via a zigzag path across topology and dynamics.

Applications of intrinsic shape to dynamical system

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The intrinsic shape of various types of sets that appear in Dynamical systems gives an important information about the behaviour of a dynamical system.

A brief introduction to intrinsic shape and a comparision with honotopy type will be given. Several results will be presented about intrinsic shape of chain reccurent set and non-saddle set generalizing previous results about attractors.

Asymptotics of self-oscillations in chains of coupled oscillators

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Consider the local dynamics of system of coupled oscillators. Under condition of sufficiently large number of oscillators a spatially distributed model is obtained. Critical cases in the problem of the stability of its solutions have infinite dimension. Special nonlinear systems of partial differential equations are constructed whose nonlocal dynamics describes the behavior of all solutions of the original system in a small neighborhood of its equilibrium state.

Attractors of nonlocal Ginzburg-Landau equation

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The integro-differential equation

$$u_t = u - (1+ic)u[\frac{1}{2\pi}\int\limits_0^{2\pi}|u|^2dx] + (a+ib)u_{xx}, \qquad (1)$$

where $u = u(t,x)$ is a complex-valued function, $a \geq 0, b, c \in R$ and
$a^2 + b^2 \neq 0$, is usually called the non-local Ginzburg-Landau equation (see, for example, [1,2,3], and also the references therein). It appeared in the study of ferromagnetism. Usually, equation (1) is considered together with periodic boundary conditions

$$u(t, x + 2\pi) = u(t, x). \qquad (2)$$

In [3], the dimension of the global attractor was estimated. These results can be substantially supplemented.

We set $a_k = 1 - ak^2$ and consider only those k, for which $a_k > 0$, т.е. $k^2 \leq m_0^2, m_0 = \left[\frac{1}{a}\right]$ or $m_0 = \left[\frac{1}{a}\right] + 1$, if $\frac{1}{a} \in N$.

Theorem 1. *The boundary value problem (1), (2) has a homogeneous cycle* V_0 :

$$u(t,x) = u_0(t) = \exp(ict + i\varphi_0),\ \varphi_0 \in R,$$

as well as m_0 *invariant varieties* V_k $(k^2 \le m_0^2)$ *of dimension 3, of dimension 3, which are formed by periodic solutions of the form*

$$u_k(t,x) = \eta_k \exp(i\sigma_k t + ikx + i\varphi_k) + \eta_{-k} \exp(i\sigma_k t - ikx + i\varphi_{-k}),$$

where $\varphi_k, \varphi_{-k} \in R$ *and are arbitrary,* $\sigma_k = -bk^2 - ca_k, a_k = 1 - ak^2 > 0,$
$k = 1, 2, \ldots, m_0, \eta_k, \eta_{-k} \ge 0$ *and they satisfy the following equality*

$$\eta_k^2 + \eta_{-k}^2 = a_k,\ k = 0, 1, \ldots, m_0.$$

Theorem 2. *All solutions of the boundary value problem (1), (2) tend to one of the manifolds over time* V_0 *or* $V_k (k = 1, \ldots, m_0)$. *Moreover, the one-dimensional manifold* V_0 *is a local attractor, and the remaining invariant manifolds* V_k *of dimension 3 are saddle.*

In other words, the global attractor of a dynamical system generated by a nonlinear boundary value problem (1), (2)

$$M = \bigcup_{k=0}^{m_0} V_k.$$

A special case arises if $a = 0$. For such case, the statement holds.

Theorem 3. *The global attractor* M_∞ *can be selected by the condition*

$$\int_0^{2\pi} |u(t,x)|^2 dx = 2\pi.$$

All the solutions of the boundary value problem (1), (2) belonging to M_∞*, in the general case, are quasiperiodic functions of the evolution variable* t.

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Bifurcations in integrable Hamiltonian systems

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One of contemporary problems in the theory of integrable Hamiltonian systems (IHS) is to classify Liouville fibrations generated by low dimensional integrable systems, i.e. in 2, 3, and 4 degrees of freedom. This includes semi-local description of the Liouville fibrations for nondegenerate singularities (rank 0), semi-local nondegenerate singularities of rank 1, 2, and 3 [1]. But such classification unavoidably requires studying bifurcations [4, 5]. This can be seen in many integrable systems depending on parameters, for instance, in integrable systems of mechanics (see [2, 3]).

Bifurcations in IHS are related with the fact that in such systems for the related Poisson action all its singular orbits (of dimension lesser than the half of the manifold dimension) are met in families. For instance, periodic orbits being 1-dimensional orbits of the induced Poisson action belong to a 1-parameter families, 2-dimensional Lagrangian tori belong to 2-parameter families, etc. This implies the following phenomenon: if one moves along the family, an orbit being more degenerate (in transverse direction) than neighboring orbits can be met and hence one may expect branching the family. Also bifurcations are met at the study of parametrized families of integrable systems, then parameters of the family play the similar role. It is important to stress that common tool to study integrable systems uses some assumptions on the linearized system on the related Poisson orbits (like to be a Cartan algebra for the related set of commuting integrals, etc). Such properties are usually violated at the bifurcation and one needs to use another tool to study the related orbit structure.

In the talk I intend to discuss these themes for 3 degree of freedom integrable Hamiltonian systems. In this case (if no outer parameter exist) the related integrable system can contain 1-parameter families of periodic orbits and 2-parameter families of Lagrangian 2-tori. The reduction procedure allows one to reduce locally near a point on the degenerate orbit to studying an integrable family of IHS of lesser dimension depending on related number of parameters. We investigate such bifurcations and after that globalize this study to get a semi-local description.

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Bifurcations in the singular pertubed second-order system with delay

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Consider second-order delay dynamical system

$$\begin{aligned} &\gamma^{-1}\dot{x} + x = x(t-T)(a + d_1 y + d_2 y^2), \\ &\dot{y} = by + cx^2. \end{aligned} \tag{1}$$

Study the dynamics of (**??**) in the neighbourhood of zero equilibrium.

Main assumption is γT is sufficiently large, i.e. $0 < \varepsilon = (\gamma T)^{-1} \ll 1$. Thus, system (**??**) is singular perturbed. The critical cases (points of bifurcation) has infinite dimension. The quainormal forms, nonlinear evolutional equations, are constructed in each critical case. Solutions of these equations determine the behaviour and main terms of asymptotics of the solutions of (**??**).

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Bifurcations of multiple attractors in a predator-prey system

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This is an update of the presentation from previous conference inculding many questions. The system of n competing predators feeding on the same prey is of the type

$$X_i' = p_i\,\varphi_i(S)\,X_i - d_i\,X_i, \quad i = 1, ..., n, \tag{1a}$$

$$S' = H(S) - \sum_{i=1}^{n} q_i\,\varphi_i(S)\,X_i, \tag{1b}$$

where the variable S represents the prey and the variables X_i represent the predators. They are, of course, non-negative. The function φ_i is assumed non-decreasing.

We consider the case where

$$H(S) = r\,S\left(1 - \frac{S}{K}\right),\ \varphi_i(S) = \frac{S}{S + A_i}, \tag{2}$$

and where the parameters r, K and A_i are positive.

The dynamics in the coordinate planes representing one of the predators and the prey is well known and there is no more than one cycle. The system has no equilibrium, where predators coexist (in non-degenerate cases). But the predators can coexist in a cyclic and complicated way. There exists multiple attractors of cyclic and different chaotic chaos including "spiral-like" chaos. This happens even in cases, where the populations do not become unrealistic low. We present new discovered phenomena and discuss the possible bifurcations of these and contours from where they could develop.

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Bursting activity in system of two predator–prey communities coupled by migration

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The report is discusses the mechanism of bursting activity for a group of biological predator–prey communities. The simplest situation is considered, when two non-identical communities live in isolated patches and are weakly coupled by migration. Coupling is predator migration at constant rate. The non-identity is the difference in the prey growth rates or predator mortalities in each patch. It should be noted here, the completely identical communities demonstrate only full synchronization with an arbitrarily small nonzero coupling and any initial conditions. By changing of the variables and characteristic time, it is easy to show the model of communities that differ in the prey birth rate or the predators mortality rates are equivalent. In the latter case, the equations of dynamics after all simplifications and replacements of the parameters have the form:

$$\begin{cases} \frac{dx_1}{dt} = x_1\left(1 - ax_1\right) - \frac{x_1 y_1}{1+hx_1}, \\ \frac{dy_1}{dt} = -c_1 y_1 + \frac{c_1 x_1 y_1}{1+hx_1} + c_1 m\left(\frac{c_1}{c_2} y_2 - y_1\right), \\ \frac{dx_2}{dt} = x_2\left(1 - ax_2\right) - \frac{x_2 y_2}{1+hx_2}, \\ \frac{dy_2}{dt} = -c_2 y_2 + \frac{c_2 x_2 y_2}{1+hx_2} + c_2 m\left(\frac{c_2}{c_1} y_1 - y_2\right), \end{cases} \tag{1}$$

where x_i and y_i are numbers or density of prey and predator, $1/a$ is prey habitat carrying capacity, c_i is rate of decline in predator numbers or mortality, h is handling time, mc_i is predator migration rate $(i = 1, 2)$. Without coupling $(m = 0)$ the model is as well known Rosenzweig–MacArthur equations with logistic growth of prey and Hollings predation functional response of II type.

The analysis of local stability for all equilibrium points and limit cycles of system (1) was performed as well as the qualitative analysis of global bifurcations of periodic solutions. As result it was shown with increasing difference between predator mortality $(c_2 - c_1)$ there are changes in the types of dynamics differing by a period of oscillation in each patch, the ratio of numbers and the degree of synchronization. Typically in the first patch there is a fast-slow limit cycle (canard) with a large period and amplitude which modulates the limit cycle with a small period in the second patch. Here, the coupling, in fact, is unidirectional so that the fast cycle of second community does not qualitatively affect the first. As a result there exists a hysteresis loop of bursting connecting the fast spiking oscillations and slow orbit. Consequently the typically phase trajectory lies on the surface of a Klein bottle or torus.

Using a singular perturbation analysis, it was found that the fast cycle emerges and disappears at certain values of the numbers of predator and prey in the first patch (according to the Andronov–Hopf bifurcation and saddle-node bifurcation). As a result, if the differences between mortality parameters are significant ($0 < c_1 \ll c_2 < 1$), then the dynamics of the system (1) contains segments of slowly resting dynamic and fast bursts of spikes. Moreover, in the resting part the dynamics of the second community, as a rule, follow the slow changes in the first community, i.e. the dynamics in different patches are synchronous. But in the fast part there is only phase synchronization between the fast-slow cycle in first patch and bursts in second. However, depending on the system parameters, spiking manifold can be differently lies relative to canard. For example, the start of bursting activity (divergent fast oscillations) coincides with the minimum numbers of prey in the first territory ($x_1 \approx 0$). After a rapid increase in the number of prey in the first patch, diverging fluctuations give way to damped in the second patch. Such dynamics correspond to the rhombus shape of spikes cluster. Another case is interesting, when the bursting activity is possible only after the full recovery of prey in the first patch ($x_1 > 0$). In this case, the spikes cluster has the shape of a triangle or a truncated rhombus.

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Cascade of period-doubling biurcations in the "generalized" FitzHugh-Nagumo system

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We study a FitzHugh-Nagumo-like sysytem of three ODEs with one fast variable corresponding to the membrane potential and two slow gating variables:

$$\begin{aligned} \varepsilon\dot{x} &= x - x^3/3 - y - z \\ \dot{y} &= a + x \\ \dot{z} &= a + x - z, \end{aligned}$$

where ε is a small parameter and the parameter a is assumed to be slightly less than one. The slow manifold of the system is described by the eqution $x - x^3/3 - y - z = 0$ and possesses folds at $x = \pm 1, y + z = \pm 2/3$.

One may observe that the system has a unique equilibrium, which is stable for sufficiently large a. However, decrease of a leads to the supercritical Andronov-Hopf bifurcation at a value $a_H = 1 - \frac{1}{4}\varepsilon + O(\varepsilon^2)$ (see e.g. [1]) . Immediately after the bifurcation the amplitude of the newborn stable periodic orbit is small and lies below the threshold of spiking. In contrast, for $a \ll a_H$ the system exhibits large-scale periodic oscillations: continuous spiking known as "canards".

In [1] the author found nemrically that dynamics near the slow surface can effectively become three-dimensional. As a result, the initial periodic state may lose stability already before the canard transition via a sequence of period-doubling bifurcations. Studying numerically the period-doubling cascades for small but fixed values of the parameter ε, M. Zaks observed that the cascade follows the Feigenbaum law with the Feigenbaum constant $4.67\ldots$, which is typical for dissipative systems. On the other hand for smaller values of ε the process switches to the Feigenbaum constant of a conservative map as, in the limit $\varepsilon \to 0$, two-dimensional Poincaré map nearly preserves the area.

The reason for such phenomenon lies in the closeness of the equilibrium to a fold of the slow manifold. Varying a the position of the equilibrium moves and reaches the fold at $a = a_H$.

In this paper we study the system in a vicinity of the pair "equilibrium-fold" and derive the asymptotic formula for the Poincaré return map. We calculate the parameter values for the first period-doubling bifurcation and also discuss more general $3d$ model with a similar bifurcation scenario.

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Chaos with positive and zero Lyapunov exponents in a three-dimensional map: discrete Lorenz-84 system

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Chaotic behavior is one of the fundamental properties of nonlinear maps [1-3]. Chaos can be most easily and reliably diagnosed using the largest Lyapunov exponent, which will be positive for the chaotic regime. Chaotic dynamics can occur in diffeomorphisms of dimension two or higher, or even in one-dimensional endomorphisms. For maps, the absence of a zero exponent in the spectrum of Lyapunov exponents is characteristic, since they are discrete. A zero exponent in the spectrum will indicate the possibility of embedding such a map in the flow.

In the frame of this work, the possibility of the appearance of chaotic attractors will be shown, the spectrum of Lyapunov exponents of which contains one positive, one close to zero, and one negative exponents. As objects of study, three-dimensional discrete oscillator will be used: a discrete Lorentz-84 oscillator [4]. The paper will present charts of Lyapunov exponents, on which areas with chaotic dynamics with zero Lyapunov exponent are localized, and characteristic phase portraits are shown. A mechanism of occurrence of chaos with a close to zero Lyapunov exponent via cascade of period-doubling bifurcations of an invariant curve will be discussed.

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Complicated dynamics in a reversible Hamiltonian system near a symmetric heteroclinic contour

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Hamiltonian systems arise as mathematical models in many branches of physics, chemistry, engineering. Such systems as their study shows have usually a rather complicated structure that leads to great difficulties in their examination. Therefore one of the fruitful method of their investigation is the study of the orbit behavior near some specific structures which can be distinguished by simple conditions. The study of a system near a homoclinic orbits or contours made up of several heteroclinic orbits and equilibria or periodic orbits is undoubtedly one of such problem.

We study the dynamics of an analytic reversible Hamiltonian system X_H with two degrees of freedom assuming the system has a heteroclinic contour involving a symmetric saddle-center equilibrium p (its eigenvalues are nonzero numbers $\pm i\omega$, $\pm\lambda$, $\omega, \lambda \in \mathbf{R}$), an orientable symmetric saddle periodic orbit γ lying in the same level of Hamiltonian $H = H(p)$ and two nonsymmetric heteroclinic orbits Γ_1, Γ_2 joining p with γ and interchanged by the involution L, $\Gamma_2 = L(\Gamma_1)$. The reversible involution L is supposed to have a smooth two-dimensional set $Fix(L)$ of its fixed points. Such a system are met in generic one-parameter families of reversible Hamiltonian systems.

Saddle periodic orbit γ belongs to a 1-parameter family γ_c of saddle periodic orbits in all close levels $H = c$ forming a symplectic cylinder. Reversible Hamiltonian systems possessing the above mentioned contour can be of two different types in dependence on how the involution acts locally near a saddle-center.

Our results demonstrate the existence in such a system:

- countable set of transverse 1-round homoclinic orbits to γ and related to them non-uniformly hyperbolic subsets;
- appearance for $c > 0$ of two transverse heteroclinic contours involving γ_c, a small Lyapunov periodic orbit l_c near p and four heteroclinic orbits $\Gamma_1^{\pm}$ and $\Gamma_2^{\pm} = L(\Gamma_1^{\pm})$ and related with them uniform hyperbolic subsets;
- a finite set of transverse 1-round homoclinic orbits to γ_c for $| c |$ close to $H(p)$ and uniformly hyperbolic sets related with them;
- a countable set of values $c_n < 0$ accumulating at $c = 0$ such that on the level $H = c_n$ the system has a tangent homoclinic orbit to γ_{c_n} and bifurcations nearby orbits related to this tangency;
- countable sets of saddle and elliptic periodic orbits.

Some other bifurcation phenomena will be discussed when generic one parameter reversible unfoldings of such a system are considered.

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Concentration of Haar measure and estimate of medians of matrix elements of real linear irreducible representations of classical compact Lie groups

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The main question addressed in this report is: "what kind real numbers appear as the median of matrix elements representations of classical matrix compact Lie groups?"

First of all recall important P. Levy result on concentration of the invariant measure on the Euclidean sphere S^{n-1} [1, 2].

Theorem (P. Levy). *Let $f\colon S^{n-1} \to \mathbb{R}$ be Lipschitz function with Lipschitz constant L and let X be a uniform measure on S^{n-1}. Then for m_f denoting the median of function f with respect to measure on S^{n-1},*

$$|Ef(X) - m_f| \leq L\sqrt{\pi/(n-2)}$$

and

$$P[|f(X) - Ef(X)| > Lt] \leq \exp(\pi - nt^2/4).$$

That is, a Lipschitz function on the sphere S^{n-1} essentially constant.

Above $Ef(X)$ is the mean of random variable $f(X)$ and we say that real number m_f is a median of function f if $P(\{f \leq m_f\}) \geq 1/2$ and $P(\{f \geq m_f\}) \geq 1/2$. It is clear that the set of median of f is a closed and bounded interval on real line $\mathbb{R}$.

The Levy mean $lm(f, \mu)$ of f with respect to measure μ is defined to be $lm(f, \mu) = (\underline{m} + \overline{m})/2$, where $\underline{m}$ is the minimum of medians of f and $\overline{m}$ the maximum of medians of f.

Our goal is to obtain the estimate of medians probability distribution of matrix elements of the real irreducible representations of the classical compact Lie groups $G = SO(n)$, $SU(n)$ and $Sp(n)$ under probabilistic Haar measure on G. More precisely, let X be distributed according to Haar measure on G and let A be fixed $n \times n$ matrix over real field $\mathbb{R}$, where $n = \dim \rho$. Assume also that $W = \mathrm{Tr}(A\rho(X))$ be matrix element of ρ considered as a random variable on G.

Our main result is

Theorem. *Medians of the matrix elements $W = \mathrm{Tr}(A\rho(X))$ of real irreducible representation ρ classical compact Lie group G with respect probability Haar measure satisfy following inequality:*

$$|m_f| \leq \sqrt{2}\,\mathrm{Tr}(AA^*)/\sqrt{\dim \rho}.$$

Here an A is linear operator on representation space and A^* is its conjugate operator.

Proof of theorem based on the orthogonality relations for matrix elements representations and the Chebyshev inequality [3].

From the geometric point of view it is impotent to find more structured subsets on which functions are concentrated. Some preliminary results of such kind see in [4, 5].

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Deformations of functions on surfaces

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Let M be a compact surface, $f \in C^{\infty}M\mathbb{R}$ be a Morse function and Γ_f its Kronrod-Reeb graph. Denote by $\mathcal{O}(f) = \{f \circ h \mid h \in \mathcal{D}(M)\}$ the orbit of f with respect to the natural right action of the group of diffeomorphisms $\mathcal{D}(M)$ on $C^{\infty}M\mathbb{R}$, and by $\mathcal{S}(f) = \{h \in \mathcal{D}(M) \mid f \circ h = f\}$ the stabilizer of this function.

S. Maksymenko [1], proved that

- if f has at least on saddle critical point, then the connected components of $\mathcal{S}(f)$ are contractible;
- othersize, every path component of $\mathcal{S}(f)$ is homotopy equivalent to the circle.

In that paper is was also shown that for **generic** Morse function f connected components of its orbit $\mathcal{O}(f)$ is homotopy equivalent to

- $(S^1)^k \times SO(3)$ for some k if M is either a 2-sphere or a projective plane;
- and to $(S^1)^k$ for some k in all other cases;

Further E. Kudryavtseva [2] extended that result proving that for **arbitrary** Morse function f there exists a free action of a certain finite group G on the torus $(S^1)^k$ such that the connected components of orbits $\mathcal{O}(f)$ are homotopy equivalent to the spaces of the form $(S^1)^k/G \times SO(3)$ if $M = S^2$ or $\mathbb{R}P^2$ and $(S^1)^k/G$ in all other cases.

The aim of the talk is to describe recent progress in the computations of homotopy types of the fundamental group $\pi_1\mathcal{O}(f)$ and the groups G.

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Dynamics of Kuramoto oscillator networks

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Patterns of phase locked oscillators are observed in many networks, ranging from neuronal populations to power grids. Despite significant interest among physicists and applied mathematicians, the emergence and hysteretic transitions between phase-locked patterns in oscillatory networks, including the celebrated Kuramoto network, have still not been fully understood. In this talk, I will review the state of the art in research on phase-locking in networks of Kuramoto oscillators and discuss new results and research trends.

Emergence of wandering stable components

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In a joint work with Sebastien Biebler, we show the existence of a locally dense set of real polynomial automorphisms of $\mathbb{C}^2$ displaying a wandering Fatou component; in particular this solves the problem of their existence, reported by Bedford and Smillie in 1991. These wandering Fatou components have non-empty real trace and their statistical behavior is historical with high emergence. The proof follows from a real geometrical model which enables us to show the existence of an open and dense set of C^r-families of surface diffeomorphisms in the Newhouse domain, each of which displaying a historical, high emergent, wandering domain at a dense set of parameters, for every $2 \le r \le \infty$ and $r = \omega$. Hence, this also complements the recent work of Kiriki and Soma, by proving the last Taken's problem in the C^∞ and C^ω-case.

Evolution of the spatial spectra of the quasi-magnetostatic Weibel turbulence in an anisotropic collisionless plasma and the relayed particle magnetization

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On the basis of numerical modeling of the nonlinear stage of the Weibel instability in a homogeneous nonrelativistic electron-ion plasma with a strong initial temperature anisotropy and comparable initial energies of electrons and ions, the evolution of the spatial spectrum of the developing quasimagnetostatic turbulence is investigated. Until very recently, analytical and numerical studies of the dynamics of this instability (see, e.g., [1, 2]) were mainly limited to the analysis of a single spatial harmonic of magnetic field or electric current, the one with the maximum growth rate at the linear stage. The dynamics of various harmonics of the field and current and their interaction at the nonlinear stage remain essentially unexplored. We started such an analysis in our recent work [3], where the Weibel instability in nonrelativistic electron-ion bimaxwellian plasma was investigated.

This report discusses peculiarities and difficulties of the analytical description of the considered evolution of the Weibel turbulence.

The calculations presented in the report were carried out with the DARWIN code, which implements the particle-in-cell method in a 5-dimensional (2D3V) phase space and is based on the non-radiative Vlasov – Darwin model for electromagnetic field dynamics. Initially, we set the Maxwellian distributions of particles by each of the velocity components, but with different temperatures parallel and orthogonal to the z-axis of Cartesian coordinates. The longitudinal temperature was the greatest, and the simulation was carried out in the xy-plane.

It is shown that after the growth of the total magnetic field RMS value stops, the exponential growth of the electron current harmonics at a certain stage before their saturation changes to a power-law one, and that the long-wave harmonics saturate later than the short-wave ones. On the whole, the dynamics of the spatial spectra of the magnetic field is largely determined by the relay processes of the trapped electrons release from decaying short-wave current filaments and subsequent trapping into growing longer-wavelength ones. This leads to a universal power law of the decay of the magnetic field (or current) spatial spectrum components that decrease in time with an exponent close to 5/2. Meanwhile, the wave number corresponding to the maximum of the magnetic field and current spectrum decreases with time approximately according to the root law. Finally, the RMS value of the inductive electric field decreases as a power-law with an exponent close to 5/3. The spectral indices of the spatial spectra of the Weibel turbulence are also established.

The question of the universality of the discovered spectral indices and the laws of the temporal evolution of the Weibel instability in a plasma with various types of anisotropy remains open.

The considered scenario can be realized in the solar (stellar) wind, coronal mass ejections on the late spectral class stars, or in laboratory conditions, e.g., in laser experiments on the ablation of solid targets, where anisotropic heating of electrons is possible.

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Fermi-like acceleration growth in nonholonomic systems

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The equations of motion in the Suslov problem [1] with two rotors, which govern the evolution of the angular velocity of a rigid body are represented as

$$\dot{u} = -vu - K(t)v - \dot{\Lambda}(t), \quad \dot{v} = u^2 + K(t)u, \tag{1}$$

where $K(t)$ and $\Lambda(t)$ are periodic functions with the same period T, which define the angular velocities of the rotors.

Of particular interest is the question of whether the reduced system has trajectories unbounded on the plane (v, u) (i.e., trajectories which leave any bounded region on the plane). In this case the angular velocity of the carrying body and hence the kinetic energy must increase indefinitely (in absolute values) with time.

The case $K(t) = 0$. Then the reduced system coincides with the reduced system describing another nonholonomic system: a Chaplygin sleigh with gyrostatic momentum. For this system the following theorem holds [2]:

Theorem 1. *Let $\Lambda(t)$ be a periodic function. Let us calculate the following average value:*

$$\langle \dot{\Lambda}^2 \rangle = \frac{1}{T}\int\limits_0^T \dot{\Lambda}^2(t)dt.$$

If at the initial time $v > 0$, then the function $v(t)$ increases indefinitely and $u(t)$ tends to zero:

$$v(t) = Ct^{\frac{1}{3}} + o\big(t^{\frac{1}{3}}\big), \quad u(t) = -C\dot{\Lambda}(t)t^{-\frac{1}{3}} + o\big(t^{-\frac{1}{3}}\big),\ C = \big(3\langle\dot{\Lambda}\rangle^2\big)^{\frac{1}{3}}. \tag{2}$$

If $K(t) \neq 0$, then for the reduced system the following theorem holds:

Theorem 2. *If the average*

$$\langle G \rangle = \frac{2}{T}\int\limits_0^T K(t)\dot{\Lambda}(t)dt > 0, \tag{3}$$

then the reduced system has trajectories unbounded in v, which have the following asymptotics:

$$v(t) = Ct^{\frac{1}{2}} + o(t^{\frac{1}{2}}),\ u(t) = -K(t) + o(t^{-\frac{1}{2}}),\ C = \sqrt{\langle G \rangle}. \tag{4}$$

If $\langle G \rangle < 0$, then there are no unbounded trajectories. The case $\langle G \rangle = 0$ requires a separate analysis.

Numerical experiments for the one-parameter family of functions:

$$\Lambda(t) = \alpha \cos t - \frac{1}{2}\sin t, \quad K(t) = \sin t, \quad \alpha = \mathrm{const}$$

show that, depending on α, the reduced system exhibits the following qualitatively different dynamical regimes.

— As $t \to +\infty$, all trajectories tend to one or several periodic solutions of the system.

— Chaotic oscillations: the system exhibits a strange attractor.

— An intermediate situation: noncompact and bounded chaotic trajectories are observed.

— Speedup: except for the unstable fixed points of the map, all trajectories are noncompact, and $v \to +\infty$ as $t \to +\infty$. Numerical experiments show that their asymptotics is described by relations.

The work of I.S.Mamaev and I.A.Bizyaev was carried out within the framework of the state assignment of the Ministry of Education and Science of Russia (1.2405.2017/4.6 and 1.2404.2017/4.6 respectively).

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Frequency-domain methods for reduction of cocycles in Hilbert spaces

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Let $\mathcal{Q}$ be a metric space with a dynamical system $\vartheta^t\colon \mathcal{Q} \to \mathcal{Q}$, $t \in \mathbb{R}$. A cocycle in a Hilbert space $\mathbb{H}$ is a family of maps $\psi^t(q, \cdot)\colon \mathbb{H} \to \mathbb{H}$, where $t \geq 0$ and $q \in \mathcal{Q}$, satisfying the following conditions:

1. $\psi^0(q, u) = u$ for every $u \in \mathbb{H}, q \in \mathcal{Q}$.
2. $\psi^{t+s}(q, u) = \psi^t(\vartheta^s(q), \psi^s(q, u))$ for all $u \in \mathbb{H}, q \in \mathcal{Q}$ and $t, s \geq 0$.
3. The map $\mathbb{R}_+ \times \mathcal{Q} \times \mathbb{H} \to \mathbb{H}$ defined as $(t, q, u) \mapsto \psi^t(q, u)$ is continuous.

With each cocycle there is the corresponding skew-product dynamical system $\pi^t\colon \mathcal{Q} \times \mathbb{H} \to \mathcal{Q} \times \mathbb{H}$ defined as $\pi^t(q, u) := (\vartheta^t(q), \psi^t(q, u))$. We study the cocycle under the following conditions:

(H1) There is a continuous linear operator $P\colon \mathbb{H} \to \mathbb{H}$, self-adjoint ($P = P^*$) such that $\mathbb{H}$ splits into the direct sum of orthogonal P-invariant subspaces $\mathbb{H}^+$ and $\mathbb{H}^-$, i. e. $\mathbb{H} = \mathbb{H}^+ \oplus \mathbb{H}^-$, such that $P\big|_{\mathbb{H}^-} < 0$ and $P\big|_{\mathbb{H}^+} > 0$.

(H2) We have $\dim \mathbb{H}^- = j < \infty$.

(H3) For $V(u) := (Pu, u)$ and some numbers $\delta > 0$, $\nu > 0$ we have

$$e^{2\nu r} V(\psi^r(q, u) - \psi^r(q, v)) - e^{2\nu l} V(\psi^l(q, u) - \psi^l(q, v)) \leq \\ \leq -\delta \int_l^r e^{2\nu s} |\psi^s(q, u) - \psi^s(q, v)|^2 ds, \quad (1)$$

for every $u, v \in \mathbb{H}$, $q \in \mathcal{Q}$ and $0 \leq l \leq r$.

We show that these conditions imply there is a subset $\mathfrak{A} = \bigcup_{q \in \mathcal{Q}} \mathfrak{A}_q$ of $\mathcal{Q} \times \mathbb{H}$ containing bounded trajectories, invariant w. r. t. π and having fibers $\mathfrak{A}_q$ homeomorphic to some subsets of the j-dimensional space $\mathbb{H}^-$. Under certain compactness assumptions imposed on P or on the cocycle these fibers become homeomorphic to entire $\mathbb{H}^-$. In other words, interesting dynamics under these conditions is only j-dimensional. This can be used to derive various extensions of some well-known low-dimensional results (such as the Poincaré-Bendixson principle for autonomous ODEs; Massera's convergence theorems for periodic ODEs [5]; Zhikov's principle of stationary point for almost periodic ODEs [3]) to high- and infinite-dimensional cases.

Similar assumptions were widely used [1, 2, 5] to study various autonomous and nonautonomous ODEs, where **(H3)** can be verified via the frequency theorem of Yakubovich-Kalman. We will show how to apply infinite-dimensional versions of the frequency theorem [4] to study periodic or almost periodic nonlinear evolutionary problems, especially parabolic and functional-differential equations [6].

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Hamiltonian generalization of Topaj – Pikovsky lattice

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We discuss the Hamiltonian model of oscillator lattice with nonlinear local coupling:

$$\begin{aligned}\dot{I}_j = -\frac{\partial\mathcal{H}}{\partial\phi_j} &= -2\varepsilon\sqrt{I_{j+1}I_j}\,(I_{j+1}-I_j)\cos(\phi_{j+1}-\phi_j) - \\ &\quad - 2\varepsilon\sqrt{I_{j-1}I_j}\,(I_{j-1}-I_j)\cos(\phi_{j-1}-\phi_j)\,, \\ \dot{\phi}_j = \frac{\partial\mathcal{H}}{\partial I_j} &= \omega_j + \beta I_j + \varepsilon\left\{3\sqrt{I_{j+1}I_j} - I_{j+1}\sqrt{\frac{I_{j+1}}{I_j}}\right\}\sin(\phi_{j+1}-\phi_j) + \\ &\quad + \varepsilon\left\{3\sqrt{I_{j-1}I_j} - I_{j-1}\sqrt{\frac{I_{j-1}}{I_j}}\right\}\sin(\phi_{j-1}-\phi_j)\,,\end{aligned} \tag{1}$$

with free boundary conditions $\phi_0 = \phi_1$, $\phi_{N+1} = \phi_N$, $I_0 = I_1$, $I_{N+1} = I_N$. Equations (1) are generated by Hamiltonian function

$$\begin{aligned}\mathcal{H}(\ldots, I_j, \phi_j, \ldots) = &\sum_{j=1}^{N}\omega_j I_j + \frac{1}{2}\beta\sum_{j=1}^{N} I_j^2 - \\ &- 2\varepsilon\sum_{j=1}^{N}\sqrt{I_{j+1}I_j}\,(I_{j+1}-I_j)\sin(\phi_{j+1}-\phi_j)\,.\end{aligned} \tag{2}$$

Equations (1) describe in the classical limit the dynamics of quantum bosonic gas in a tilted periodic lattice [1]. I_j are intensities of oscillations in potential wells, ϕ_j are phases of oscillations.

Frequencies are distributed linearly: $\omega_{j+1} - \omega_j$=1. Since there are only differences of phases on right-hand-sides of equations (1), we write equations for phase shifts $\psi_j = \phi_{j+1} - \phi_j$:

$$\begin{aligned}\dot{\psi}_j =&1 + \beta\left(I_{j+1} - I_j\right) + \varepsilon\left\{3\sqrt{I_{j+2}I_{j+1}} - I_{j+2}\sqrt{\frac{I_{j+2}}{I_{j+1}}}\right\}\sin\psi_{j+1}+\\ &+\varepsilon\left\{3\sqrt{I_{j-1}I_j} - I_{j-1}\sqrt{\frac{I_{j-1}}{I_j}}\right\}\sin\psi_{j-1}-\\ &-\varepsilon\left\{6\sqrt{I_{j+1}I_j} - I_{j+2}\sqrt{\frac{I_{j+2}}{I_{j+1}}} - I_{j+1}\sqrt{\frac{I_{j+1}}{I_j}}\right\}\sin\psi_j.\end{aligned} \tag{3}$$

System (1)-(**??**) has an invariant manifold $I_j = I = const\ \forall j$, on which the dynamics of phases is described by Topaj – Pikovsky lattice [2, 3] of locally coupled phase oscillators. Furthermore, the system (1)-(**??**) has an involution, that reduces to the Topaj – Pikovsky involution on the invariant manifold:

$$\mathbf{R} : I_j \mapsto I_{N-j+1},\ \psi_j \mapsto \pi - \psi_{N-j}. \tag{4}$$

The dynamics on the invariant manifold is not conservative, but that does not contradict to the Hamiltonicity of the system (1). We show by the numerical simulation, that asymptotic trajectories on the invariant manifold are saddle ones with sum of Lyapunov exponents equal to zero.

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Homoclinic orbits for conservative surface diffeomorphisms

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Let S be a closed surface furnished with an area form ω. For $1 \le r \le \infty$, denote $\mathrm{Diff}^r_\omega(S)$ the set of C^r diffeomorphisms of S preserving ω, endowed with the C^r-topology. In a join work with Martín Sambarino (Universidad de la República, Montevideo) we prove that there exists a residual set $\mathcal{R} \subset \mathrm{Diff}^r_\omega(S)$ such that if $f \in \mathcal{R}$, there exist hyperbolic periodic points, and every such point has a transverse homoclinic intersection.

Hopf bifurcation and stability of whirl and whip in rotor systems

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Rotors of high-speed turbo machines are commonly supported by hydrodynamic journal bearings. Many authors wrote about an effect of self-oscillations for this class of bearings, which in this context are usually referred as вЂњoil whirlвЂќ and вЂњoil whipвЂќ. This effect has been extensively studied experimentally as well as analytically [1-13]. At the center position of the rotor loses its stability and a stable limit-cycle appears (oil whirl). The stability loss of the equilibrium position of a rigid rotor at the speed has been widely investigated both analytically and numerically by many authors [1, 6, 7-13]. In this paper we consider the problem of stability/instability effect oil whirl in the motion of the rotor in the fluid. The instability problem of rotor/seal system has been extensively analyzed by the linearization of model around the equilibrium position of the rotor. Such authors as [1, 5, 6, 7, 8, 9] wrote, that this method is not useful, since the nature of the whirling motion after onset of instability cannot be analyzed using only the linearized model. Non-linear analysis of the problem is very difficult [1, 6, 7, 9]. An analysis of the self-excited oscillations of a rotor is presented. We consider a symmetric Jeffcott rotor mounted at both rigid ends . The seal fluid force is assumed to be acting on the disk of the shaft [1, 2, 3, 10, 12]. It is shown that Hopf bifurcation theory may be used to investigate small-amplitude periodic solutions of the nonlinear equations of motion for rotor speeds.

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Instability induced by prey-taxis in a prey-predator model

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The possibility of spatial structure formation in the activator–inhibitor systems, if diffusion coefficient for inhibitor is considerably greater than that for activator, was shown by Turing. Later, a number of papers established that this condition is unnecessary for the Turing instability. Conditions for the emergence of spatial and spatiotemporal patterns after flow-induced instabilities [1] of spatially uniform populations were derived by Malchow [2, 3] and illustrated by patterns in a minimal phytoplankton–zooplankton model. Instabilities in the uniform distribution can arise, if phytoplankton and zooplankton move with different velocities, regardless of which one is faster. This mechanism of generating patchiness is more general than the Turing mechanism, which depends on strong conditions on the diffusion coefficients. Present study deal with a prey-predator model for spatiotemporal dynamics of phytoplankton, zooplankton and nutrients. The system is described by reaction-diffusion-advection equations in a one-dimensional vertical column of water in the surface layer. Advective term of the predator equation represents the vertical movements of zooplankton with velocity, which is assumed to be proportional to the gradient of phytoplankton density. This study aimed to determine the conditions under which these movements (taxis) lead to the spatially heterogeneous structures generated by the system in the case of equal diffusion coefficients of all model components.

Necessary conditions for the flow-induced instability were obtained through linear stability analysis. Depending on the parameters of the model local kinetics, increasing the taxis rate leads to Turing or wave instability. This fact is in good agreement with conditions for the emergence of spatial and spatiotemporal patterns derived by other authors. While the taxis exceeding a certain critical value, the wave number corresponding to the fastest growing mode remains unchanged. This value determines the type of spatial structure.

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Integrable and Non-Integrable Equations of the Korteweg-de Vries Hierarchy

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Here we consider the generalized Korteweg-de Vries (gKdV) equation in the form

$$\partial u/\partial t + su^n \partial u/\partial x + \partial^3 u/\partial x^3 = 0$$

or

$$\partial u/\partial t + s|u|^p \partial u/\partial x + \partial^3 u/\partial x^3 = 0,$$

where n is integer, $p > 0$ – is an arbitrary constant, and $s = \pm 1$. The Korteweg-de Vries (KdV, $s = 1, n = 1$) equation and modified Korteweg-de Vries (mKdV, $s = \pm 1, n = 2$) equation are famous members of this series of equations. They are integrable and thoroughly investigated.

The gKdV equations with higher order of the nonlinearity, $n > 2$, may appear in the application to the hydrodynamics of stratified fluid [1]. Some versions of the gKdV equation contain non-integer values of p; for instance, the power $p = 1/3$ is present in the Schamel equation applicable to ion-acoustic waves which interact with resonant electrons [2]. The log-KdV equation for solitary waves in FPU lattices can be mentioned in addition [3]. In all papers cited above the main attention was paid to the soliton dynamics, their stability and interactions. Dynamics of periodic and modulated wave packets in KdV-like systems is less studied. Two problems are discussed here.

1. Dispersionless limit of the generalized KdV equation is a generalized Hopf equation. The general explicit procedure to find the Fourier spectrum is described. In the case of a sinusoidal initial condition all expressions are found in closed form. These results are presented in [4]. The asymptotic shape of the spectrum of the breaking Riemann wave is found [5].

2. If the wave amplitude is small (wave dispersion prevails), the standard approach to investigate the stability of weakly modulated wave trains is to derive the nonlinear Schrödinger equation (NLS) and to determine its type. For the classic KdV equation the modulations are described by the defocusing NLS equation, and therefore waves packet are stable [6]. In the case of the mKdV equation with $s = +1$ a wave train is modulationally unstable, what leads to the generation of rogue waves [7]. In this work such analysis is extended to the gKdV equation. It is discussed in the context of nonlinear mechanisms of rogue wave formation. These results are summarized in the recent publications [8].

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Local and Global Inverse Problems for Difference Equations

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We present the new proof of the local inverse problem for systems of linear difference equations in the neighborhood of the infinity. This proof is based on almost complex structures. Using solution of the local problem we apply holomorphic vector bundles with meromorphic additive shift to studying of generalized Riemann–Hilbert–Birkhoff problem for difference systems.

As the application of this approach we obtain a generalization of Birkhoff's existence theorem. We prove that for any admissible set of characteristic constants and monodromy there exists a system

$$Y(z+1) = A(z)Y(z), \tag{1}$$

which has the given monodromy and characteristic constants and rational matrix $A(z)$.

Linearization and integrability of nonlinear non–autonomous oscillators

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In this talk we consider the following family of nonlinear oscillators

$$y_{zz} + f(z,y)y_z^2 + g(z,y)y_z + h(z,y) = 0, \tag{1}$$

where f, g and h are sufficiently smooth functions and $g(z,y) \not\equiv 0$. Particular cases of (**??**) often appear in various applications in mechanics, physics and biology (see, e.g. [1, 2]). We study linearizability conditions for (**??**) into

$$w_{\zeta\zeta} + \beta w_\zeta + \alpha w = 0, \tag{2}$$

via the following nonlocal transformations

$$w = F(z,y), \quad d\zeta = G(z,y)dz. \tag{3}$$

Here $\alpha, \beta \neq 0$ are arbitrary parameters and F and G are sufficiently smooth functions satisfying $F_y G \neq 0$. Linearization of (**??**) into (**??**) via (**??**) was studied previously only for some particular cases of (**??**), namely $F_y = G_y = 0$ and $F_z = 0$ (see [3, 4, 5] and references therein). Here we consider the general case of transformations (**??**) and provide linearizability conditions for (**??**) via (**??**) in the explicit form. We show that in the linearizable case of (**??**) not only can we obtain the general solution of the corresponding equation with the help of the general solution of (**??**), but we also can explicitly construct a first integral for (**??**) via a known first integral for (**??**). We also demonstrate that there are several interesting examples of both autonomous and nonautonomous oscillators that can be linearized via (**??**) with $F_z \neq 0$. In particular we consider a generalization of the Duffing–van der Pol oscillator, a cubic Liénar oscillator with linear damping and some other examples and construct their general solutions in the parametric form along with their first integrals.

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Non-stationary hyperbolic attractors in chaotic driven maps

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In this talk we consider driven maps having the form

$$x(i+1) = F(x(i), u(i)), \tag{1}$$

where $x \in R^n$, F is a n-dimensional vector, $i \in Z$ is a discrete time and the function $u : Z \to R^m$ is a set of driving parameters changing the structure of the map at each time iteration.

Four cases are possible.

The case 1. In the trivial case u =const the map (1) becomes an ordinary autonomous map for which all standard notations of attractors and bifurcations are applicable.

The case 2. In the case when a driving parameter $u(i)$ is a periodic function of discrete time $u(i) = u(i+p)$ with a period $p \in Z^+$ the dynamics of the map (1) is defined by the autonomous map $x(j+1) = \hat{F}(x(j), i_0)$, where $i_0 = 1, 2, \ldots, p$, $j = pi$ is a new discrete time and $\hat{F}$ is the composition of the sequential maps.

The case 3. A driving parameter $u(i)$ is an arbitrary bounded function of discrete time. To study a particular case of an attracting set with hyperbolic properties we use the next definition of non-stationary hyperbolic attractor [1].

Definition 1. *Let $G : (\|x\| \leq x^*, x^* = const)$ be an absorbing domain of the map $F(x(i), u(i))$, $FG \subset G$, $\forall i \in Z^+$. Let at each point $x_0 \in G$ the similar pairs of stable and unstable invariant cones K^s and K^u be defined. Denote the linearization of the map F in the point x_0: $L(x_0, i) = D_x F(x_0, u(i))$, where D_x is a differential with respect to x. Let the next conditions be fulfiled. The operator L (the operator L^{-1}) expands any vector V_0^u (V_0^s, respectively) released from x_0 and lying in the unstable cone K^u (the stable cone K^s, resp.) for any $x_0 \in G$ and $i \in Z^+$. Then the set of points in G on which the map $F(x(i), u(i))$ eventually acts for unboundly increasing i is called a non-stationary hyperbolic attractor.*

In this talk we study the problem of the existence of a non-stationary hyperbolic attractor for the following two-dimensional Lurie-type map [2]

$$F: \quad \begin{array}{l} x(i+1) = x(i) + y(i) + ag(x(i)) \equiv X(x, y), \\ y(i+1) = \lambda u(i)(y(i) + bg(x(i))) \equiv Y(u, x, y). \end{array}$$

where a, b, λ are positive parameters and $g(x)$ is a piecewise-linear function of cubic type

$$g(x) = \begin{cases} 2 + 2x, & x < -\frac{1}{2}, \\ -2x, & |x| \leq \frac{1}{2}, \\ -2 + 2x, & x > \frac{1}{2}. \end{cases}$$

For this map according to Def. 1 we rigorously prove the existence of the non-stationary hyperbolic attractor.

The case 4. The driving parameter $u(i)$ is dynamically defined by the map

$$u(i+1) = f(u(i)). \tag{2}$$

In this case one can join the maps (1) and (2) in one autonomous map defined in the extended phase space and having a master-slave structure where the map (2) serves the master equation.

Any attractor of the obtained autonomous map becomes stationary, and the master-slave structure simplifies the study of the joint map hyperbolic properties.

In our talk we give en example of such map and prove hyperchaotic properties of its attractor.

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Mathematical Models of Rogue Waves

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The rogue wave problem originates in the ocean-related application, though today it is understood in broader sense. Abnormally large waves which occur on the sea surface have been discovered not long ago, see reviews and the discussion of promising researches in [1, 2]. The nonlinear modulational instability have been suggested as a regular mechanism which can alter the probability distribution function for wave heights and result in a larger likelihood of extreme events. This mechanism is related to the formation of nonlinear coherent wave patterns, which possess their own dynamic features. Within the frameworks of idealized evolution equations which can be analyzed with the help of the Inverse Scattering Transform (the Korteweg – de Vries equation, KdV, and the nonlinear Schrödinger equation and their generalizations), the simplest long-living coherent patterns correspond to solitons and envelope solitons respectively.

In this paper we discuss the effects of the dynamics of ensembles of solitons in long-wave models (KdV equation and modified KdV, mKdV, and Gardner equations of the focusing type), which may lead to the generation of extremely large waves. These problems belong to so-called 'soliton turbulence', though instead of the consideration of the density of the 'soliton gas' within the kinetic approach, we focus on the wave amplitudes. The role of multiple soliton and breather interactions in the formation of very high waves is disclosed. The discovered scenario depends crucially on the soliton polarities and breather phases, and is not described by kinetic models.

In particular, conditions of optimal (synchronized) collisions of any number of solitons and breathers are studied within the framework of the Gardner equation (GE) with positive cubic nonlinearity, which in the limits of small and large amplitudes tends to the classic and the modified Korteweg–de Vries equations respectively. To this end the N-soliton-M-breather solution is considered for any natural numbers N and M. The solution is constructed with the help of the Darboux transform, following the technique suggested in [3]. The wave amplitude in the focal point is calculated exactly. It exhibits a linear superposition of partial amplitudes of the solitons and breathers (see details in [4, 5]). The crucial role of the choice of proper soliton polarities and breather phases on the cumulative wave amplitude in the focal point is demonstrated. Solitons are most synchronized when they have alternating polarities. The straightforward link to the problem of synchronization of envelope solitons and breathers in the focusing nonlinear Schrödinger equation is discussed (then breathers correspond to envelope solitons propagating above a condensate).

The soliton dynamics in the focal point is essentially nonlinear and may suffer from weak disturbances, including inaccuracy of a numerical code. This effect strengthens when the number of colliding solitons grows [6]. As a result, solitons of opposite polarities with close velocities may transform to breathers, which seem to be more stable nonlinear wave structures.

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Minimal generating sets and the structure of Wreath product of groups with non-faithful action.

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We denote by $d(G)$ the minimal number of generators of the group G [1, 4]. A diffeomorphism $h : M \to M$ is said to be f-preserving if $f \circ h = f$. This is equivalent to the assumption that h is invariant for each level-set, i.e. $f^{-1}(c)$, $c \in P$ of f, where P denotes either the real line R or the circle S^1.

Let G be a group. The *commutator width* of G [3], denoted by $cw(G)$, is defined to be the least integer n, such that every element of G' is a product of at most n commutators if such an integer exists, and otherwise is $cw(G) = \infty$. The following Lemma imposes the Corollary 4.9 of [2].

Lemma 1. *An element of form* $(r_1, \ldots, r_{p-1}, r_p) \in W' = (B \wr C_p)'$ *if and only if the product of all* r_i *(in any order) belongs to* B'*, where* $p \in \mathrm{N}$*,* $p \geq 2$*, where* $r_i = h_i g_{a(i)} h^{-1}_{ab(i)} g^{-1}_{aba^{-1}(i)}$*,* $h, g \in B$ *and* $a, b \in C_p$*.*

Lemma 2. *For any group* B *and integer* $p \geq 2$*, if* $w \in (B \wr C_p)'$*, then* w *can be represented by making use of the following wreath recursion*

$$w = (r_1, r_2, \ldots, r_{p-1}, r_1^{-1} \ldots r_{p-1}^{-1} \prod_{j=1}^{k} [f_j, g_j]),$$

where $r_1, \ldots, r_{p-1}, f_j, g_j \in B$ *and* $k \leq cw(B)$*.*

Theorem 1. *If the orders of cyclic groups* $\mathbb{C}_{n_i}$*,* $\mathbb{C}_{n_j}$ *are mutually coprime* $i \neq j$*, then the group* $G = C_{i_1} \wr C_{i_2} \wr \cdots \wr C_{i_m}$ *admits two generators, namely* β_0*,* β_1*.*

Let $\mathop{\wr}\limits_{j=0}^{n} C_{i_j}$ be generated by β_0 and β_1 and $\mathop{\wr}\limits_{l=0}^{m} C_{k_l} = \langle \alpha_0, \alpha_1 \rangle$. Denote an order of g by $|g|$.

Theorem 2. *If* $(|\alpha_0|, |\beta_0|) = 1$ *and* $(|\alpha_1|, |\beta_1|) = 1$*, or if* $(|\alpha_0|, |\beta_1|) = 1$ *and* $(|\alpha_1|, |\beta_0|) = 1$*, then there exists generating sets of two elements for the wreath-cyclic group* $G = (\mathop{\wr}\limits_{j=0}^{n} C_{i_j}) \times (\mathop{\wr}\limits_{l=0}^{m} C_{k_l})$*, where* i_j *are orders of* C_{i_j}*.*

We have found an upper bound for the generator number of G'. Let $\mathcal{A}$ be a group and $\mathcal{B}$ a permutation group, i.e. a group $\mathcal{A}$ acting upon a set X, where the active group $\mathcal{A}$ can act not faithfully.

Theorem 3. *If* $W = (\mathcal{A}, X) \wr (\mathcal{B}, Y)$*, where* $|X| = n$*,* $|Y| = m$ *and active group* $\mathcal{A}$ *acts on* X *transitively, then*

$$d\left(G'\right) \leq (n-1)d(\mathcal{B}) + d(\mathcal{B}') + d(\mathcal{A}').$$

We consider when the active group can be either finite or infinite and consider a center of such group. This consideration is a generalisation of Theorem 4.2 from the book [2]. Let $X = \{x_1, x_2, \ldots, x_n\}$ be an $\mathcal{A}$-space. If a non-faithful action by conjugation determines a shift of copies of $\mathcal{B}$ from the direct product $\mathcal{B}^n$, then we do not have the standard wreath product $(\mathcal{A}, X) \wr \mathcal{B}$ which is a semidirect product of $\mathcal{A}$ and $\prod\limits_{x_i \in X} \mathbb{B}$, i.e. $\mathcal{A} \ltimes_{\varphi} (\mathcal{B})^n$.

Corollary 1. *A center of the group* $(\mathcal{A}, X) \wr \mathcal{B}$ *is the direct product of the normal closure of the center of the diagonal of* $Z(\mathcal{B}^n)$*, i.e.* $(E \times Z(\triangle(\mathcal{B}^n)))$*, trivial an element, and the intersection of* $(\mathcal{K}) \times E$ *with* $Z(\mathcal{A})$*. In other words, we have*

$$Z((\mathcal{A}, X) \wr \mathcal{B}) = \langle (1;\ \underbrace{h, h, \ldots, h}_{n}),\ e,\ Z(\mathcal{K}, X) \wr \mathcal{E} \rangle \simeq (Z(\mathcal{A}) \cap \mathcal{K}) \times Z(\triangle(\mathcal{B}^n)),$$

where $h \in Z(\mathcal{B})$*,* $|X| = n$*.*

For the restricted wreath product with n *non-trivial coordinates, we have* $Z((\mathcal{A}, X) \wr \mathcal{B}) = \langle (1;\ \ldots, h, h, \ldots, h, \ldots),\ e,\ Z(\mathcal{K}, X) \wr \mathcal{E} \rangle \simeq (Z(\mathcal{A}) \cap \mathcal{K}) \times Z(\triangle(\mathcal{B}^n))$*.*

In case of unrestricted wreath product, we have $Z((\mathcal{A}, X) \wr \mathcal{B}) = \langle (1;\ \ldots, h_{-1}, h_0, h_1, \ldots, h_i, h_{i+1}, \ldots,),\ e,\ Z(\mathcal{K}, X) \wr \mathcal{E} \rangle \simeq (Z(\mathcal{A}) \cap \mathcal{K}) \times Z(\tilde{\triangle}(\mathcal{B}))$*.*

Remark 1. *The quotient group of a restricted wreath product $G = Z \wr_X Z$ by a commutator subgroup is isomorphic to $\mathbb{Z} \times \mathbb{Z}$. Making use of previous conditions, we have if $G = A \wr_X B$, then $G/G' = A/A' \times B/B'$. If $G = Z_n \wr Z_m$, where $(m,\ n) = 1$, then $d(G/G') = 1$. If $G = Z \wr Z$ is an unrestricted regular wreath product, then $G/G' \simeq Z \times E \simeq Z$.*

The minimal set of generators for the fundamental group $\pi_1(O_f, f)$ of orbits of one function f with respect to the action of the group of diffeomorphisms of non-moving ∂M has been found here.

Theorem 4. *The group $H \simeq \mathbb{Z} \ltimes (\mathbb{Z})^n = \langle \rho,\ \tau \rangle$ with defined above homomorphism in $AutZ^n$ has two generators and non trivial relations, namely*

$$\rho^n \tau \rho^{-n} = \tau^{-1},\ \rho^i \tau \rho^{-i} \rho^j \tau \rho^{-j} = \rho^j \tau \rho^{-j} \rho^i \tau \rho^{-i},\ 0 < i, j < n.$$

This group admits another presentation which makes use of generators and relations, namely

$$\left\langle \rho, \tau_1, \ldots, \tau_n \left| \rho \tau_{i(\ mod\ n)} \rho^{-1} = \tau_{i+1(\ mod\ n)}\ ,\ \ \tau_i \tau_j = \tau_j \tau_i,\ i,\ j \le n \right.\right\rangle.$$

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Morse theory and rigidity for transversely affine foliations

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This is a work in progress, joint work with Gilbert Hector (University of Lyon I).

Codimension one taut foliations on 3-manifolds have been studied for many years. Their classification on hyperbolic 3-manifolds has a mysterious finite aspect, but still very few is known. In this talk, we consider classification of taut foliations on surface bundles with pseudo-Anosov monodromy, which is a typical example of hyperbolic 3-manifolds.

On a torus bundle M over S^1 with linear Anosov monodromy map f, taut foliations are essentially classified by Ghys-Sergiescu [GS]: Any foliation without compact leaves on M is isotopic to either of the stable or unstable foliation of f. On a surface bundle M over S^1 whose fiber is an orientable closed surface Σ of genus > 1, the classification becomes much more complicated. For example, Cooper-Long-Reid [CLR] produced uncountably many minimal foliations which are close to $\mathcal{E}$ and mutually non-isotopic by some surgery on the bundle foliation. On the other hand, Nakayama [Na] generalized Ghys-Sergiescu's theorem in the context of transversely affine foliations under some conditions. Here, a transversely orientable codimension one foliation is transversely affine if it is defined by a 1-form ω such that $d\omega = \eta \wedge \omega$ for some closed 1-form η. In Nakayama's result, (un)stable foliations of linear Anosov maps are replaced with Meigniez's example [Me], so-called suspension foliation of pseudo-Anosov map.

Our main results are generalizations of Meigniez's example and Nakayama's theorem: Given $f \in \mathrm{Diff}_+(\Sigma)$ and $\sigma \in H^1(\Sigma)$ with $f^*\sigma = \lambda\sigma$ for some $\lambda\ (\neq 1) > 0$, we construct a foliation $\mathcal{F}_\sigma$ on the surface bundle with monodromy f by modifying Meigniez's construction with Moser's technique. These foliations $\mathcal{F}_\sigma$ are transversely affine, and share good properties with Meigniez's examples. The following generalizes Nakayama's theorem.

Theorem 1. *Let $f \in \mathrm{Diff}_+(\Sigma)$ be a pseudo-Anosov map and let M_f be the Σ-bundle whose monodromy is f. Let $\mathcal{E}$ be the bundle foliation on M_f. If $b_1(M_f) = 1$, then any orientable transversely affine foliation without compact leaves whose tangent plane field is homotopic to $T\mathcal{E}$ is isotopic to $\mathcal{F}_\sigma$ for some real eigenvector $\sigma \in H^1(\Sigma)$ of f^*.*

Since $\mathcal{F}_\sigma$ can be isotoped to a foliation which is almost tangent to $\mathcal{E}$, we have the following consequence.

Theorem 2. *Let f, M_f and $\mathcal{E}$ be as in Theorem 1. If $b_1(M_f) = 1$, then, for any $\varepsilon > 0$, any orientable transversely affine foliation without compact leaves whose tangent plane field is homotopic to $T\mathcal{E}$ is ε-coarse isotopic to $\mathcal{E}$ in the sense of Gabai [Ga].*

The main step of the proof of Theorem **??** is to isotope given foliation $\mathcal{F}$ to a foliation transverse to given 1-dimensional flow transverse to the surface fibers. This is done by eliminating certain tangent points of $\mathcal{F}$ to $\mathcal{E}$ by isotopies, based on a one-parameter family version of the Morse theory and Cerf's theorem. The argument is related to the argument of Roussarie-Quê [QR] and Blank-Laudenbach [BL] based on Morse theory for foliations without holonomy.

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On 2-dimensional expanding attractors of A-flows on 3-manifolds

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We consider closed 3-manifolds supporting A-flows with 2-dimensional expanding attractors. The natural question is what closed 3-manifolds admits A-flows with 2-dimensional expanding attractors. The main results are the following statements.

Theorem 1. Given any closed 3-manifold M^3, there is an A-flow f^t on M^3 such that the non-wandering set $NW(f^t)$ consists of a non-orientable two-dimensional expanding attractor and trivial basic sets.

Theorem 2. There is a nonsingular A-flow f^t on a 3-sphere S^3 such that the non-wandering set $NW(f^t)$ contains an orientable two-dimensional expanding attractor.

The study was implemented in the framework of the Basic Research Program at the National Research University Higher School of Economics (HSE) in 2019.

On 4-dimensional flows with wildly embedded invariant manifolds of a periodic orbit

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Qualitative study of dynamical systems reveals various topological constructions naturally emerged in the modern theory. For example, Cantor set with cardinality of continuum and Lebesgue measure zero as expanding attractor or contracting repeller. Also, a curve in 2-torus with irrational winding number, which is not a topological submanifold but is injectively immersed subset, can be found being invariant manifold of Anosov toral diffeomorphism's fixed point.

Another example of intersection of topology and dynamics is the Artin-Fox arc [**?**] appeared in work by D. Pixton [2] as the closure of a saddle separatrix of a Morse-Smale diffeomorphism on the 3-sphere. A wild behaviour of the Artin-Fox arc complicates the classification of dynamical systems, it does not admit already a combinatorial description like to Peixoto's graph [3] for 2-dimensional Morse-Smale flows.

It is well known that there are no wild arcs in dimension 2. In dimension 3 they exist and can be realize as invariant set for a discrete dynamics, in different from regular 3-dimensional flows, which do not possess wild invariant sets. The dimension 4 is very rich. Here wild objects appear both for discrete and continuous dynamics. Despite the fact that there are no wild arcs in this dimension, there are wild objects of co-dimension 2. So the closure of 2-dimensional saddle separatrix can be wild for 4-dimensional Morse-Smale system (diffeomorphism or flow). Such examples recently were constructed by V. Medvedev and E. Zhuzoma [4]. T. Medvedev and O. Pochinka [5] shown as wild Artin-Fox 2-dimension sphere appears as closure of heteroclinic intersection of Morse-Smale 4-diffeomorphism.

In the present paper we prove that the suspension under a non-trivial Pixton's diffeomorphism provides a 4-flow with wildly embedded 2-dimensional invariant manifold of a periodic orbit. Moreover, we show that there are countable many different wild suspensions.

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On Interactions Between Flexural and Torsion Vibrations of a Bar

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In the paper we consider a boundary-value problem for equations of flexural-torsional vibrations of a bar, described by the system of two differential equations in the domain $Q = \{0 < x < l,\, 0 < t < T\}$:

$$E\,I\,\frac{\partial^4 y}{\partial x^4} + \rho\,A\,\frac{\partial^2 y}{\partial t^2} - \varepsilon\,\rho\,A\,\frac{\partial^2 \theta}{\partial t^2} = 0\,, \tag{1}$$

$$-G\,C\,\frac{\partial^2 \theta}{\partial x^2} - \varepsilon\,\rho\,A\,\frac{\partial^2 y}{\partial t^2} + \rho\,(I + A\,\varepsilon^2)\,\frac{\partial^2 \theta}{\partial t^2} = 0\,, \tag{2}$$

where $y(x,t)$ is the lateral displacement of the bar, $\theta(x,t)$ is the turning angle of the bar cross-section, E is the Young modulus, I is a polar inertia moment of the cross section with respect to its gravity center, ρ is a density of the bar material, A is the area cross section, G a shear modulus, C is geometrical rigidity of free torsion and ε is the distance from the gravity center to the center of torsion of the bar (see [1] and references there in).

We underline that in equation (2) we neglect by the sectional moment of inertia of the bar's cross-section.

These equations ought to be provided by initial conditions:

$$y(x,0) = y_0(x)\,, \quad \frac{\partial y}{\partial t}(x,0) = y_1(x)\,, \quad 0 \le x \le l\,, \tag{3}$$

$$\theta(x,0) = \theta_0(x)\,, \quad \frac{\partial \theta}{\partial t}(x,0) = \theta_1(x)\,, \quad 0 \le x \le l\,. \tag{4}$$

Further let us suppose that ε is small parameter: $0 < \varepsilon \ll 1$. In this case system (1)-(2) is closely connected with the problem of influence of small distinction between the gravity center and the center of torsion of the cantilever of scanning probe microscope on precision of measurement of the local properties of the solid body surface.

In order to answer on this question first of all one ought to solve system (1)-(2) with homogeneous boundary conditions:

$$y|_{x=0} = 0\,, \quad \frac{\partial y}{\partial x}|_{x=0} = 0\,, \quad \frac{\partial^2 y}{\partial x^2}|_{x=l} = 0\,, \quad \frac{\partial^3 y}{\partial x^3}|_{x=l} = 0\,, \quad 0 \le t \le T\,, \tag{5}$$

$$\theta|_{x=0} = 0\,, \quad \frac{\partial \theta}{\partial x}|_{x=0} = 0\,, \quad 0 \le t \le T\,. \tag{6}$$

In this preliminary investigation we construct solution of this problem in the framework of the perturbation theory:

$$y(x,t) = \sum_{k=0}^{\infty} \varepsilon^k\, y^{(k)}(x,t)\,, \quad \theta(x,t) = \sum_{k=0}^{\infty} \varepsilon^k\, \theta^{(k)}(x,t)\,. \tag{7}$$

Under $\varepsilon = 0$ input system (1)-(2) is split on two independent equations namely initial approximation $y^{(0)}(x,t)$ for the lateral displacement of the bar obeys to the next equation:

$$E\,I\,\frac{\partial^4 y^{(0)}}{\partial x^4} + \rho\,A\,\frac{\partial^2 y^{(0)}}{\partial t^2} = 0\,. \tag{8}$$

Equation (8) we solve exactly with initial conditions (3) and boundary conditions (5).

Initial approximation $\theta^{(0)}(x,t)$ for the turning angle of the bar cross-section obeys to the following equation:

$$-G\,C\,\frac{\partial^2 \theta^{(0)}}{\partial x^2} + \rho\,I\,\frac{\partial^2 \theta^{(0)}}{\partial t^2} = 0\,. \tag{9}$$

Equation (9) we solve exactly with initial conditions (4) and boundary conditions (6).

Equations for higher order approximations $y^{(k)}(x,t)$ and $\theta^{(k)}(x,t)$ we obtain substituting asymptotic series (7) into input system (1)-(2). After that we solve these equations with the same boundary conditions (5)-(6) but with zero initial conditions.

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On limit cycles and resonances in the systems close to nonlinear Hamiltonian

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The purpose of this report to give the short review of my basic results, the concerning systems close to nonlinear two-dimensional Hamiltonian.

- Limit cycles
- Resonances in periodic case:
  - bifurcations in non-degenerate zones
  - bifurcations in degenerate zones
- Resonances in quasi-periodic case

On Modeling of One Unstable Bifurcation in the Dynamics of Vortex Structures

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This report is devoted to the results of phase topology research on a generalized mathematical model, which covers such two problems as the dynamics of two point vortices enclosed in a harmonic trap in a Bose-Einstein condensate [1] and the dynamics of two point vortices bounded by a circular region in an ideal fluid [2]. This model leads to a completely Liouville integrable Hamiltonian system with two degrees of freedom, and for this reason, topological methods used in such systems can be applied.

In this talk, we analytically derive equations that define the parametric family of bifurcation diagrams of the generalized model, including bifurcation diagrams of the specified limiting cases. The dynamics of the bifurcation diagram in a general case is shown using its implicit parameterization. A stable bifurcation diagram, related to the problem of dynamics of two vortices bounded by a circular region in an ideal fluid, is observed for particular parameters' values. Interactive visualization of the bifurcation diagram was made by A. A. Shadrin based on the equations of a bifurcation diagram and reduction to a system with one degree of freedom in the general case [3] and [4].

New bifurcation diagrams are obtained and three-into-one and four-into-one tori bifurcations are observed for some values of the physical parameters of the model. The three-into-one tori bifurcation was previously encountered in the works of M. P. Kharlamov in studying the phase topology of the integrable Chaplygin-Goryachev-Sretensky case in the dynamics of a rigid body [5] and as one of the features in the form of a 2-atom of a singular layer of Liouville foliation in the works of A. T. Fomenko, A. V. Bolsinov, S. V. Matveev [6]. In the work of A. A. Oshemkov and M. A. Tuzhilin [7], devoted to the splitting of saddle singularities, such a bifurcation turned out to be unstable and its perturbed foliations, one of which is realized in the integrable model under consideration, are given.

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On non-conservative perturbations of Hamiltonian systems containing quasi-periodic parameters

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We consider non-conservative quasi-periodic perturbations of 2-dim Hamiltonian systems that include terms depending on both t and the phase variables:

$$\begin{aligned} \dot{x} &= \frac{\partial H(x,y)}{\partial y} + \varepsilon[g_0(x,y) + a(\omega_1 t, ..., \omega_m t)g_1(x,y)]; \\ \dot{y} &= -\frac{\partial H(x,y)}{\partial x} + \varepsilon[f_0(x,y) + a(\omega_1 t, ..., \omega_m t)f_1(x,y)]. \end{aligned} \tag{1}$$

Here H, g_0, f_0, g_1, f_1 are nonlinear and assumed to be smooth while a and b are contionious and quasi-periodic; ε is a small parameter. We examine a certain domain $D = \{(x,y)|h_- \le H(x,y) \le h_+\} \subset \mathbf{R}$ filled with closed phase curves of the unperturbed system. Resonance levels of energy are determined by the condition $n\omega(h_{n,\mathbf{k}}) = k_1\omega_1 + ... + k_m\omega_m, n, k_1, ..., k_m \in \mathbf{Z}$, where $\omega(h)$ is the natural frequency of the unperturbed system. In the $\sqrt{\varepsilon}$-neighborhood of a fixed resonance level $h = h_{n\mathbf{k}}$ we obtain the following 2-dim averaged system

$$\begin{aligned} \dot{u} &= A(v) + \sqrt{\varepsilon}\sigma(v)u; \\ \dot{v} &= b_1 u + \sqrt{\varepsilon} b_2 u^2. \end{aligned} \tag{2}$$

$A(v), \sigma(v)$ are $2\pi/n$–periodic functions. In contrast to the previous studies [1,2], for the pertubation under consideration function $\sigma(v)$ is signalternative. This may result in the emergence of limit cycles in (2) that correspond to quasi-periodic solutions in the initial system with $m+1$ frequencies. The conditions of such tori existence are established. We use the equation

$$\ddot{x} + x + x^3 = \varepsilon((p_1 - x^2)\dot{x} + (1 + x\dot{x})\sin t \sin \sqrt{5}t) \tag{3}$$

to illustrate the study.

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On the classes of stable isotopic connectivity of polar cascades on a torus.

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The problem of the existence of an arc with no more than a countable (finite) number of bifurcations connecting structurally stable systems (Morse-Smale systems) on manifolds is included in the list of fifty Palis-Pugh problems [6] under number 33. The report will present a solution this problem for polar gradient-like diffeomorphisms of a torus.

In 1976, S. Newhouse, J. Palis, F. Takens [3] introduced the concept of a stable arc connecting two structurally stable systems on a manifold. Such an arc does not change its quality properties with little movement. In the same year, S. Newhouse and M. Peixoto [4] proved the existence of a simple arc (containing only elementary bifurcations) between any two Morse-Smale flows. From the result of the work of J. Fleitas [1] it follows that a simple arc constructed by Newhouse and Peixoto can always be replaced by a stable one. For Morse – Smale diffeomorphisms given on manifolds of any dimension, examples of systems that cannot be connected by a stable arc are known. In this connection, the question naturally arises of finding an invariant that uniquely determines the equivalence class of the Morse – Smale diffeomorphism with respect to the connection relation by a stable arc (*is a component of stable connection*).

Consider the class G of polar gradient-like diffeomorphisms on a torus with a fixed nonwandering set. The diffeomorphisms $f_0, f_1 \in G$ are smoothly isotopic to the identity map of the 2-sphere and, therefore, can be connected by some arc $\{f_t : S^2 \to S^2, t \in [0,1]\}$. However, *stability* of such an arc with a finite number of bifurcation values $0 < b_1 < \dots < b_k < 1$ is characterized by the fact that all of its points are structurally stable diffeomorphisms, with the exception of a finite number of bifurcation points, which typically pass through saddle-node bifurcations or flip.

In this report, a stable arc will be constructed connecting any two cascades from the class in question. Note that it was shown in [5] that polar cascades on a two-dimensional sphere are always connected by an arc without bifurcations. For a two-dimensional torus, the situation is different due to the fact that the closures of the invariant manifolds of saddle points of the polar cascade are circles belonging to any previously defined homotopy class. It follows directly from this that in the general case there is no arc without bifurcations between the systems under consideration. Nevertheless, the authors of this paper established the following result.

Theorem 1. *Theorem Any diffeomorphisms $f_0, f_1 \in G$ belong to the same class of stable isotopy connection. Moreover, there exists a stable arc connecting them, all the bifurcation points of which are saddle-nodes.*

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On the classification of homoclinic attractors of three-dimensional flows

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This report covers some issues of classification of strange homoclinic attractors of three-dimensional dynamical systems with continuous-time (flows). Strange attractors are called homoclinic if they contain a specific saddle equilibrium together with its unstable manifold. The classification of homoclinic attractors is based on the properties of the equilibrium state belonging to the attractor. These properties are determined by eigenvalues of the equilibrium. Depending on the signs of real parts of the eigenvalues, the saddle equilibrium states of three-dimensional flows are of only two types: (2,1) – with two-dimensional stable and one-dimensional unstable invariant manifolds, and (1,2) – with one-dimensional stable and two-dimensional unstable manifolds. If a saddle equilibrium has a pair of complex-conjugated eigenvalues, then it is called a saddle-focus. Topologically, the saddle-focus equilibrium topologically is not distinguished from the saddle (with real eigenvalues). However, concerning the dynamics, the saddle focus is fundamentally different from the saddle [1]. Another important characteristic of saddle equilibrium states is the sum of the real parts of the eigenvalues closest to the imaginary axis, but lying on it on opposite sides. Depending on the described characteristics, strange homoclinic attractors can be of six different types: Shilnikov attractor containing a saddle focus (1,2); figure-eight spiral attractor containing a saddle focus (2,1) with the Shilnikov homoclinic loop of the saddle focus when the saddle value is positive; a figure-eight spiral attractor with a saddle-focus loop (2,1), whose saddle value is negative; attractor of the Lorenz type containing a saddle (2,1) with a positive saddle value; Lyubimov-Zaks-Rovella attractor containing a saddle (2,1) with a negative saddle value; and the Shilnikov saddle attractor containing the saddle equilibrium (1,2).

The idea of classifying homoclinic attractors according to the type of equilibrium state was proposed in [3]. It was framed as a saddle charts method and applied to the class of three-dimensional Henon maps in [2]. In this report the classification of homoclinic attractors based on the saddle charts method is applied to the three-dimensional flows of the following form:

$$\begin{cases} \dot{x} = y + g_1(x, y, z), \\ \dot{y} = z + g_2(x, y, z), \\ \dot{z} = Ax + By + Cz + g_3(x, y, z), \end{cases} \tag{1}$$

where A, B и C – parameters of the system and $g_i, i = 1, 2, 3$ – non-linear functions satisfying to

$$g_i(0,0,0) = \frac{\partial g_i}{\partial x}(0,0,0) = \frac{\partial g_i}{\partial y}(0,0,0) = \frac{\partial g_i}{\partial z}(0,0,0) = 0,\ i = 1, 2, 3.$$

whose linearization matrix is represented in the Frobenius form, and the eigenvalues are determined by the coefficients A, B and C. In the parameters space A, B and C, a saddle chart (extended bifurcation diagram) is constructed, where 7 regions corresponding to attractors of various types are distinguished. It is noted that a wide class of three-dimensional flows can be reduced to the class of systems under consideration.

The report also discusses problems related to the pseudohyperbolicity of homoclinic attractors of three-dimensional flows. According to the theory of Turaev and Shilnikov chaotic attractors are called pseudohyperbolic if any its trajectory has a positive Lyapunov exponent, and this property persists after small perturbations of the system [4]. It is proved that in three-dimensional flows only two types of homoclinic attractors can be pseudohyperbolic: Lorenz-like attractors containing a saddle equilibrium state with a two-dimensional stable manifold whose saddle value is positive; as well as Shilnikov saddle attractors containing a saddle equilibrium with a two-dimensional unstable manifold. The remaining attractors inevitably belong to a class of quasiatractors by Afraimovich-Shilnikov (such attractors either contain stable periodic orbit with narrow absorbing domains or such orbits appear after arbitrarily small perturbations).

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On the Critical Cases of Stability in Impulsive Systems

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Impulsive differential equations (impulsive systems for short) correspond to a smooth evolution that at certain times changes instantaneously, or one could also say abruptly. There are many applications of these equations to mechanical and natural phenomena. We refer to [1] for an extensive list of references. In the report, an algorithm for computing the first Lyapunov value in a critical case is proposed for a second-order periodic impulsive system

$$\dot{x} = Ax + \sum_{|m|\geq 2} f_m x^m, \quad t \neq \tau_k, \qquad x(t^+) = Bx(t) + \sum_{|m|\geq 2} g_m x^m, \quad t = \tau_k, \tag{1}$$

where $x(t^+) = \lim_{s\to t+0} x(s)$, $m = (m_1, m_2) \geq 0$, $|m| = m_1 + m_2$, $x = (x_1, x_2)^T$, $x^m = x_1^{m_1} x_2^{m_2}$, A, $B \in R^{2\times 2}$, $\det B \neq 0$, $f_m, g_m \in R^2$, $\tau_k = k\theta$, $k = 0, 1, 2, \ldots$, $0 < \theta$. The series on the right-hand side of the system are assumed to be absolutely convergent in some neighborhood of zero. The system is a periodic one of period θ.

We suppose that the linearization of the system (1)

$$\dot{x} = Ax, \quad t \neq \tau_k, \qquad x(t^+) = Bx(t), \quad t = \tau_k.$$

Without loss of generality, we will assume that the linearization monodromy matrix has the canonical form $M = e^{\theta A}B = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}$, $\alpha, \beta \in \mathbb{R}$, $\alpha^2 + \beta^2 = 1$. Following an approach suggested in [2], we will perform a linear change of variables $y = \Psi(t)x$ in the system (1), where

$$\Psi(t) = \Psi(\tau_k^+)e^{-tA}, \quad \Psi(\tau_k^+) = e^{\theta A}\Psi(\tau_k), \quad \tau_k < t \leq \tau_{k+1}, \quad k = 0, 1, \ldots, \quad \Psi(0) = I.$$

The θ-periodic nonsingular matrix $\Psi(t)$ is a piecewise smooth on $\mathbb{R}$. Thus the transformation $y = \Psi(t)x$ is a Lyapunov transformation.

In new variables, the system (1) takes the form

$$\dot{y} = \sum_{|m|\geq 2} \tilde{f}_m(t) y^m, \quad t \neq \tau_k, \qquad y(t^+) = My(t) + \sum_{|m|\geq 2} e^{\theta A} g_m y(t)^m, \quad t = \tau_k, \tag{2}$$

where $\tilde{f}_m(t)y^m = \Psi(t) f_m(\Psi^{-1}(t)y)^m$ are θ-periodic functions with respect to t.

Denote by $y(t, y_0)$ the solution of the differential equation of the impulsive system (2), which satisfies the initial condition $y(0) = y_0$. Calculating the coefficients of the expansion of the solution in a series from the initial data in a neighborhood of the origin $y(t, y_0) = y_0 + \sum_{m\geq 2} s_m(t) y_0^m$, $s_m(0) = 0$, to the third degree inclusive, we find an estimate of the value $y(\theta, y_0)$. Substituting this result into the formula of the impulse operator of the system (2), we obtain an approximation of the Poincare map

$$y(\theta^+, y_0) = My_0 + \sum_{|m|=2}^{3} p_m y_0^m + \ldots,$$

where $p_m = \tilde{g}_m(\theta) + Ms_m(\theta)$, if |m|=2, and $p_{30} = \tilde{g}_{30}(\theta) + 2s^{(1)}_{20}\tilde{g}_{20}(\theta) + s^{(2)}_{20}\tilde{g}_{11}(\theta)$, $p_{21} = \tilde{g}_{21}(\theta) + 2s^{(1)}_{11}(\theta)\tilde{g}_{20}(\theta) + (s^{(1)}_{20}(\theta) + s^{(2)}_{11}(\theta))\tilde{g}_{11}(\theta) + 2s^{(2)}_{20}(\theta)\tilde{g}_{02}(\theta)$, $p_{12} = \tilde{g}_{12}(\theta) + 2s^{(1)}_{22}(\theta)\tilde{g}_{20}(\theta) + (s^{(2)}_{02}(\theta) + s^{(1)}_{11}(\theta))\tilde{g}_{11}(\theta) + 2s^{(2)}_{11}(\theta)\tilde{g}_{02}(\theta)$, $p_{03} = \tilde{g}_{03}(\theta) + 2s^{(2)}_{02}(\theta)\tilde{g}_{02}(\theta) + s^{(1)}_{02}(\theta)\tilde{g}_{11}(\theta)$. Due to the nonlinearity of the differential equation in (2), for the values of $s_m(\theta)$ we obtain very simple recurrence relations: $s_m(\theta) = \int_0^\theta \tilde{f}_m(t)\,dt$ for $|m| = 2$; $s_m(\theta) = \int_0^\theta \big(\tilde{f}_m(t) + r_m(t)\big)\,dt$ for $|m| = 3$, where the functions $r_m(t)$ depend only on $s_n(t)$ with $|n| < |m|$, more precisely, $r_{30} = 2s^{(1)}_{20}\tilde{f}_{20} + s^{(2)}_{20}\tilde{f}_{11}$, $r_{21} = 2s^{(1)}_{11}\tilde{f}_{20} + (s^{(1)}_{20} + s^{(2)}_{11})\tilde{f}_{11} + 2s^{(2)}_{20}\tilde{f}_{02}$, $r_{12} = 2s^{(1)}_{22}\tilde{f}_{20} + (s^{(2)}_{02} + s^{(1)}_{11})\tilde{f}_{11} + 2s^{(2)}_{11}\tilde{f}_{02}$, $r_{03} = 2s^{(2)}_{02}\tilde{f}_{02} + s^{(1)}_{02}\tilde{f}_{11}$.

Thus the problem of stability of the trivial solution of the initial impulsive system is reduced to the problem of stability of the zero fixed point of the smooth mapping $\tilde{P}(u) = Mu + \sum_{|m|=2}^{3} p_m u^m$, $p_m = (p^{(1)}_m, p^{(2)}_m)^T \in \mathbb{R}^2$. To study this problem, we bring it to the normal form until terms of the third degree. It is convenient to pass to the complex conjugate variables $z = u_1 + iu_2$, $\overline{z} = u_1 - iu_2$. In the new variables we obtain the scalar equation

$$\tilde{z} = F(z) = e^{i\gamma}z + \sum_{|m|=2}^{3} G_m z^{m_1}\overline{z}^{m_2} = e^{i\gamma}z + \mathcal{G}_2(z,\overline{z}) + \mathcal{G}_3(z,\overline{z}), \quad e^{i\gamma} = \alpha + i\beta. \tag{3}$$

We eliminate the quadratic terms in (**??**) using an almost identical change of variables $z = H(w) = w + H_2(w,\overline{w})$, $H_2 = \sum_{|m|=2} h_m w^{m_1}\overline{w}^{m_2}$, $h_m \in \mathbb{C}$.

We compute the inverse mapping up to cubic terms $w = H^{-1}(z) = z - H_2(z,\overline{z}) + Q_3(z,\overline{z}) + \dots$. Complex coefficients of a homogeneous polynomial are determined by the formulas $q_{30} = 2h^2_{20} + h_{11}\overline{h}_{02}$, $q_{21} = 3h_{20}h_{11} + |h_{11}|^2 + 2|h_{02}|^2$, $q_{12} = 2h_{20}(\overline{h}_{11} + h_{02}) + h_{11}(\overline{h}_{20} + h_{11})$, $q_{03} = 2h_{02}\overline{h}_{20} + h_{11}h_{02}$.

In the new variables, the mapping takes the form $\tilde{w} = H^{-1} \circ F \circ H(w) = e^{i\gamma}w + \sum_{|m|\geq 3} W_m w^{m_1}\overline{w}^{m_2}$. Given the presence of third-order resonances $i(m_1 - m_2)\gamma = \pm i\gamma$, $|m| = 3$, and discarding, on the basis of the Poincare-Dulac theorem, nonresonant monomials of the third degree and all monomials of higher degrees, we obtain the so-called model map

$$\hat{w} = e^{i\gamma}w(1 + \mathcal{A}|w|^2),$$

where $\mathcal{A} = G_{21}e^{-i\gamma} + \dfrac{e^{-i\gamma} - 2}{e^{2i\gamma} - e^{i\gamma}}G_{20}G_{11} + \dfrac{2|G_{02}|^2}{e^{3i\gamma} - 1} + \dfrac{|G_{11}|^2}{e^{i\gamma} - 1}$.

The sign of the first Lyapunov quantity $L = \mathrm{Re}\mathcal{A}$ determines the stability character of the zero solution of the initial impulse system (1) according to Lyapunov, namely, the solution is stable asymptotically if $L < 0$ and unstable if $L > 0$.

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On topological classification and realization of Morse-Smale flows on the sphere

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A Morse-Smale diffeomorphism $f : S^n \to S^n$ (or a flow f^t) on the sphere S^n is a structurally stable diffeomorphism (flow) whose non-wandering set belongs to finite number of periodic hyperbolic orbits, including fixed points.

A wide set of Morse-Smale flows admit combinatorial description of topological equivalence classes, while similar cascades do not (see, for example, [1] for references). We discuss the difference and define a class of Morse-Smale cascades to which it is possible to borrow the topological invariants from the flows.

Describe a combinatorial invariant for a class G of Morse-Smale diffeomorphisms on the sphere S^n of dimension $n \geq 4$ without heteroclinic intersection of invariant manifolds of saddle periodic points introduced in [2] and called a colored graph.

Let Ω_f be a non-wandering set of the diffeomorphism $f \in G$ and $\Omega^i_f = \{p \in \Omega_f |\ dim\ W^u_p = i\}$, $i \in \{0, 1, n-1, n\}$.

For any saddle point σ of the diffeomorphism $f \in G$ the closure $cl\ W^\delta_\sigma$ of its invariant manifold W^δ_σ, $\delta \in \{s, u\}$ of dimension $(n-1)$ consists of the union of W^δ_σ and exactly one periodic point (a sink if $\delta = u$ and a source otherwise). A union $\mathcal{L}_f = (\bigcup\limits_{p \in \Omega^1_f} cl\ W^s_p) \cup (\bigcup\limits_{q \in \Omega^{n-1}_f} cl\ W^u_q)$ cuts the sphere S^n in $k = |\Omega^1_f \cup \Omega^{n-1}_f| + 1$ connected components (where $|P|$ is a cardinality of the set P). Denote these components by $D_1, \dots, D_k$ and put $\mathcal{D}_f = \bigcup\limits_{i \in 1}^{k} D_i$.

A colored graph of the diffeomorphism $f \in G$ is the graph Γ_f with the following properties:

1) a set Γ^0_f of vertices of the graph Γ_f is isomorphic to the set $\mathcal{D}_f$, a set Γ^1_f of edges is isomorphic to the set $\mathcal{L}_f$ by an isomorphism $\xi : \Gamma^0_f \cup \Gamma^1_f \to \mathcal{D}_f \cup \mathcal{L}_f$;

2) vertices v_i, v_j are joined by an edge $e_{i,j}$ if and only if the correspondent domains $D_i = \xi(v_i), D_j = \xi(v_j)$ have the common boundary;

3) an edge $e_{i,j}$ have a color s (u) if $\xi(e_{i,j}) = W^s_p$ $(\xi(e_{i,j}) = W^u_q)$ for some points $p, q \in \Omega_f$.

One can show that the colored graph Γ_f of any $f \in G$ is a tree.

Endow the graph Γ_f by an automorphism $P_f : \Gamma_f \to \Gamma_f$ such that $\xi P_f = f\xi$.

Theorem 1. *Diffeomorphisms $f, f' \in G$ are topologically conjugated if and only if there exists an isomorphism $\zeta : \Gamma_f \to \Gamma_{f'}$ preserving color of edges such that $P_{f'} = \zeta P_f \zeta^{-1}$.*

Theorem 2. *For any 2-colored tree Γ enriched by an automorphism P there is a diffeomorphism $f \in G$ such that Γ_f is isomorphic to Γ by means of isomorphism $\zeta : \Gamma_f \to \Gamma$ preserving color of edges such that $P = \zeta P_f \zeta^{-1}$.*

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Optimal self-similar metrics of expansive homeomorphisms and expanding continuous maps

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It is known since the 1980s that any expansive homeomorphism on a metrizable compactum possesses some Lyapunov or adapted metric. This means that the homeomorphism contracts (resp. expands) local stable (resp. unstable) "manifolds" of a small radius in this metric. Simultaneously, an analogous result was obtained for positively expansive continuous maps on compacta, that is, such a map expands small distances in a suitable metric. The author [1] have sharpened this result in the case of homeomorphism: there exists a Lyapunov metric such that the homeomorphism on local stable (resp. unstable) "manifolds" is approximately representable on a small scale as a contraction (resp. expansion) with constant coefficient λ_s (resp. λ_u^{-1}) in $(0;1)$. Precisely speaking, $\rho(f(x), f(y))/\rho(x,y)$ tends to λ_s (resp. λ_u^{-1}) for two points x, y on one local stable (resp. unstable) "manifold", when $\rho(x,y) \to 0$, where f is the homeomorphism under discussion and ρ is a metric constructed. Also, the homeomorphism together with its inverse are Lipschitz with constants λ_u^{-1} and λ_s^{-1}, respectively, with respect to the metric constructed. Moreover, for homeomorphisms with local product structure the lower bounds for the contraction constants λ_s and expansion constants λ_u are attained simultaneously for some "optimal" metric that satisfies all the conditions described. These results can be immediately transferred to the case of positively expansive continuous maps, though this fact was not pointed out in [1].

Recently, A. Artigue [2] have constructed the so-called self-similar metrics. For the case of homeomorphism f, the metric is called self-similar if $\max\{\rho(f(x), f(y)), \rho(f^{-1}(x), f^{-1}(y))\} = \lambda^{-1}\rho(x,y)$ with some constant $\lambda \in (0;1)$, provided $\rho(x,y)$ is small enough. In particular, on a small scale, the homeomorphism contracts (resp. expands) local stable (resp. unstable) "manifolds" with exact constant λ (resp. λ^{-1}). Respectively, for the case of positively expansive maps, the equality defining self-similar metric takes the form $\rho(f(x), f(y)) = \lambda^{-1}\rho(x,y)$.

Thus, we deal with asymptotic estimates for rate of contraction/expansion, while Artigue states the exact equalities. On the other hand, we consider distinct constants λ_s, λ_u for contraction and expansion, and discuss simultaneous attainability of their lower bounds at some metric, while Artigue simply considers case where these constants are equal.

In the present talk, we adapt Artigue's approach to sharpen our results and to introduce self-similar metrics with distinct contraction and expansion constants λ_s and λ_u. In fact, the basic idea is only to use the sup operation instead of traditional summation in the well-known formula for Lyapunov metric (such an expression was written in [3] and when used in [2]) and in analogous formulas in [1] that were based on the latter!

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Phase change for perturbations of Hamiltonian systems

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We consider a small perturbation of a Hamiltonian system with one degree of freedom that has a separatrix loop. We also assume that the perturbation is such that the solutions strarting outside the separatrix loop approach it and eventually cross it. For study of such systems see, e.g., [1] and references therein.

We are interested in the change of phase while approaching the separatrix. A parameter called the pseudo-phase ([2]) describes the phase at the moment of separatrix crossing. In [2] a formula for the dependence of the pseudo-phase on the initial conditions was obtained for slow-fast Hamiltonian systems. We show that a similar formula also holds for our case. The main tool we use is the averaging method.

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Puiseux series, invariant algebraic curves and integrability of planar polynomial dynamical systems

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Establishing integrability of an ordinary differential equation or a system of ordinary differential equations is an important problem from theoretical and practical points of view. The system of first–order ordinary differential equations

$$x_t = P(x, y), \quad y_t = Q(x, y), \quad P(x, y),\, Q(x, y) \in \mathbb{C}[x, y] \tag{1}$$

with coprime polynomials $P(x, y)$ and $Q(x, y)$ is called integrable with the first integral $I(x, y) \not\equiv$ const defined in a domain D of full Lebesgue measure in $\mathbb{C}^2$ if the function $I(x, y)$ remains constant along any solution $(x(t), y(t))$ in D. The following equation $\mathcal{X}I = 0$ holds whenever $I(x, y)$ is of class at least C^1 in D. In this relation $\mathcal{X} = P\partial_x + Q\partial_y$ is the vector field associated to system (1). If there exists a function $R(x, y)$ such that the product of the differential form $d\omega = P(x, y)dy - Q(x, y)dx$ and $R(x, y)$ makes the form exact, then this function is called an integrating factor of the differential form and dynamical system (1).

Suppose a differential system under consideration possesses the first integral $I(x, y)$ that is a Liouvillian function; then we shall say that dynamical system (1) is *Liouvillian integrable.* A function is Liouvillian if it can be expressed as a finite superposition of algebraic functions, quadratures and exponential of quadratures over the field of rational functions $\mathbb{C}(x, y)$. It is known [1] that dynamical system (1) is Liouvillian integrable if and only if it has an integrating factor $R(x, y)$ given by

$$R(x, y) = \exp\left\{\frac{g(x, y)}{f(x, y)}\right\} \prod_{j=1}^{r} F_j^{s_j}(x, y), \tag{2}$$

where $g(x, y)$, $f(x, y)$, $F_1(x, y)$, ..., $F_r(x, y)$ are bivariate polynomials with coefficients from the field $\mathbb{C}$ and s_1, ... ,$s_r \in \mathbb{C}$. The algebraic curve $F_j(x, y) = 0$ given by the polynomial $F_j(x, y)$ in expression (**??**) is an invariant algebraic curve of dynamical system (1). In other words, the polynomial $F_j(x, y)$ satisfies the following partial differential equation $\mathcal{X}F_j = \lambda_j(x, y)F_j$, where $\lambda_j(x, y)$ is a bivariate polynomial called the cofactor of the algebraic curve $F_j(x, y) = 0$. Analogously, the function $E(x, y) = \exp\{g(x, y)/f(x, y)\}$ is an exponential invariant of dynamical system (1). Consequently, the problem of establishing Liouvillian integrability or non–integrability of a dynamical system can be reduced to the problem of constructing all irreducible invariant algebraic curves of $\mathcal{X}$ and all exponential invariants of $\mathcal{X}$. The main difficulty in finding irreducible invariant algebraic curves lies in the fact that bounds on the degrees of $F_j(x, y)$ are as a rule unknown in advance.

The aim of the talk is to present a general method of constructing all irreducible invariant algebraic curves of dynamical system (1) [2, 3]. The main idea of the method is to use the factorization of invariant algebraic curves in the algebraically closed field of Puiseux series. We shall derive the general structure of irreducible invariant algebraic curves and their cofactors for any polynomial dynamical system of the form (1).

As an application of our results we shall solve completely the problem of Liouvillian integrability for a number of physically relevant dynamical systems including the famous Duffing and Duffing–van der Pol oscillators [2, 3]. In addition, we shall demonstrate that a similar method is applicable in the case of time–dependent polynomial dynamical systems in the plane [4].

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Regularity of attractors of transversally similar Riemannian foliations

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Recall that a transformation $f \in Diff(N)$ of a q-dimensional Riemannian manifold (N, g) is called a similarity if $f^*g = \lambda g$ where λ is a positive constant. The set of all similarities of a Riemannian manifold (N, g) forms the Lie group $Sim(N, g)$.

A foliation (M, F) of a codimension q on n-dimensional manifold M, where $0 < q < n$, is called a *transversally similar Riemannian* foliation if the transformations of foliated transversal coordinates are local similarities of some q-dimensional Riemannian manifold (N, g) whose connectivity is not assumed.

A subset of a manifold M is said to be saturated if it is a union of leaves of the foliation (M, F). A closed saturated nonempty subset $\mathcal{M}$ of M for which there exists an open neighborhood $\mathcal{U}$ such that the closure of every leaf from $\mathcal{U} \setminus \mathcal{M}$ contains $\mathcal{M}$, is called an *attractor* of this foliation. The neighborhood $\mathcal{U}$ is defined by the above condition, is called the basin of the attractor $\mathcal{M}$ and is denoted by $Attr(\mathcal{M})$.

If, moreover, $Attr(\mathcal{M}) = M$, then the attractor $\mathcal{M}$ is called global.

An attractor $\mathcal{M}$ of a foliation (M, F) is called *regular* if it is a smooth submanifold of M.

Recall that a minimal set of a foliation (M, F) is referred to as a nonempty closed saturated subset $\mathcal{K}$ of M such that every leaf belonging to $\mathcal{K}$ is dense in $\mathcal{K}$.

Minimal sets and attractors of foliations (M, F) largely determine the topology of (M, F). Therefore, the investigation of the existence and the structure of minimal sets and attractors is one of the main problems of both topological dynamics and qualitative theory of foliations.

Attractors that are minimal sets of conformal and Weyl foliations were investigated, in particular, in [1], [2] and [3].

The purpose of this work is to describe the structure of global attractors of transversally similar Riemann foliations of an arbitrary codimension on n-dimensional manifolds.

The following statement is one of the main results of this work.

Theorem *Every complete transversally similar Riemannian foliation (M, F), which is not Riemannian, has a regular global attractor.*

The application of this theorem made it possible to describe the global structure of such foliations. In particular, it is proved that transversally similar non-Riemannian foliations exist only on non-compact manifolds. Also, the leaf closures are not submanifolds of M, in general.

A leaf of a foliation (M, F) is called proper if it is an embedded submanifold of M. A foliation (M, F) is referred to as proper if all its leaves are proper. In the case where (M, F) is a proper non-Riemannian transversally similar Riemannian foliation, its global attractor is a closed leaf.

Examples are constructed.

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Smale – Williams solenoids in autonomous model of coupled oscillators with "figure-eight" homoclinic bifurcations

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We continue the research of autonomous model [1] with uniformly hyperbolic chaotic attractor of Smale – Williams type. It is composed of two self-oscillators with saddle equilibrium at the origin. Each oscillator is governed by equations:

$$\begin{aligned} \dot{x} &= u, \\ \dot{u} &= \left(1 - x^2\right) x + \left[L - \left(1 - x^2\right)^2\right] u, \end{aligned} \tag{1}$$

and demonstrates "gluing" of the limit cycles into the asymptotic trajectories of the saddle equilibrium at $L \approx 0.3197$. At the same parameter value a "figure-eight" pair of bi-asymptotic trajectories of the saddle exists. We investigate two coupled subsystems (1) with coordinates (x, y) and velocities (u, v). Equations in complex variables $z = x + iy$ and $w = u + iv$ are:

$$\begin{aligned} \dot{z} &= w, \\ \dot{w} &= \left(1 - |z|^2\right) z + \left[L - \left(1 - |z|^2\right)^2\right] w + \varepsilon w^M. \end{aligned} \tag{2}$$

The term εw^M describes auxiliary coupling. We discuss examples of system (2) with $M = 2$ and $M = 3$. Let us introduce an angular variable θ as an argument of complex variable $z \propto \exp i\theta$. If a tipical trajectory is close to the saddle equilibrium at $z = 0$, $w = 0$ so that the amplitude $|z|$ is small, then the angular variable θ expands M times due to auxiliary coupling εw^M. If one constructs a proper Poincaré cross-section of the flow (2), for example by the surface $|z|^2 = 1$ (trajectories crossing outwards), which is far from saddle equilibrium, the angular variable θ undergoes an expanding circle map $\theta_{n+1} = M\theta_n + \text{const} \pmod{2\pi}$ after each iteration of Poincaré return map. If there is strong contraction of phase space in all other directions in Poincaré map, the Smale – Williams solenoid emerges with factor of angular expansion $M = 2$ or 3.

We investigate system (2) by means of numerical simulation. We obtain atlases of dynamical regimes for Poincaré maps of (2) with expansion factors $M = 2$ and $M = 3$. We qualify parameters at which Smale – Williams attractors exist with recently developed numerical technique [2]. We check numerically that average expansion of angular variable is close to M. Namely we accumulate values of angular variable θ_n in small intervals $\left[\frac{2\pi}{N}k, \frac{2\pi}{N}(k+1)\right]$ of the circle $[0, 2\pi)$ until smooth distribution, find averages $\langle \exp i\theta \rangle_k$ for every interval and calculate the sum $\sum_{k=0}^{N-1} \arg \frac{\langle \exp i\theta \rangle_{k+1}}{\langle \exp i\theta \rangle_k}$. If obtained sum is close to $2\pi M$, while there are no empty intervals and negative values of $\arg \frac{\langle \exp i\theta \rangle_{k+1}}{\langle \exp i\theta \rangle_k}$, we confirm that trajectory belongs to Smale – Williams solenoid. We show that the domain of Smale – Williams attractors is large and continuous on the parameter plane (L, ε) for models (2) with $M = 2$ or 3. We spot other regimes by calculation of Lyapunov exponents. We report that birth of Smale – Williams solenoid is preceded by Feigenbaum transition to non-hyperbolic chaos. The details of the transition between non-hyperbolic and hyperbolic attractors will be discussed elsewhere.

In addition we check the hyperbolicity of attractor at tipical values of parameters by numerical test of the angles between stable and unstable subspaces with technique developed in [3]. Absence of zero angles gives us reason to consider the attractor uniformly hyperbolic.

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Scenarios of transition to chaos in the discrete–time predator–prey model

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The study of biological communities, such as predator–prey or host–parasite systems, is one of the most important environmental problems. Predator–prey interactions are crucial to formation of the species composition in a community and their dynamics. In particular, predator–prey interactions can cause fluctuations in the numbers of both interacting species and can amplify such fluctuations if they exist due to other causes. In this work, we present a new look at the problem of complex dynamics that can arise between a prey and a predator. This paper investigates scenarios of transition to chaos in the predator–prey model with age structure for prey. We use a slight modification of the Nicholson–Bailey model to describe the interaction between predator and prey. We assume the population size is regulated by decreasing juvenile survival rate with growth of age class sizes. The model considered may be written as a system of three equations:

$$\begin{cases} X_{n+1} = rY_n \exp(-bZ_n), \\ Y_{n+1} = \exp(-\alpha X_n - \beta Y_n)X_n + vY_n, \\ Z_{n+1} = crY_n(1 - \exp(-bZ_n)), \end{cases} \tag{1}$$

where n is a reproductive season number; X and Y are the population size of juveniles and adults of prey (hosts), respectively; Z is the number of predators (parisitoids); r is the birth rate of prey (hosts); v is the survival rate of prey adults; b is the attack rate of the predator; c is a measure of the "conversion" of hosts (prey) into predators the following year. The survival rate of immature individuals of prey is selected as the Ricker model: $exp(-\alpha X_n - \beta Y_n)$, where α and β are the parameters describing the intensity of intrapopulation competition.

We made the analytical and numerical research of the mathematical model. For making the numerical experiments, we have elaborated the software systems to construct of bifurcation diagrams, attraction basins, and charts of dynamic modes, eigenvalues and Lyapunov exponents.

Conditions for sustainable coexistence of interacting species are described. It is shown that the coexistence of species becomes possible if there are a transcritical or saddle–node (tangential) bifurcations. Due to the saddle–node bifurcation there is bistability in the system of interacting species: predator either coexists with prey or dies depending on the initial conditions.

It is shown, with changing parameters' values and transition through the stability domain boundary the stability loss of the fixed point may occur according to both scenarios: the period doubling and the Neimark–Sacker bifurcation. Consequently depending on the values of the model parameters, the transition to chaos can be realized through the period doubling or through the destruction of the invariant curve.

The characteristic of the first scenario is that the eigenvalues pair of a fixed point is almost always imaginary (a fixed point is a saddle–focus). Therefore, the loss of stability of the 2–cycle is rarely accompanied by a period doubling cascade. The 2–cycle loses stability due to the Neimark–Sacker bifurcation. As a result, two invariant curves are formed around each periodic point. Further, there are two possible options for complicating of the attractor. First, a single doubling of the invariant curve occurs and a two–component strange attractor of the torus–chaos type emerges. Secondly, when parameters pass through resonant cycles, the invariant curve is strongly deformed and complexly wraps around the initial saddle cycle and a super–spiral attractor is formed. In both cases, a strange homoclinic attractor of period 2 emerges. Then the model trajectories are strongly mixed. As a result, in addition to periodic points, the new completely non–periodic attractor contains fixed points as well.

The second scenario is associated with the formation of a non–orientable Shilnikov funnel based on a single invariant curve formed according to the classical Neimark–Sacker bifurcation scenario. As a consequence of the period doubling bifurcation, the curve becomes a saddle, and in its neighborhood there is a pair of stable invariant curves of period 2. Unlike the classical or orientable case, these curves do not wrap around the initial curve, but are located in its neighborhood. With a further change in the bifurcation parameter, two super–spiral attractors are formed on the basis of the doubled invariant curve. Their fusion forms a non–homoclinic attractor of the torus–chaos type, which does not contain a fixed point.

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Self-reliance of attractors

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It is known that there is the locally residual set of C^3 diffeomorphisms with maximal attractors of positive Lebesgue measure [1]. We can improve this result without understanding the proof. So called self-reliance of attractors allows us to conclude that there is an open set of C^1-systems with thick attractors.

The idea of self-reliance is basically about standard topological tricks with G_δ sets, and will be clear after few examples.

This technique also can be used for the research of Milnor attractors of C^1-Anosov diffeomorphisms [2], and to establish connection between unstable Milnor attractors and so-called Takens Last Problem, introduced in [3].

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Small Perturbations of Smooth Skew Products and Sharkovsky's Theorem

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1. Let $I = I_1 \times I_2$ be a closed rectangle in the plane (I_1, I_2 are closed intervals of the straight line $\mathbf{R^1}$, $I_k = [a_k,\, b_k]$ for $k = 1,\, 2$). We consider a map $F : I \to I$ satisfying the equality

$$F(x,\, y) = (f(x) + \mu(x,\, y),\, g(x,\, y)) \quad \text{for any } (x,\, y) \in I. \tag{1}$$

We suppose that the map (1) is C^1-smooth on the rectangle I, and the map $f : I_1 \to I_1$ satisfies the following conditions:

(i_f) $f(\partial I_1) \subset \partial I_1$, where $\partial(\cdot)$ is the boundary of a set;

(ii_f) f is Ω-stable in the space of C^1-smooth maps of the interval I_1 into itself with the invariant boundary.

We suppose also that for a C^1-smooth function μ (of variables x and y) the following property holds:

(i_μ) the equalities $\mu(x,\, a_2) = \mu(x,\, b_2) = 0$ are correct for every $x \in I_1$; and the equalities $\mu(a_1,\, y) = \mu(b_1,\, y) = 0$ are correct for every $y \in I_2$.

Moreover, we consider functions $\mu = \mu(x,\, y)$ so small that the following inequality is valid:

(ii_μ) $||\mu||_{1,\,(1,\,1)} < \varepsilon$, where ε is found for an arbitrary $\delta > 0$ by the condition of f- Ω-stability in the space of C^1-smooth self-maps of the interval I_1 with the invariant boundary, and $||\cdot||_{1,\,(1,\,1)}$ is the standard C^1-norm of the linear normalized space of C^1-smooth maps of the rectangle I into the straight line $\mathbf{R^1}$ (that contains the interval I_1).

Let $C^1_\omega(I)$ be the space of C^1-smooth maps (1) such that f satisfies conditions (i_f) – (ii_f), and μ satisfies conditions (i_μ) – (ii_μ).

This talk is the presentation of results of the paper [1]. We prove here that the Sharkovsky's order is reserved for maps from the space $C^1_\omega(I)$. But the proof of this claim requires a large preliminary work. Therefore, first of all, we prove existence of the invariant set (under the map $F \in C^1_\omega(I)$) of continuous pairwise disjoint curvelinear fibers over the points of the nonwandering set $\Omega(f)$ of the map f. Then we rectify these fibers, deduce the map under consideration on the above set to the skew product of interval maps on the set $\Omega(f) \times I_2$, and apply Kloeden's result on the preservation of the Sharkovsky's order for continuous skew products of interval maps (see [2]).

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Stability by linear approximation of time scale dynamical systems

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We study systems on time scales that are generalizations of classical differential or difference equations and appear in numerical methods. We consider linear systems and their small nonlinear perturbations. In terms of time scales and of eigenvalues of matrices we formulate conditions, sufficient for stability/instability by linear approximation. We use techniques of central upper Lyapunov exponents (a common tool of the theory of linear ODEs) to study stability of solutions. We develope a new technique to demonstrate that methods of non-authonomous linear ODE theory may work for time-scale dynamics.

The talk is based on joint paper with Sergey Kryzhevich [1].

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Structures of a parabolic problem with spatial variable transformation

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A mixed boundary value problem for a nonlinear parabolic equation in a circle

$$v_t + v - D\triangle v + K\gamma \sin w\, Qv = K\gamma((\cos w(\cos Qv - 1) - \sin w(\sin Qv - Qv)),$$

is considered with Neumann conditions on the boundary for $r = r_1$

$$\frac{\partial v(r_1, \varphi, t)}{\partial r} = 0,$$

periodicity conditions

$$v(r, \varphi, t) = v(r, 2\pi + \varphi, t),$$

boundedness conditions at the origin

$$|v(0, \varphi, t)| \le c < \infty$$

and initial condition

$$v(r, \varphi, 0) = v_0(r, \varphi).$$

Lemma [1]. The linear operator L has eigenfunctions

$$X_{km}(r, \varphi) = \{J_k(\lambda^c_{km} r)\cos k\varphi, J_k(\lambda^s_{km} r)\sin k\varphi\},$$

which correspond to eigenvalues

$$\lambda^c_{km} = D\left(\frac{\mu_{km}}{r_1}\right)^2 + (-1)^k K\gamma \sin\omega + 1,$$

$$\lambda^s_{km} = D\left(\frac{\mu_{km}}{r_1}\right)^2 + (-1)^{k+1} K\gamma \sin\omega + 1, k = 0, 1, 2, \ldots, \; m = 1, 2, \ldots,$$

where $J_k(x)$ — Bessel function, μ_{km} — solutions of the equation

$$J'_k(\mu_{km}) = 0, \; k = 0, 1, 2, \ldots, \; m = 1, 2, \ldots.$$

To analyze the structure of the solution, depending on the parameter D, it is necessary to evaluate the eigenvalues.

Denote by $\Lambda = -K\gamma \sin w$. Choose $\Lambda = \Lambda(K, \gamma) < -1$.

At $D_1 = \frac{-1-\Lambda}{\left(\frac{\lambda^c_{11}}{r_1}\right)^2}$, λ^c_{11} can change sign when decreasing D. As a result of bifurcation, a pair of spatially inhomogeneous stationary solutions branches off from the zero solution.

Theorem [2]. There is $\delta_0 > 0$ such that if $0 < D - D_1 < \delta_0$, then the equation has two asymptotically stable solutions:

$$v^{\pm}(r, \varphi, D) \approx \pm\left(\frac{D - D_1}{c_1(D)}\right)^{1/2} J_1(\lambda^c_{11} r)\cos\varphi +$$

$$+\frac{1}{2!}\left(\frac{D-D_1}{c_1(D)}\right)\frac{\Lambda}{2}ctg\,\omega((\lambda_{10}^c-2\lambda_{11}^c)^{-1}+(\lambda_{12}^c-2\lambda_{11}^c)^{-1}\cos 2\varphi)J_1^2(\lambda_{11}^c r)\pm$$

$$\pm\frac{1}{3!}\left(\frac{D-D_1}{c_1(D)}\right)^{3/2}(\lambda_{13}^c-3\lambda_{11}^c)^{-1}\left(\frac{\Lambda}{4}-\frac{3}{4}\Lambda^2 ctg^2\,\omega(\lambda_{12}^c-2\lambda_{11}^c)^{-1}\right)\cdot$$

$$\cdot J_1^2(\lambda_{11}^c r)J_3(\lambda_{11}^c r)\cos 3\varphi,$$

where

$$c_1(D)=\left[\frac{\Lambda}{8}-\frac{1}{4}(\Lambda ctg\,\omega)^2\left((\lambda_{10}^c-2\lambda_{11}^c)^{-1}+\frac{1}{2}(\lambda_{12}^c-2\lambda_{11}^c)^{-1}\right)\right]J_1^2(\lambda_{11}^c r)<0.$$

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Studying of the spatial distribution of the chlorophyll "a" in the Bering Sea on the basis of satellite data

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The concentration of chlorophyll "a" is the one of the parameters allowing to estimate a condition of ecosystems of the ocean. Chlorophyll "a" is the main pigment of cages of phytoplankton providing photosynthesis process. The amount of photosynthetic primary production which is the speed of producing organic substance in the course of photosynthesis which defines the general overall bio-productivity of the ocean depends on quantity and intensity of functioning chlorophyll "a".

To observe the phytoplankton (more precisely, "chlorophyll-a") and its spatial distribution are developed special space sensors, scanners of color of the sea such as the SeaWiFS (Sea-View Wide Field-of-View Sensor) on the Seastar satellite and also the MERIS spectroradiometers (Medium Spectrometer with image resolution) on the Envisat and MODIS (moderate resolution spectrometer) satellites on the Aqua and Terra satellites.

The regularity of collection of data over the entire area of the World Ocean allows to distinguish features of dynamics of a chlorophyll "a" on various water areas, compare them, to reveal long-term tendencies of change. The results of satellite monitoring do not contain direct information about phytoplankton, but give an opportunity to judge his state on the basis of indicators of content of a chlorophyll in the upper water layer of the ocean. Monitoring of distribution of concentration of a chlorophyll has important practical value for fishery because phytoplankton is the food base of zooplankton and fishes.

For a research the region of the Bering Sea limited to coordinates 45° -75° NL, 160° EL-155° WL was chosen. The Bering Sea is rich in nutrients for phytoplankton, the is rather biologically various, certain areas of the sea are abundant of different types of fishes. The open area allowing to allowing to analyze the patterns of formation of the lower trophic levels of the marine ecosystem has been chosen.

From satellite data are used concentration of a chlorophyll, temperature and illumination on a surface. Data of May 2014 are processed. The size of the spatial cell (points) is 4 x 4 km, the time interval is 1 day.

The averaging of satellite characteristics on time and space (the surface of the sea) are constructed. If averaging over space has a small variability in time, the averaging on time is highly dynamic depending on the spatial coordinates. The results are compared to similar researches in the Okhotsk and Japanese seas.

Transition to estimates of a bio-productivity of the Bering Sea by satellite information with application of mathematical models of dynamics of plankton is supposed.

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Superradiant mode transition in a heterolaser via the formation of a self-consistent population-inversion grating

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A talk is devoted to an intriguing physical phenomenon observed in our numerical modeling of a superradiant lasing in a low-Q symmetric cavity [1, 2], namely, a spontaneous breaking of the mirror symmetry of counter-propagating waves which results in the asymmetric profiles of the field, polarization and population inversion of an active medium. The phenomenon is owing to the nonlinear population inversion grating which is generated self-consistently by the inhomogeneous counter-propagating waves under CW pumping. Such coherent dynamical effects take place in the case of very dense active medium and low-Q cavities when a photon (cavity) lifetime is much shorter than a polarization (optical dipole) lifetime of an active center.

Here, on the basis of numerical solution to the Maxwell-Bloch equations, we describe in detail the steady and dynamic spontaneous symmetry breaking of the structure of the field, polarization and population inversion of an active medium with almost homogeneously broadened spectral line placed into a symmetric combined distributed feedback (DFB) Fabry-Perot cavity. Under the conditions of this breaking in the steady-state or weakly modulated regimes of superradiant lasing, the spatial profiles of the counter-propagating waves become strongly asymmetric and differ essentially from the symmetric profiles of known so-called cold and hot modes, calculated at zero or quasistationary homogeneous population inversion, respectively.

It is shown that the asymmetric (with respect to the cavity center $\zeta = 0$) superradiant lasing is typical below or near the non-stationary lasing threshold. The symmetry breaking occurs during the long transient stage of moving to steady or slow self-modulated lasing and exists even without DFB. We discuss a range of the laser parameters where the phenomenon is present and suggest possible designs of the semiconductor heterolasers of this kind.

In general, in any non-stationary superradiance regime, a dynamic spontaneous symmetry breaking can take place for the profiles of mode fields and the consistent population inversion profile averaged over a long enough time interval containing several characteristic sets of pulses of all lasing modes. The appearing asymmetry can be metastable and the regions of the maximum inversion of the medium and the minimum intensity of the mode field can displace alternately to one side or another from the cavity center. Such spontaneous switchings of metastable laser states can cause temporal changes in the average emission intensity and in the correlation properties of laser pulses, these averages being considerably different for opposite ends of the laser. We attract attention to same mathematical problems of the theory of spontaneous symmetry breaking in both cases of the steady-state and non-stationary superradiant lasing.

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Symmetry breaking in a system of two coupled microbbuble contrast agents

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In this work we study a dynamical system that describes behavior of two different interacting microbbuble contrast agents. Contrast agents are micro-meter size gas-filled bubbles, which are encapsulated into a visco-elastic shell [1, 2]. Such bubbles can be used for various biomedical applications, for example, for enhancing ultrasound visualization of blood flow. It is known that contrast agents can demonstrate complex dynamics and its type is important for applications [1, 3].

The dynamics of two intercating bubbles is described by a non-autonomous sysem of four differential equations (or an equivalent autonomous system of five equations). If the equilibrium radii of both bubbles are the same then the system describing their dynamics is invariant with respect to the symmetry: $R_1 \leftrightarrow R_2, \dot{R}_1 \leftrightarrow \dot{R}_2$, where $R_1(t)$ and $R_2(t)$ denote the first and second bubbles' radii respectively and dot is the derivative with respect to time. This symmetry leads to the appearance of the three-dimensional invariant manifold $R_1 = R_2, \dot{R}_1 = \dot{R}_2$, the orbits lying in which can only be periodic or chaotic with one positive Lyapunov exponent. Solutions embedded in this manifold are characterized by completely in-phase (synchronous) oscillations of both bubbles. Some of these solutions can be asymptotically stable (attractive). Various synchronous (periodic, chaotic) and asynchronous (periodic, quasiperiodic, chaotic and hyperchaotic) states were studied recently in work [4].

The main aim of this talk is to study the influence of destruction of the synchronization manifold on various dynamical regimes in the system. We introduce a perturbation of the equilibrium radius of one of the bubbles which leads to the symmetry breaking. Since synchronous attractors are essentially defined by presence of the symmetry, it is natural to assume that they are in general more sensitive to the symmetry breaking. We show that multistable states consisting of synchronous and asynchronous attractors often transit to monostable states via crisis of a previously synchronous state. Asynchronous states, especially hyperchaotic ones, are in general more stable with respect to symmetry breaking perturbations. However we also demonstrate that in some cases symmetry breaking in a monostable synchronous state does not lead to qualitative changes in the dynamics. Further we consider different transition scenarios of symmetry breaking that can lead to both death of old multistabile states or to the birth of new multistable ones.

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The Baum-Connes conjecture localised at the unit element of a discrete group

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The Baum—Connes conjecture, a central topic in noncommutative topology, relates two natural objects associated with a discrete group. The first one is topological in nature and involves a classifying space for proper actions, the second one is analytical and involves the K-theory of the reduced group C^*-algebra. One of the main features of the Baum—Connes conjecture is that it implies the Novikov conjecture about the homotopy invariance of higher signatures of oriented manifolds.

In this talk we first give an introduction to the topic, then we present a version of the conjecture that we constructed in collaboration with S. Azzali and G. Skandalis. It is called localised at the unit element of a discrete group. This localised conjecture is defined using von Neumann algebras and has some interesting properties especially in the relation with the Novikov conjecture.

The Depth of the Centre for Continuous Maps on Dendrites

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By *continuum* we mean a compact connected metric space. *Dendrite* is a locally connected continuum without subsets homeomorphic to the circle. Dendrite with a finite set of end points is called *a finite tree*. *A graph* is a continuum if it can be written as the finite union of arcs such that every two of them meet at their end points.

The depth of the centre for continuous maps on one-dimensional continua is studied in [1] – [8]. The depth of the centre of a continuous map on a closed interval and the circle is at most 2 (see, e.g., [1], [2]). In [6] J. Mai, T. Sun proved that the depth of the centre of a continuous map on a finite tree and a graph is at most 2. In [7] H. Kato showed that for any countable ordinal number λ there are a dendrite X and a continuous map $f : X \to X$ such that the depth of the centre of f is λ. In [8] T. Sun and H. Xi proved that the depth of a centre for a continuous map $f : X \to X$ on dendrite X with finite branch points is at most 3. Moreover they constructed a dendrite X with finite branch points and a continuous map $f : X \to X$ such that the depth of the centre of f is 3.

Let X be a dendrite, $f : X \to X$ be a continuous map.

Denote by $E^{(0)}(X)$ the set of end points of a dendrite X. For any ordinal number $\lambda \geq 1$ we define $E^{(\lambda)}(X)$ as follows:
if $\lambda = \alpha + 1$, then we denote by $E^{(\lambda)}(X)$ the set of limit points of the set $E^{(\alpha)}(X)$;
if λ is a limit ordinal number, then we set $E^{(\lambda)}(X) = \bigcap\limits_{\alpha<\lambda} E^{(\alpha)}(X)$.
There is a countable ordinal number γ such that $E^{(\gamma)}(X) = E^{(\gamma+1)}(X)$. The minimal such γ is called *the rank of a dendrite X*.

In the report the relationship between the depth of the centre of a continuous map on a dendrite and a rank of a dendrite is studied.

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The flux of magnetic helicity for the mean magnetic field equation

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The mean magnetic field equation describes the process of generating a magnetic field on a larger scale due to turbulent pulsations on a smaller scale, see [R] for the basic definitions. As a rule, the equation of the magnetic helicity flux is studied on a larger scale, since the helicity for the mean field is well defined. One may assume that the large scale velocity field admits a fast transport of the helicity density.

In [S-S] the basic equation, described the flux of the magnetic helisity, is applied for cosmological magnetic fields. In this framework a first-order approximation of the total flux equation, the approximative equation for a flux of magnetic helicity, is introduced in [S-S-S]. This equation is not complete, an extra term in this equation, using local formula for quadratic helisity [A-C-S], is proposed in [A-DJCL-S]. Our goal is to construct a hierarchical (infinite-dimensional) equation for the total flux of magnetic helicity, which contains fluxes of momenta. To solve this problem I will present calculations based on the preprint [A-V].

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The Generalized Integral Minkowski Inequality and L_p -norms of Some Special Functions

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In this report we prove two theorems concerning L_p -norms of some functions $f(x)$:

$$\|f(x)\|_{L_p([A,B])} \equiv \left[\int_A^B |f(x)|^p\,dx\right]^{\frac{1}{p}}, \qquad p > 1\,,$$

namely,

Theorem 1. Let $I_0(z)$ is the modified Bessel function of the first kind and zero order [1] then the following inequality is true:

$$\|I_0(\cos x)\|_{L_p([0,2\pi])} \leq \int_0^{2\pi} I_0^{\frac{1}{p}}(p\cos x)\,dx\,.$$

Theorem 2. Let $\beta > \alpha > 0$ and

$$M(\alpha,\beta,x) = \frac{1}{B(\alpha,\beta-\alpha)}\int_0^1 e^{xy}y^{\alpha-1}(1-y)^{\beta-\alpha-1}\,dy$$

is a confluent hypergeometric function [1] where

$$B(\xi,\eta) = \int_0^1 x^{\xi-1}(1-x)^{\eta-1}\,dx$$

is the beta function [1] then the following inequality is true:

$$\|M(\alpha,\beta,x)\|_{L_p([-1,1])} \leq \int_0^1 \frac{x^{\alpha-1}(1-x)^{\beta-\alpha-1}}{B(\alpha,\beta-\alpha)}\left(2\,\frac{\sinh p\,x}{p\,x}\right)^{\frac{1}{p}} dx$$

The corner stone for proof of these theorems is the generalized integral Minkowski inequality for arbitrary continuous function $f(x,y)$ [2]:

$$\left\{\int_a^b\left[\int_c^d |f(x,y)|\,dy\right]^p dx\right\}^{\frac{1}{p}} \leq \int_c^d\left[\int_a^b |f(x,y)|^p\,dx\right]^{\frac{1}{p}} dy\,.$$

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Three-dimensional Poincare cross-sections in the model of oscillatory interaction of different-scaled structures in solids

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As is well known the properties of solids are the result of the joint influence of structures of various scales. The report presents the model of the oscillatory interaction of different-scaled structures in solids.

In the model exchange of energy between structural levels is described as evolution of the dynamic system consisting of subsystems that interact defined by laws:

$$\Phi(x,y,z,w)=\begin{cases} x_{n+1}=x_n-k_{xy}px_n^2+k_{yx}qy_n^2+x_{in} \\ y_{n+1}=y_n+k_{xy}px_n^2-(k_{yx}+k_{yz})qy_n^2+k_{zy}rz_n^2 \\ z_{n+1}=z_n+k_{yz}qy_n^2-(k_{zy}+k_{zw})rz_n^2+k_{wz}sw_n^2 \\ w_{n+1}=w_n+k_{zw}rz_n^2-(k_{wz}+k_{out})sw_n^2 \end{cases},$$

where x, y, z and w are dynamic variables characterizing the energy of scale structural levels, and k_{ij}, p, q, r and s are distributing coefficients, that have a clear interpretation depending on a physical nature of the system.

Solution of the system was obtained numerically. Stability of phase trajectories was computed by methods of Lagrange and Lyapunov; it was shown that the region of existence of stable trajectories is limited.

It obvious, evolution of this dynamical system is described by various types of attractors in the four-dimensional phase space. For determined this types in the computer realization of the model, visualization of attractors is provided. For this, the trajectory of the system in phase space was dissecting by a three-dimensional analogue of the Poincare cross-sections.

Trajectories obtained in that cross-sections, as obviously, have dimension on one less then initial attractors, and can be visualize (fig.1). It is by these visualizations in the model is determined character of the evolution of the system for different values of the control parameters.

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Transcendental First Integrals of Dynamical Systems

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In this paper, we examine the existence of transcendental first integrals for some classes of systems with symmetries. We obtain sufficient conditions of existence of first integrals of second-order nonautonomous homogeneous systems that are transcendental functions (in the sense of the theory of elementary functions and in the sense of complex analysis) expressed as finite combinations of elementary functions.

The results of the present paper develop previous studies, including some applied problems of the rigid-body dynamics (see [1, 2, 3], where complete lists of transcendental first integrals expressed as finite combinations of elementary functions were obtained). Later, this fact allowed one to perform an analysis of all phase trajectories and to indicate rough properties that are preserved for systems of a more general form. The complete integrability of such systems is associated with hidden symmetries.

As is well known, the concept of integrability, generally speaking, is quite vague. It is necessary to consider the sense in which it is meant (i.e., a certain criterion that allows one to conclude that trajectories of a dynamical system have an especially "attractive and simple structure"), and in which class of functions first integrals are taken, and so on (see also [4, 5]).

In this paper, we accept an approach in which the class of first integrals consists of elementary transcendental functions. Here the transcendence is meant not only in the sense of the elementary functions (e.g., trigonometric) but in the sense of complex analysis, i.e., as functions of a complex variable possessing essential singular points. In this case these functions must be formally continued in the complex domain.

Of course, in the general case, the construction of any integration theory of such nonconservative systems (even of low dimension) is quite difficult. But in some cases where the systems studied possess additional symmetries, one can find first integrals as finite combinations of elementary functions.

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Unfolding a Bykov attractor: from an attracting torus to strange attractors

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We present a comprehensive mechanism for the emergence of strange attractors in a two-parametric family of differential equations acting on a three-dimensional sphere. When both parameters are zero, its flow exhibits an attracting heteroclinic network (Bykov network) made by two

1-dimensional and one 2-dimensional separatrices between two hyperbolic saddles-foci with different Morse indices. After slightly increasing both parameters, while keeping the one-dimensional connections unaltered, we focus our attention in the case where the two-dimensional invariant manifolds of the equilibria **do not intersect**.

Under some conditions on the parameters and on the eigenvalues of the linearisation of the vector field at the saddle-foci, we prove the existence of many complicated dynamical objects, ranging from an attracting quasi-periodic torus, Newhouse sinks to Hénon-like strange attractors, as a consequence of the Torus Bifurcation Theory (developed by Afraimovich and Shilnikov).

Under generic and checkable hypothesis, we conclude that any analytic unfolding of a Hopf-zero singularity (within the appropriate class) contains strange attractors.

Riemannian manifolds and laminations

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A foliated space is a topological space endowed with a partition into connected manifolds, called the *leaves* of the foliated space, that locally looks like a product space in a coherent way. To each such space, we can associate a dynamical system, which can be realized in different ways.

In [3], the author together with Álvarez López and Candel studied the foliated properties of the *smooth Gromov space*, which is the subspace of the Gromov space of pointed proper metric spaces that only consists of pointed, complete and connected Riemannian n-manifolds. Subsequently, this was used to give an answer to a modified version of the *realization problem* in foliation theory[1, 2]. The precise statement of the main results are the following.

Theorem 3. *The smooth Gromov space is a Polish space, and the subspace consisting of locally non-periodic manifolds is a foliated space.*

Theorem 4. *Every connected, complete Riemannian manifold of bounded geometry can be realized in a compact foliated space without holonomy. The foliated space can be chosen so that it is a matchbox manifold.*

An interesting characteristic of the methods used to prove this results is that one could try to modify them in order to study the realization of manifolds in foliated spaces satisfying further dynamical properties. In this talk we will present our previous research as well as the current research regarding the realization of manifolds in foliated spaces with a dense set of compact leaves.

The talk is based on ongoing research with Prof. Álvarez López and Nozawa.

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