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Conference

Book of abstracts

"Topological methods in dynamics and related topics. Shilnikov workshop"

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Topological Methods in Dynamics and Related Topics. Shilnikov Workshop.

Book of Abstracts

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CONTENTS

To the 85th anniversary of L.P. Shilnikov

This year, we celebrate the 85th anniversary of the birth of the outstanding Russian mathematician L.P. Shilnikov (1934-2011), and our joint conference "Topological Methods in Dynamics and Related Topics. Shilnikov WorkShop" is also devoted to this memorable date. It has been 8 years since he passed away. However, the ideas of Shilnikov and his school still have a very strong influence on his disciples and colleagues.Probably, this can explain the fact that over these years they obtained a large number of significant results in the theory of dynamical systems, some of which can be safely attributed as important discoveries. Some of these achievements will be also discussed at the conference.

Leonid Pavlovich Shilnikov is one of the founders of the mathematical theory of dynamical chaos and the theory of global bifurcations of multidimensional systems. His works are widely known in the world, and his results recognized as classical are included in all modern textbooks on dynamical systems. He worked all his life in the Gorky State University (now Lobachevsky University of Nizhny Novgorod). He was a brilliant follower of the famous Andronov school on nonlinear dynamics and, in fact, he created his own direction which is known nowadays as the Shilnikov school on dynamical systems.

Leonid Pavlovich Shilnikov
(17.12.1934–26.12.2011)

The scientific results by L.P. Shilnikov impress with their breadth and depth. Already in his first works (in the 60's) he founded the base of the theory of global bifurcations of multidimensional dynamical systems in quite nontrivial way generalizing the classical two-dimensional theory by Andronov and Leontovich. As one of the most significant and impressive results of Shilnikov, one can remark his discovery of the complex structure of orbits near a saddle-focus homoclinic loop. This discovery actually marked the formation of the theory of spiral chaos, one of the famous kinds of dynamical chaos. Many of deep and pioneering results were obtained by L.P.Shilnikov in the theory of homoclinic chaos including the complete solution of the Poincaré-Birkhoff problem on the structure of the set of orbits near a transverse homoclinic orbit to a saddle periodic orbit and to a saddle invariant torus. Shilnikov regarded this result as a very important one, and he always stressed that the existence of a transverse Poincaré homoclinic orbit is the universal criterion of Chaos. Also it is worth mentioning his discovery of the phenomenon of Ω-explosion (while studying bifurcations of a system containing a saddle-saddle equilibrium with several homoclinic loops), as well as a series of fundamental results on bifurcations of homoclinic tangencies obtained in collaboration with his students N. Gavrilov, S. Gonchenko, and D. Turaev. The principally important place in the scientific heritage of L.P. Shilnikov is occupied by his results on the structure of the Lorenz attractor (obtained together with his students V. Afraimovich and V. Bykov) as well as his results (obtained along with his students V. Afraimovich, A. Morozov and V. Lukyanov) which, in fact, form a solid foundation of the modern mathematical theory of synchronization. It is necessary to note also the fundamental contribution of Shilnikov to such branches of the theory of dynamical systems as the theory of torus-chaos, the theory of pseudohyperbolic attractors, the theory of bifurcations of type "blue sky catastrophe" and many others.

Let us point out that L.P. Shilnikov was never affected by “bourbakism”: the statements of his theorems can always be easily verified, and all the obtained results provide strong mathematical tools which are widely used by “proper” specialists from various fields of natural science.

What I did in dynamics

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I started in mathematics in 1969, if one means an official presentation as a speaker on the V International Conference in Nonlinear Oscillations in Kiev [1]. That concerned a generalization onto nonautonomous almost periodic systems of a result by L.S. Pontryagin [2] on the dimension of the closure of an almost periodic orbit of a vector field given on a n-dimensional smooth manifold. That talk and the paper in Proceedings was based on my diploma paper made under the tutorship by L.P. Shilnikov when graduating the Gorky State University in 1968. It was very funny that a rather close in topic talk was presented by famous mathematician M.Cartwright at the same conference.

Professor L.M. Lerman

After graduating Gorky State University I started working in the Institute of Applied Mathematics and Cybernetics in the Department of Differential Equations headed by Prof. E.A. Leontovich-Andronova. The topic of nonautonomous dynamics was continued in my PhD thesis made under the tutorship by L.P.Shilnikov during 1971-1975 [3]. There were three main type of results in this thesis.

First, the definition of structurally stable nonautonomous vector fields given on a smooth closed manifold was introduced using uniform homeomorphisms in the extended phase space as an equivalency relation. On this base a class of structurally stable vector fields on a closed 2-dimensional surface was constructed. Here the notion of an exponential dichotomy for integral curves of the nonautonomous vector field play an essential role. It was published a paper [4], where, however, an approach to the construction of the structurally stable nonautonomous vector fields was different than in [3]. The point is that working under the thesis I understood that the approach of [4] was not flexible and then I changed it in the thesis. Also the inequalities of the Morse-Smale type were found in the thesis connecting a topology of the ambient manifold and the possible collection of integral curves with various types of exponential dichotomy. This was similar to the well known result by Smale [5]. These results unfortunately were not published up to now by my own personal reasons. Only recently, a one-dimensional version of this theorem was proved for a vector field on the circle [6]. This theorem was also applied there to the case of almost periodic vector field on the circle.

The second cycle of problems in the thesis was the generalization onto the nonautonomous case the theory of Poincaré homoclinic orbits and the description of the structure of nearby integral curves. These results were published in [7]. In particular, this situation arises when a one degree of freedom Hamiltonian system with a homoclinic orbit to a saddle is perturbed by an almost periodic force. This case was considered later in [9] and [10]. This result is closely connected with the Shilnikov work on homoclinic torus dynamics [11] and was initiated by him for my thesis.

The third topic of the thesis was a theorem connecting a structurally stable nonautonomous vector field on a closed surface M with some autonomous Morse-Smale vector field on the cylinder $M \times I$. This result was also not published, its 1-dimensional variant is in [6].

After defending PhD thesis in 1975 I proceeded some time working in nonautonomous dynamics. I became interested with "real life" applications and problems related with genuine nonautonomous dynamics and addressed to several physicists trying to find such problems in a hope to apply results and tools developed in the thesis. By "genuine" I mean nonautonomous vector fields without almost periodic dependence on time, when this vector field cannot be extended by means of the Bebutov-like construction to the autonomous vector field on some smooth closed manifold. My attempts were worthless and I got cold to this topic. Nevertheless, I introduced then the class of vertically hyperbolic skew products over the torus shifts and hypothesized they should be structurally stable under vertical perturbations. This was the topic of my talk at the Conference on Differential Equations in Kishinev in 1979. Similar results more deeply were elaborated by Bronshtein and his group [8].

In 1980 in Gorky the Conference "Stochasticity Seminar" was organized by the Institute of Applied Physics of USSR Academy of Sciences. At this conference our close friend Valya Afraimovich introduced us with Yan Umanskiy to the Moscow physicist V.M. Eleonsky who became later my coauthor for many years and a friend. Also we made the acquaintance with his collaborator and former PhD student N.E. Kulagin who also became my friend and co-author and we work together till now. Eleonsky formulated a problem that was of great interest of him that time. He made several papers on that topic with his co-authors from the F.V.Lukin Institute of Physical Problems in Zelenograd [12, 13]. The problem was about "phase portraits" of integrable Hamiltonian systems.

It is now well known that two breakthroughs were made in mathematics that time. The first was about systems with chaotic dynamics. This was started in 1961 with the famous talk by S.Smale at the I International Conference on Nonlinear Oscillations in Kiev in 1961 on that was named later as "Smale horseshoe". In the USSR this topic soon afterwards became very popular and developed deeply after brilliant works by D.V.Anosov, Ya.G. Sinai, L.P. Shilnikov, A.B. Katok, M.I. Brin, Ya.B. Pesin, V.S. Afraimovich and many others. This "hyperbolic revolution" (the term introduced by D.V. Anosov) was prepared by the previous great acievements in Dynamical Systems in Russia and the USSR due to works by A.A. Lyapunov, A.A. Andronov, L.S. Pontryagin, N.M. Krylov, N.N. Bogolyubov, E.A. Leontovich-Andronova, A.G. Maier, V.V. Nemytsky, and many others. This was very vividly described by D.V. Anosov [14].

The second breakthrough occurred at approximately the same time in the field that later was named as Integrable Hamiltonian PDEs. It was initiated by the famous work [15] where the inverse scattering method was elaborated. Though PDEs to which this methods was applied (KdV, Nonlinear Schrödinger, sin-Gordon, Landau-Lifshits) were known much earlier (for instance, Boussinesq, 1877, Korteweg-de Vries, 1895), the method and its development using algebro-geometric approach allowed to prove the integrability of these equations in one-dimensional spatial media. Studying integrable PDEs revived the interest to finite-dimensional Hamiltonian systems which appear naturally at studying special solutions like traveling waves, stationary solutions, etc., as well as in investigations of hierarchies of integrable equations, etc. In this way, many old integrable models were rediscovered and revived, like integrable models in mechanics, hydrodynamics, etc.

The interest of V.M. Eleonsky was just related with the traveling wave equations generated by the Landau-Lifshits equations. The problem concerns the following more general question: we know that some Hamiltonian system is integrable, can we understand its orbit behavior? This question was of interest for us with Yan Umanskiy, my old friend and fellow student, since problems of similar type were popular in our group in the Institute of Applied Mathematics and Cybernetics of the Gorky State University were we worked that time (that is not surprising if one remembers the history of our department headed by E.A. Leontovich-Andronova).

We became work with Umanskiy in this direction applying the method that seemed to us very perspective. Namely, we studied the structure of a foliation given by orbits of the related Poisson action. This action is generated by commuting integrals of the integrable Hamiltonian system. We started with the local study in a neighborhood of a singular point but very soon we realized that it is more correctly to study the extended neighborhood of the singular point. The point is that the integrability forces action orbits leaving the neighborhood of the singular point to return there, if the levels are compact. We made several papers in this topic [16, 17, 18].

Very soon the known Moscow mathematician A.T. Fomenko became interested with similar problems [19]. Being a good geometer, he started from the observation that, for an integrable Hamiltonian system in two degrees of freedom, any compact connected nondegenerate level of the Hamiltonian carries the foliation defined by the second integral. Singularities of this foliation on the level form smooth curves as a rule (sometimes they can be two-dimensional but this case is rare). He assumed these curves to be nondegenerate in the transverse directions, what led him to the notion of Bottian integral and he studied the foliation of the level on this base. The most spectacular results in this direction were later obtained by Nguyen Tien Zung [22, 23]. In fact, what we did with Umanskiy and the work by Fomenko and his school [20, 21] went in parallel, we always referred on their papers though this was not reciprocal till some time, to our regret...

Finally, we were lucky to construct complete invariants of topological and iso-energetical equivalence of integrable Hamiltonian systems with two degrees of freedom in their extended neighborhoods of their singular point. These results were published in [24, 25, 26] and later were gathered in the monograph [27] written by the proposition by V.I. Arnold.

My answer to the questions set up by Eleonsky was the paper [28] devoted to the description of the phase portrait of the traveling wave solutions for the Landau-Lifshits equation. It was earlier proved by Veselov [29] that this system is integrable in theta functions Prima. However, the complete orbit behavior was hard to extract from this result. This system is equivalent to the mechanical system describing the motion of a particle on the sphere in the linear potential. The related bifurcation diagrams were constructed by Kharlamov [30]. A perturbation of this system when the Hamiltonian of the Landau-Lifshits equation becomes a quartic polynomial instead of quadratic one was studied in [31], this system turned out to be non-integrable at some region of parameters with the existence of multi-hump solitons (multi-round homoclinic orbits). Here I applied a formula derived in our paper [32] being an extension of the Melnikov formula onto the autonomous Hamiltonian case in two degrees of freedom. Its multidimensional variant was given in [33].

In 1994 Ya.L. Umanskiy left for the USA. After his leaving I made only one paper dedicated to integrable Hamiltonian systems with three degrees of freedom [34]. In that paper I studied extended neighborhoods of singular points in such systems and bifurcations which arise unavoidably in these systems. The topic of bifurcations in integrable systems turned out very interesting and today I proceed studying bifurcations in integrable Hamiltonian systems.

During the work on integrable Hamiltonian systems I naturally became interested with the structure of nonintegrable Hamiltonian systems since the study of systems with complicated dynamics was the main stream of the research in our Department. The first kick for me was the result by R.L. Devaney [35] who carried over the result by Shilnikov [36] about a homoclinic loop of a saddle-focus to the Hamiltonian case. The problem laid initially in the fact that formally Shilnikov's results cannot formally be applied to a Hamiltonian 4-dimensional case, since Shilnikov required for the saddle-focus some inequality to hold (nonzero saddle value) that is always not fulfilled for a Hamiltonian case. So, one needs to understand how Shilnikov's results can be modified. Devaney used the local normal form of Moser valid for a saddle-focus (a quadruple of complex eigenvalues with nonzero real parts) and was able to distinguish a hyperbolic subsystem in a neighborhood of a transverse homoclinic orbit to a saddle-focus which is described by a suspension over Bernoulli scheme with two symbols.

My contribution to this problem was twofold [37, 38]. It was described the whole invariant subset in a neighborhood of a transverse homoclinic orbit on the singular level of the Hamiltonian H by means of the symbolic system with countably many states (see also, [39]). Also I was able to prove that if one varies the value of the Hamiltonian, when approaching zero, the system countably many times passes through bifurcations of which the first is the appearance of a parabolic point for the Poincaré map on some strip in a neighborhood of a homoclinic point, creation of an elliptic point and a saddle, doubling of the elliptic point, doubling cascade, etc. with the formation of new pair of states in the related symbolic system.

After that I studied an orbit structure and bifurcations when varying values of H for a Hamiltonian system that has on some level of a Hamiltonain two saddle-foci with two transverse heteroclinic orbits to them forming a heteroclinic connection [40]. This problem, besides its own interest, was induced by my new interest to problems of PDEs. [1]

One more problem, I solved that time, was the study of the orbit structure for a Hamiltonian system with two degrees of freedom near a homoclinic orbit to a non-hyperbolic equilibrium of the saddle-center type. Such an equilibrium is structurally stable under Hamiltonian perturbations but, in contrast to the case of homoclinic orbits to saddle-focus or saddle, existence of a homoclinic orbit to such equilibrium is a degeneration of codimension 2, though it can be of codimension 1 within the class of reversible Hamiltonian systems that is frequent in applications. The main point of interest for me was to find conditions when the existence of such a loop would lead to the complicated (chaotic) behavior of orbits. By that time I knew one paper by Conley [43], who considered the structure near a homoclinic orbit to a saddle-center in an analytic Hamiltonian system with 2 degrees of freedom, but, in my opinion, no essential results were obtained in that paper.

In [44] I found a quite useful for applications sufficient condition in order all Lyapunov small periodic orbits filling the center manifold near a saddle-center have four transverse homoclinic orbits on their corresponding levels of the Hamiltonian. This naturally implies the system be nonintegrable and chaotic [44]. That paper received many references (96) and has many followers, among them are A.Mielke, P.Holmes, C.Grotta Ragazzo, A.Celletti, K.Yagasaki, others. This result was extended in papers with my PhD student O.Koltsova [45, 46]. There we reformulated the sufficient conditions in the form of the scattering problem for a system linearized on the homoclinic orbit, studied bifurcations, and extended this to the case of n degrees of freedom for a Hamiltonian system with 1-elliptic equilibrium (having one pair of simple pure imaginary eigenvalues and other with nonzero real parts) and a homoclinic orbit to it. In the latter problem the Moser normal form near the equilibrium was absent and we developed the method of the scattering problem for the linearized system at the homoclinic orbit. The method allowed us to give the condition of the existence of four transverse homoclinic orbits for all Lyapunov periodic orbits on the center manifold. This gave the non-integrability criterion for the multi-dimensional case under consideration.

Results on nonintegrable Hamiltonian systems, a part of results on integrable dynamics, along with the investigation of some applied problems (Landau-Lifshits equations, stationary Swift-Hohenberg equation and nonlocal sin-Gordon equation) made up the content of my Habilitation thesis (Doctor of Science) defended in 1999 in the Nizhny Novgorod University. Just after the defence I left for Berlin to work in the group by B. Fiedler in the Free University of Berlin. I worked

[1] The orbit structure of a 2 DOF Hamiltonian system near a bunch of transverse homoclinic orbits to a saddle was described by Turaev and Shilnikov [41, 42].

there 1 year, after that I returned to Nizhny Novgorod and realized that I should change the job, since the salary in the Institute of Applied Mathematics, where I worked since 1968, was that time so low that it was impossible to live. I became a Professor at the Nizhny Novgorod University, in the Department of Differential Equations and Calculus. I work in the University till now. One of my permanent lecture courses were ordinary differential equations for students of the second year. The lecturing led me to the necessity to publish my lectures since I regarded that the stuff presented was more agreed to my tastes. It was done in 2016 [48].

Another application of these ideas was a joint paper [49] on the existence of homoclinic orbits to invariant low dimensional KAM tori on the center manifold for a Hamiltonian system being a perturbation of an integrable 3 DOF system with a homoclinic orbit to a center-center-saddle equilibrium.

Later the methods and results on homoclinic orbits to nonhyperbolic equilibria were also extended for 3 degrees of freedom Hamiltonian systems with homoclinic orbits to a 1-elliptic periodic orbit. It was the paper with my other PhD student A. Markova [50]. This is equivalent to the study of a symplectic diffeomorphism on a 4-dimensional symplectic manifold such that the diffeomorphism has a 1-elliptic fixed point (two eigenvalues on the unit circle and reals out of it) and a homoclinic orbit to the fixed point.

One more line of my research was the study of slow-fast Hamiltonian systems. This was also inspired by my collaboration with Eleonsky and his group [51]. In a model, being a nonlocal generalization of the sin-Gordon equation, they found a situation, when the system for traveling waves being a Hamiltonian in 2 degrees of freedom with a saddle-center, has homoclinic orbits to it (they correspond to soliton-like solutions of the model). At some limiting case this system becomes singularly perturbed (slow-fast) such that its slow subsystem (being 1 DOF Hamiltonian) has a saddle with a homoclinic orbit. A question then arose, whether the full system has homoclinic orbits to the corresponding saddle-center. That time I worked at the Free University of Berlin in the group by B. Fieldler where V. Gelfreich, former PhD student by Prof. V.F. Lazutkin, had a postdoc position (today he is a Professor at the Warwick University, UK). I told him about this problem and we discussed it a lot. The result of our discussions were several papers [52, 53, 54]. In these papers we discovered that for an analytic slow-fast Hamiltonian system near its slow manifold, if the related fast system is fast rotating, the system can be reduced up to exponentially small error to a more simple system. The Hamiltonian of this simplified system is split into fast rotation and the second function to which the fast variables (x, y) enter only in the combination $I = x^2 + y^2$. Such form of the Hamiltonian gives integrability up to exponentially small error, if the system has two degrees of freedom (one slow and one fast). One more result was published in [55] where we found approximate formula for splitting separatrices of the saddle-center in an analytic slow-fast Hamiltonian system.

In this way I got interested with a general geometric framework in which slow fast Hamiltonian systems can be defined coordinate-freely. We were also lucky to demonstrate mathematically rigorously the appearance of Painlevé equations at the description of orbits of a slow fast system passing near a disruption point on a slow manifold [56]. Though it was known by the date in concrete systems, the invariant description seemed us be useful.

As I mentioned, the collaboration with the group of Eleonsky inspired my interest to the study of solutions to PDEs. One direction of this research was to understand how rich can be a set of patterns for nonlinear elliptic equations that were obtained either as stationary solutions of evolutionary PDEs or some other ways. My role in these research was to find mathematical backgrounds to substantiate the simulations made by the team. In this way, studying solutions to nonlinear elliptic equations of the type of localized wall with a periodic modulation, I came to the method of a center manifold for a formal evolution equation derived from the elliptic one. Such a setting usually leads to an incorrect initial problem but it turned out that if this center submanifold is of a finite dimension and some conditions for nonlinearities hold then the restriction of the equation to such class of solutions give true solutions. Later, digging the literature I discovered that this idea came to brains of K. Kirschgässner earlier [57] and it was strictly realized by his pupil A.Mielke [58]. Nevertheless, this gave me a direction of thinking. We made several papers in this direction, see, for instance, [59, 60].

The main interest in our group around Shilnikov was understanding the orbit behavior in systems with complicated dynamics. When I became interested with PDEs, Shilnikov started the discussion on a possibility to extend his results on Poincaré homoclinic orbits onto infinite dimensional systems. The paper [61] was an answer to these questions. In contrast to the case of a finite dimensional diffeomorphism we did not assume the existence of an inverse map, since for PDEs, in particular, of the parabolic type, it is not naturally. Nevertheless, we obtained the description of the invariant subset near a homoclinic orbit.

Another direction of research arose after my discussions with the well known physicist M. Rabinovich concerning a structure of the Swift-Hohenberg equation which he studied with his collaborators [62]. For me it was very exciting that the related equation for its stationary solutions in one-dimensional geometry on $\mathbb{R}$ gave an ordinary differential equation which is reducible to a Hamiltonian system with 2 degrees of freedom. This was more funny since the evolutionary SH equation itself is of parabolic type and his temporal solutions decay to stationary ones (for some boundary conditions). Since I had rather big experience in the study of such systems, I tried to understand some unclear physicist' reasons to substantiate bifurcation like appearance of localized solutions, Eckhaus instability [63] of periodic patterns, etc. I drew to this problem my PhD student Lev Glebsky (now a Professor at the University of San Luis Potoci, Mexico) and we realized that all these features of the equation can be explained on the base of the Hamiltonian Hopf Bifurcation and its generalization [64]. Exploiting the additional reversibility of the system [65], we were able to show existence of localized solution, multi-round localized solutions, bifurcations of periodic patterns, the threshold character of appearance of localized solutions, etc. Temporal stability of localized and periodic solutions in the more general case of spatially reversible PDEs was studied in our paper [66].

Since the Swift-Hohenberg equation describes the patterns not only in 1-dimensional setting but in 2D, 3D geometry, it was very natural to try to find such patterns using mathematical tool. We made several papers [67, 68, 69] in this direction starting since the search for radially symmetric patterns where methods of finite-dimensional systems can be else used. Combining methods of nonautonomous dynamics, local description near equilibria at infinity and numerical methods we found several types of radial patterns in the Swift-Hohenberg equation on the plane. Later these results were extended and stood on a rigorous mathematical foundation by Sandstede with coauthors [70].

Several years ago being in Suzdal at the International Conference on Differential Equations and Dynamical Systems I discussed these problems with A.I. Nazarov (Saint Petersburg University and SP branch of the Steklov Mathematical Institute). He told me that they are able to prove existence of majority of patterns known by the date using purely variational methods that are based on concentration arguments and symmetry considerations, at least for some nonlinear elliptic equations relevant for applications. We agreed to collaborate and try to make something in this direction. As a result, the paper [71] has appeared. We hope proceeding this collaboration to catch other interesting equations.

Recently we, disciples and colleagues of L.P. Shilnikov (1934-2011), gathered and commented the main papers by L.P. Shilnikov which made his name famous worldwide. The created book is our great respect and gratitude to our teacher and friend [72].

Many interesting problems stay unsolved and much work has to be done to understand some of them...

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To the 75th anniversary of A.D. Morozov

In the theory of dynamical systems, there is a number of "ever green" topics which have been in focus for centuries. These include, first of all, the theory of conservative (Hamiltonian) and close-to Hamiltonian systems. Such systems emerged almost immediately after the differential equations and Newton's mechanics came into existence (17th century). For a long period of time, such systems were generated almost solely by celestial mechanics which essentially was always a solid theoretical basis and a means of "justification" of the interest to conservative systems. Presently, such systems are omnipresent in applications and there is no need to list the problems, where these are encountered. By now, the theory of conservative (more precisely, Hamiltonian) systems evolved into a huge chapter of the theory of dynamical systems including as its integral part the theory of systems close to integrable (KAM theory) together with the theory of systems with complex behavior.

Professor A.D. Morozov

However, by early 60s, the theory of quasi-conservative systems, i.e., of systems close to Hamiltonian yet non-Hamiltonian, was only emerging. Theory of averaging by Krylov and Bofolyubov could give some results concerning the existence of quasi-periodical solutions in perturbed systems but it could not provide any kind of a picture of global system behavior, even under the assumption of smaller dimension or for restrictions on some areas of the phase space. Generally, the question is formulated as follows: what is the dynamics of quasi-conservative systems?

By essence, almost everything is known on dynamics of such systems only in case of systems with one degree of freedom, i.e. systems of the form

$$\ddot{x} = f(x), \quad x \in R$$

As V.I. Arnold wrote [1]: "for a qualitative study of such an equation, one view at the graph of its potential energy is quite enough." However, in the case of the so-called systems with 1.5 degrees of freedom, e.g. when the function $f = f(x,t)$ is periodic in time, everything becomes immeasurably more complicated. Here, instead of the differential equation, it needs already to consider the Poincaré map, i.e. a planar map that is constructed by orbits of the corresponding non-autonomous system for the period. There are no universal methods for studying dynamics of such systems (except, perhaps, certain numerical methods). Moreover, even to now, it is not absolutely clear how to study this problem in the case of a complex (stochastic) dynamics observed at numerics. Nevertheless, for periodically perturbed systems of form $\ddot{x} = f(x) + \varepsilon g(x,t)$ or $\ddot{x} = f(x) + \varepsilon g(x,\dot{x},t)$ with small ε, this problem becomes much more researchable.

Simplifying both the factual side of the problem and its historical aspects, we can say that attempts of solving equations of the form

$$\ddot{x} = f(x) + \varepsilon g(x, t)$$

have led to the creation of the theory of area-preserving maps, whose main elements are the famous KAM-theory, the theory of exponentially small splittings, the theory of conservative chaos, etc., some of which are also extended to multidimensional systems.

In Fig.1 we show, for comparison, phase portraits for the autonomous system (when $\varepsilon = 0$) and for its Poincaré map (when ε is sufficiently small). The phase portrait in the left figure is very simple: there are 3 equilibria, two centers and a saddle; the saddle has two separatrix loops, and the centers are surrounded by closed orbits. The picture for the Poincaré map is much more complicated: there are 3 fixed points (one saddle and two elliptic points), the invariant manifolds of the saddle no longer coincide, but intersect (in general, the angles between the splitted manifolds will be of the order of ε [2]), neighborhoods of the elliptic points have a "KAM structure". The latter means that these neighborhoods are filled with a continuum of closed invariant curves whose relative measure tends to 1 as $\varepsilon \to 0$. Besides, inside neighborhoods of the elliptic points, there is a countable set of resonant zones with garlands of alternating elliptic and saddle periodic points, and separatrices of the latter have exponentially small (in ε) splitting in the case of analytic systems.

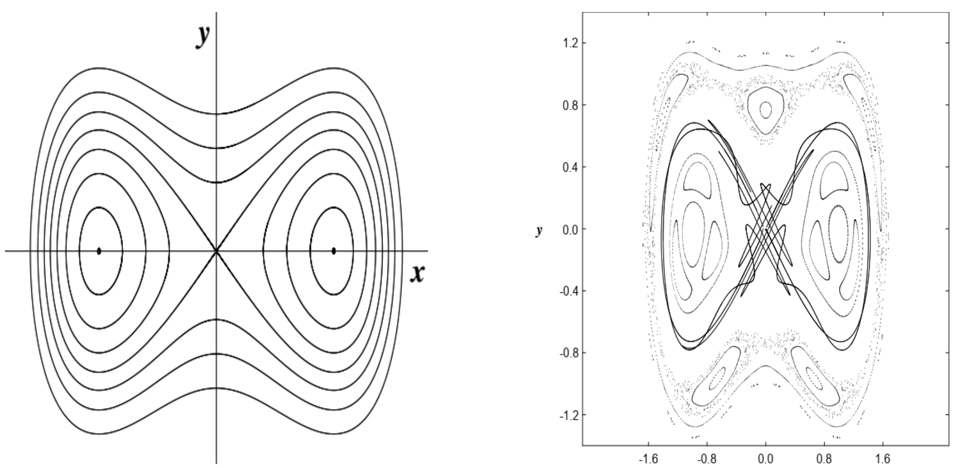

Figure 1:

In general details, this picture (right figure) became known only in the early 1960s, thanks to the famous KAM theory created by that time [3, 4, 5]. This theory generated great interest among specialists all over the world. At the same time, it brought many new questions and problems to light. Simplifying the details, one of these problems can be formulated as follows: what will happen to the dynamics of the Poincaré map when considering perturbations of more general types, e.g. such as

$$\ddot{x} = f(x) + \varepsilon g(x, \dot{x}, t).$$

In other words, what is the dynamics of quasi-conservative systems?

This problem, for a system with 1.5 degrees of freedom (i.e., periodic perturbations of a Hamiltonian system with one degree of freedom) was formulated by L.P. Shilnikov, and was essentially solved by A.D. Morozov. This solution constitutes the dominant portion of the subject matter of his PhD thesis "On the theory of equations of Duffing type close to nonlinear conservative ones" (1975) [6]. The main results were published in a series of papers [7, 8, 9, 13]. Besides, these results and their development can be found in Morozov's monographs [10, 11].

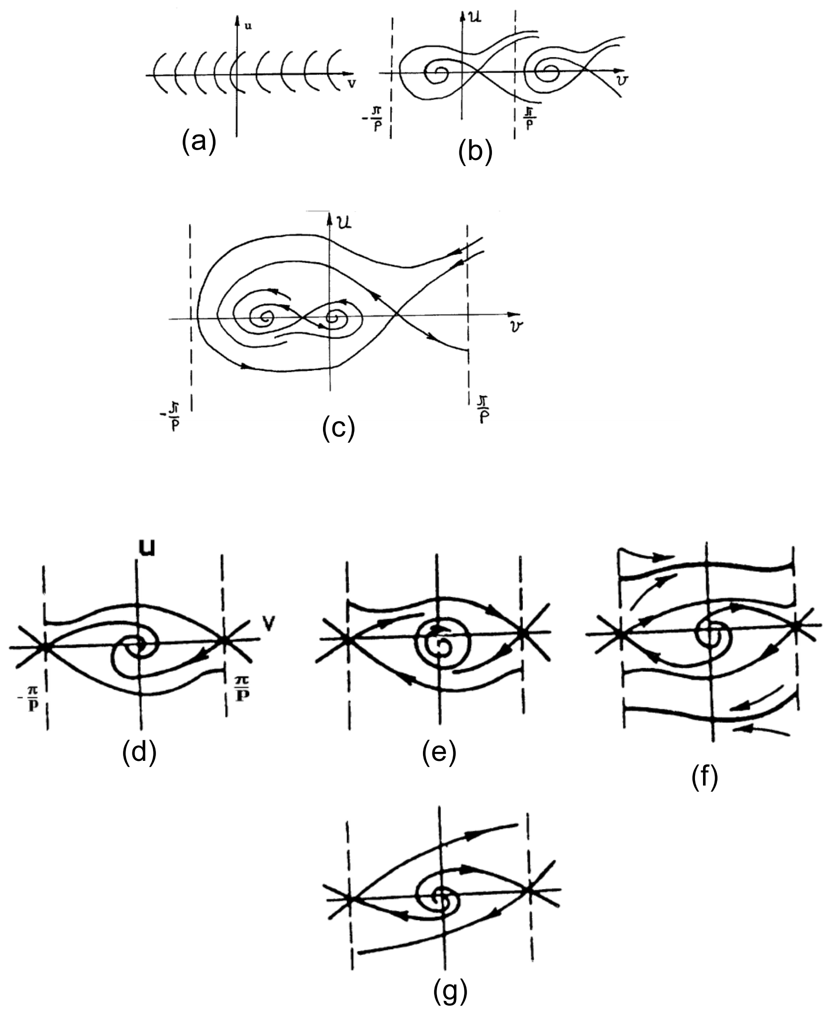

Figure 2: Examples of resonances of various types: (a) a passable resonance; (b)–(c) two types of half-passable resonances; (d)–(g) four types of impassable resonances.

In short, it can be noted that in Morozov's papers, the following important questions on quasi-conservative system dynamics were answered:

- What happens to conservative resonances in case of non-conservative perturbations?
- What can be said about the number of periodical trajectories at autonomous non-conservative perturbations?
- What happens to close invariant curves (tori)?

- What is the “dissipation” influence to the trajectories behavior in the vicinity of the unperturbed separatrix?

This short essay does not allow even for a brief overview of all of A.D. Morozov’s results in this area, but one of those should be mentioned. It concerns the classification of resonances, which can be classified as degenerate and non-degenerate. In their turn, non-degenerate resonances are subdivided into three types, namely, “passable”, “semi-passable” and “non-passable”. Further, these results were substantially complemented and generalized on the case of the systems with two degrees of freedom.

Some idea of such resonances can be obtained by looking at Fig. 2, in which the behaviour of phase curves of an averaged system is shown. Here one can also understand which bifurcations lead to a change in type of resonance.

Further, these results were substantially supplemented and generalized to the case of systems with more number of degrees of freedom, degenerate resonances were examined in detail, the influence of quasiperiodic perturbations on Hamiltonian systems was studied, and much more.

A.D. Morozov started his scientific work during his undergraduate years in Gorky (nowadays, Nizhny Novgorod) State University under the supervision of L.P. Shilnikov. After graduation in 1967, he joined the famous department at the Institute of Applied Mathematics and Cybernetics organized by A.A. Andronov, who was the first head of the department; in 1967, the department was led by E.A. Leotovich-Andronova. It was in 1967 that L.P. Shilnikov published his well-known paper [12] containing the full solution of the Poincaré-Birkhoff problem on the structure of a neighborhood of homoclinic Poincaré curve (i.e., of the trajectory which is twice asymptotic to a saddle periodic orbit). Thus no wonder that the first Morozov’s scientific results concerned these topics, namely, they contained the study of singularities of homoclinic structures in piecewise smooth and relay systems. These results were presented at two conferences [14, 15], at the Congress on Theoretical and Applied Mechanics, Moscow (1968), and on the V International Conference on Nonlinear Oscillations, Kiev (1969).

Soon, however, Morozov (as agreed upon with Shilnikov, his supervisor) has shifted the focus of his studies. This was caused by the emergence of new relevant problems following the success of KAM theory. Since that time on, his main interest area has been the multi-dimensional quasi-conservative systems. Here, as noted above, Morozov was very successful and obtained a large number of world-class results. In 1975, under supervision of L.P. Shilnikov, he defended his PhD thesis at GSU [6], and in 1990 his DSc thesis [16] (the defense was held at MSU at mechanics and mathematics faculty, which is informally considered to be the leading school for differential equations in Russia).

For more than 40 years, A.D. has been engaged in university teaching along with active and fruitful studies. In 1976, Morozov (by Shilnikov’ recommendation) joined the mechanics and mathematics faculty of the Gorky State University, where he worked, at first, as an assistant professor, and, since 1992, as a professor. From 2000 to 2015 he also held the chair of differential equations and mathematical analysis.

During this time, he delivered numerous lecture courses, both general and specialized. Among the latter, the following original courses can be distinguished: "Mathematical Methods in Nonlinear dynamics”, ”Fractals and chaos in dynamical systems”, ”Theory of nonlinear resonance”, etc. Based on the materials of these courses, manuals, textbooks and monographs were published, in particular, [21], [20],[11]. In total, the list of Morozov’s publications contains more than 150 titles, he published 10 books, two of them in the USA. Under his supervision, five PhD theses were defended.

As noted above, Morozov’s studies concern, primarily, the investigation of close-to-Hamiltonian dynamical systems. References to the main publications in this area can be found, for example, in his books [10, 11]. Besides his fundamental results on the resonance structure in quasi-Hamiltonian systems, the following results should be mentioned. A version of Gilbert's problem for pendulum-type equations was solved by him while studying the structure of resonant zones [17, 18, 19]. In collaboration with his students (A. Karabanov,[22], R. Kondrashev, [23], S. Korolev [24]), some classes of systems with two degrees of freedom close to nonlinear integrable ones were investigated. In collaboration with O. Kostromina, the equations of Duffing type having homoclinic "figure-8" loop under asymmetric perturbations were studied. In particular, the principal types of homoclinic structures for quadratic and cubic tangencies of separatrices of the fixed saddle point of the Poncaré map were established [25]. In collaboration with K.E. Morozov, non-autonomous (transitory and quasi-periodic) systems were examined [26]-[29].

In his scientific works, Morozov paid considerable attention to development of computer-based methods of studying the dynamic systems of various nature. For this purpose, in collaboration with T. Dragunov, he has implemented WinSet multi-purpose computer program for visualization of invariant sets of dynamical systems [30]-[31].

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ABSTRACTS

The flux of magnetic helicity for the mean magnetic field equation

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The mean magnetic field equation describes the process of generating a magnetic field on a larger scale due to turbulent pulsations on a smaller scale, see [1] for the basic definitions. As a rule, the equation of the magnetic helicity flux is studied on a larger scale, since the helicity for the mean field is well defined. One may assume that the large scale velocity field admits a fast transport of the helicity density.

In [2] the basic equation, described the flux of the magnetic helisity, is applied for cosmological magnetic fields. In this framework a first-order approximation of the total flux equation, the approximative equation for a flux of magnetic helicity, is introduced in [3]. This equation is not complete, an extra term in this equation, using local formula for quadratic helisity [5], is proposed in [4]. Our goal is to construct a hierarchical (infinite-dimensional) equation for the total flux of magnetic helicity, which contains fluxes of momenta. To solve this problem I will present calculations based on the preprint [6].

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On the Critical Cases of Stability in Impulsive Systems

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Impulsive differential equations (impulsive systems for short) correspond to a smooth evolution that at certain times changes instantaneously, or one could also say abruptly. There are many applications of these equations to mechanical and natural phenomena. We refer to [1] for an extensive list of references. In the report, an algorithm for computing the first Lyapunov value in a critical case is proposed for a second-order periodic impulsive system

$$\dot{x} = Ax + \sum_{|m|\geq 2} f_m x^m, \quad t \neq \tau_k, \qquad x(t^+) = Bx(t) + \sum_{|m|\geq 2} g_m x^m, \quad t = \tau_k, \tag{1}$$

where $x(t^+) = \lim_{s\to t+0} x(s)$, $m = (m_1, m_2) \geq 0$, $|m| = m_1 + m_2$, $x = (x_1, x_2)^T$, $x^m = x_1^{m_1} x_2^{m_2}$, A, $B \in R^{2\times 2}$, $\det B \neq 0$, $f_m, g_m \in R^2$, $\tau_k = k\theta$, $k = 0, 1, 2, \ldots$, $0 < \theta$. The series on the right-hand side of the system are assumed to be absolutely convergent in some neighborhood of zero. The system is a periodic one of period θ.

We suppose that the linearization of the system (1)

$$\dot{x} = Ax, \quad t \neq \tau_k, \qquad x(t^+) = Bx(t), \quad t = \tau_k.$$

Without loss of generality, we will assume that the linearization monodromy matrix has the canonical form $M = e^{\theta A}B = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}$, $\alpha, \beta \in \mathbb{R}$, $\alpha^2 + \beta^2 = 1$. Following an approach suggested in [2], we will perform a linear change of variables $y = \Psi(t)x$ in the system (1), where

$$\Psi(t) = \Psi(\tau_k^+)e^{-tA}, \quad \Psi(\tau_k^+) = e^{\theta A}\Psi(\tau_k), \quad \tau_k < t \leq \tau_{k+1}, \quad k = 0, 1, \ldots, \quad \Psi(0) = I.$$

The θ-periodic nonsingular matrix $\Psi(t)$ is a piecewise smooth on $\mathbb{R}$. Thus the transformation $y = \Psi(t)x$ is a Lyapunov transformation.

In new variables, the system (1) takes the form

$$\dot{y} = \sum_{|m|\geq 2} \tilde{f}_m(t) y^m, \quad t \neq \tau_k, \qquad y(t^+) = My(t) + \sum_{|m|\geq 2} e^{\theta A} g_m y(t)^m, \quad t = \tau_k, \tag{2}$$

where $\tilde{f}_m(t)y^m = \Psi(t) f_m(\Psi^{-1}(t)y)^m$ are θ-periodic functions with respect to t.

Denote by $y(t, y_0)$ the solution of the differential equation of the impulsive system (2), which satisfies the initial condition $y(0) = y_0$. Calculating the coefficients of the expansion of the solution in a series from the initial data in a neighborhood of the origin $y(t, y_0) = y_0 + \sum_{m\geq 2} s_m(t) y_0^m$, $s_m(0) = 0$, to the third degree inclusive, we find an estimate of the value $y(\theta, y_0)$. Substituting this result into the formula of the impulse operator of the system (2), we obtain an approximation of the Poincare map

$$y(\theta^+, y_0) = My_0 + \sum_{|m|=2}^{3} p_m y_0^m + \ldots,$$

where $p_m = \tilde{g}_m(\theta) + Ms_m(\theta)$, if |m|=2, and $p_{30} = \tilde{g}_{30}(\theta) + 2s_{20}^{(1)}\tilde{g}_{20}(\theta) + s_{20}^{(2)}\tilde{g}_{11}(\theta)$, $p_{21} = \tilde{g}_{21}(\theta) + 2s_{11}^{(1)}(\theta)\tilde{g}_{20}(\theta) + (s_{20}^{(1)}(\theta) + s_{11}^{(2)}(\theta))\tilde{g}_{11}(\theta) + 2s_{20}^{(2)}(\theta)\tilde{g}_{02}(\theta)$, $p_{12} = \tilde{g}_{12}(\theta) + 2s_{22}^{(1)}(\theta)\tilde{g}_{20}(\theta) + (s_{02}^{(2)}(\theta) + s_{11}^{(1)}(\theta))\tilde{g}_{11}(\theta) + 2s_{11}^{(2)}(\theta)\tilde{g}_{02}(\theta)$, $p_{03} = \tilde{g}_{03}(\theta) + 2s_{02}^{(2)}(\theta)\tilde{g}_{02}(\theta) + s_{02}^{(1)}(\theta)\tilde{g}_{11}(\theta)$. Due to the nonlinearity of the differential equation in (2), for the values of $s_m(\theta)$ we obtain very simple recurrence relations: $s_m(\theta) = \int_0^\theta \tilde{f}_m(t)\,dt$ for $|m| = 2$; $s_m(\theta) = \int_0^\theta \big(\tilde{f}_m(t) + r_m(t)\big)\,dt$ for $|m| = 3$, where the functions $r_m(t)$ depend only on $s_n(t)$ with $|n| < |m|$, more precisely, $r_{30} = 2s_{20}^{(1)}\tilde{f}_{20} + s_{20}^{(2)}\tilde{f}_{11}$, $r_{21} = 2s_{11}^{(1)}\tilde{f}_{20} + (s_{20}^{(1)} + s_{11}^{(2)})\tilde{f}_{11} + 2s_{20}^{(2)}\tilde{f}_{02}$, $r_{12} = 2s_{22}^{(1)}\tilde{f}_{20} + (s_{02}^{(2)} + s_{11}^{(1)})\tilde{f}_{11} + 2s_{11}^{(2)}\tilde{f}_{02}$, $r_{03} = 2s_{02}^{(2)}\tilde{f}_{02} + s_{02}^{(1)}\tilde{f}_{11}$.

Thus the problem of stability of the trivial solution of the initial impulsive system is reduced to the problem of stability of the zero fixed point of the smooth mapping $\tilde{P}(u) = Mu + \sum_{|m|=2}^{3} p_m u^m$,

$p_m = (p_m^{(1)}, p_m^{(2)})^T \in \mathbb{R}^2$. To study this problem, we bring it to the normal form until terms of the third degree. It is convenient to pass to the complex conjugate variables $z = u_1 + iu_2$, $\overline{z} = u_1 - iu_2$. In the new variables we obtain the scalar equation

$$\tilde{z} = F(z) = e^{i\gamma}z + \sum_{|m|=2}^{3} G_m z^{m_1}\overline{z}^{m_2} = e^{i\gamma}z + \mathcal{G}_2(z,\overline{z}) + \mathcal{G}_3(z,\overline{z}), \quad e^{i\gamma} = \alpha + i\beta. \tag{3}$$

We eliminate the quadratic terms in (3) using an almost identical change of variables $z = H(w) = w + H_2(w,\overline{w})$, $H_2 = \sum_{|m|=2} h_m w^{m_1}\overline{w}^{m_2}$, $h_m \in \mathbb{C}$.

We compute the inverse mapping up to cubic terms $w = H^{-1}(z) = z - H_2(z,\overline{z}) + Q_3(z,\overline{z}) + \dots$. Complex coefficients of a homogeneous polynomial are determined by the formulas $q_{30} = 2h_{20}^2 + h_{11}\overline{h}_{02}$, $q_{21} = 3h_{20}h_{11} + |h_{11}|^2 + 2|h_{02}|^2$, $q_{12} = 2h_{20}(\overline{h}_{11} + h_{02}) + h_{11}(\overline{h}_{20} + h_{11})$, $q_{03} = 2h_{02}\overline{h}_{20} + h_{11}h_{02}$.

In the new variables, the mapping takes the form $\tilde{w} = H^{-1} \circ F \circ H(w) = e^{i\gamma}w + \sum_{|m|\geq 3} W_m w^{m_1}\overline{w}^{m_2}$. Given the presence of third-order resonances $i(m_1 - m_2)\gamma = \pm i\gamma$, $|m| = 3$, and discarding, on the basis of the Poincare-Dulac theorem, nonresonant monomials of the third degree and all monomials of higher degrees, we obtain the so-called model map

$$\hat{w} = e^{i\gamma}w(1 + \mathcal{A}|w|^2),$$

where $\mathcal{A} = G_{21}e^{-i\gamma} + \dfrac{e^{-i\gamma} - 2}{e^{2i\gamma} - e^{i\gamma}}G_{20}G_{11} + \dfrac{2|G_{02}|^2}{e^{3i\gamma} - 1} + \dfrac{|G_{11}|^2}{e^{i\gamma} - 1}$.

The sign of the first Lyapunov quantity $L = \mathrm{Re}\mathcal{A}$ determines the stability character of the zero solution of the initial impulse system (1) according to Lyapunov, namely, the solution is stable asymptotically if $L < 0$ and unstable if $L > 0$.

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The Heisenberg calculus of a singular foliation

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(Work in progress with Eric van Erp, Omar Mohsen and Robert Yuncken)

The hypoellipticity question extends to quite singular "sum of squares" operators. Examples can be found in the work of Kolmogorov (e.g. the operator $\partial_x^2 + x\partial_y$), also in Bismut's hypoelliptic Laplacians. The answer is given by the celebrated theorem of Hoermander, which states that the bracket generating condition induces hypoellipticity.

In order to understand better this result, we are looking for the Geometry behind it. The first ingredient is a filtration of the module of vector fields on a manifold M, naturally induced by the order of the terms of the given operator. The associated grading gives a (singular) bundle of nilpotent "osculating" groups, which accommodate the principal symbol (either as a distribution or as a C*-algebra multiplier). Due to the bracket generating condition, the Schwarz kernel of the operator is a distribution on the pair groupoid. We observe that all this is very nicely put together using singular foliations. The filtration gives rise to a very singular foliation parametrized by t. Its associated holonomy groupoid is a kind of deformation to normal cone, which recovers the osculating group(s) at $t = 0$. So the given operator admits a "longitudinal" description.

Frequency-domain methods for reduction of cocycles in Hilbert spaces

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Let $\mathcal{Q}$ be a metric space with a dynamical system $\vartheta^t \colon \mathcal{Q} \to \mathcal{Q}$, $t \in \mathbb{R}$. A cocycle in a Hilbert space $\mathbb{H}$ is a family of maps $\psi^t(q, \cdot) \colon \mathbb{H} \to \mathbb{H}$, where $t \geq 0$ and $q \in \mathcal{Q}$, satisfying the following conditions:

1. $\psi^0(q, u) = u$ for every $u \in \mathbb{H}, q \in \mathcal{Q}$.
2. $\psi^{t+s}(q, u) = \psi^t(\vartheta^s(q), \psi^s(q, u))$ for all $u \in \mathbb{H}, q \in \mathcal{Q}$ and $t, s \geq 0$.
3. The map $\mathbb{R}_+ \times \mathcal{Q} \times \mathbb{H} \to \mathbb{H}$ defined as $(t, q, u) \mapsto \psi^t(q, u)$ is continuous.

With each cocycle there is the corresponding skew-product dynamical system $\pi^t \colon \mathcal{Q} \times \mathbb{H} \to \mathcal{Q} \times \mathbb{H}$ defined as $\pi^t(q, u) := (\vartheta^t(q), \psi^t(q, u))$. We study the cocycle under the following conditions:

(H1) There is a continuous linear operator $P \colon \mathbb{H} \to \mathbb{H}$, self-adjoint ($P = P^*$) such that $\mathbb{H}$ splits into the direct sum of orthogonal P-invariant subspaces $\mathbb{H}^+$ and $\mathbb{H}^-$, i. e. $\mathbb{H} = \mathbb{H}^+ \oplus \mathbb{H}^-$, such that $P\big|_{\mathbb{H}^-} < 0$ and $P\big|_{\mathbb{H}^+} > 0$.

(H2) We have $\dim \mathbb{H}^- = j < \infty$.

(H3) For $V(u) := (Pu, u)$ and some numbers $\delta > 0$, $\nu > 0$ we have

$$e^{2\nu r} V(\psi^r(q, u) - \psi^r(q, v)) - e^{2\nu l} V(\psi^l(q, u) - \psi^l(q, v)) \leq \\ \leq -\delta \int_l^r e^{2\nu s} |\psi^s(q, u) - \psi^s(q, v)|^2 ds, \quad (1)$$

for every $u, v \in \mathbb{H}$, $q \in \mathcal{Q}$ and $0 \leq l \leq r$.

We show that these conditions imply there is a subset $\mathfrak{A} = \bigcup_{q \in \mathcal{Q}} \mathfrak{A}_q$ of $\mathcal{Q} \times \mathbb{H}$ containing bounded trajectories, invariant w. r. t. π and having fibers $\mathfrak{A}_q$ homeomorphic to some subsets of the j-dimensional space $\mathbb{H}^-$. Under certain compactness assumptions imposed on P or on the cocycle these fibers become homeomorphic to entire $\mathbb{H}^-$. In other words, interesting dynamics under these conditions is only j-dimensional. This can be used to derive various extensions of some well-known low-dimensional results (such as the Poincaré-Bendixson principle for autonomous ODEs; Massera's convergence theorems for periodic ODEs [5]; Zhikov's principle of stationary point for almost periodic ODEs [3]) to high- and infinite-dimensional cases.

Similar assumptions were widely used [1, 2, 5] to study various autonomous and nonautonomous ODEs, where **(H3)** can be verified via the frequency theorem of Yakubovich-Kalman. We will show how to apply infinite-dimensional versions of the frequency theorem [4] to study periodic or almost periodic nonlinear evolutionary problems, especially parabolic and functional-differential equations [6].

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The Baum-Connes conjecture localised at the unit element of a discrete group

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The Baum-Connes conjecture, a central topic in noncommutative topology, relates two natural objects associated with a discrete group. The first one is topological in nature and involves a classifying space for proper actions, the second one is analytical and involves the K-theory of the reduced group C^*-algebra. One of the main features of the Baum-Connes conjecture is that it implies the Novikov conjecture about the homotopy invariance of higher signatures of oriented manifolds.

In this talk we first give an introduction to the topic, then we present a version of the conjecture that we constructed in collaboration with S. Azzali and G. Skandalis. It is called localised at the unit element of a discrete group. This localised conjecture is defined using von Neumann algebras and has some interesting properties especially in the relation with the Novikov conjecture.

Spiral chaos in three-dimensional flows

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This work is devoted to topical issues in the theory of spiral chaos of three-dimensional flows, i.e. the theory of strange attractors associated with the existence of homoclinic loops to the equilibrium of saddle-focus type. The mathematical foundations of this theory were laid in the 60th in the famous works of L.P. Shilnikov, and a lot of important and interesting results have been accumulated on this subject to date. However, for the most part, these results were related to applications. Perhaps for this reason the theory of spiral chaos lacked internal unity – until recently it seemed to consist of separate parts. We make some attempt in this work to unify this theory based on the combination of its two fundamental principles – Shilnikov's theory and universal scenarios of spiral chaos, i.e. those elements of the theory that remain valid for any models, regardless of their origin. In this work we discuss spiral chaos phenomena in both classical systems (Rossler and Arneodo-CoulletTresser) and several models from applications.

First, we consider Rossler system [1] which possibly is the most known model that shows spiral chaos:

$$\dot{x} = -y - z, \dot{y} = x + ay, \dot{z} = bx - cz + xz. \tag{1}$$

Spiral chaos in this system appears due to the Shilniklov's scenario. This scenario can be observed in one parameter families with control parameter a. For $a < a_1$ an asymptotically stable equilibrium exists. At $a = a_1$ this equilibrium undergoes a supercritical Andronov-Hopf bifurcation: it becomes a saddlefocus a1,2), and a stable limit cycle l is born. Then, at $a > a_2$, the limit cycle becomes of a focal type and a two-dimensional unstable manifold begins to wind up on it, forming a configuration resemble a whirlpool. This whirlpool tightens all trajectories from an absorbing domain except for one stable separatrix. With further increasing a the cycle l loses its stability, e.g. under a cascade of period doubling bifurcations, while the size of the whirlpool grows. Finally, at $a = a_3$, a homoclinic loop appears and a strange attractor, containing this loop occurs. We will call such attractors homoclinic attractors.

The second model is Arneodo-Coullet-Tresser system [2, 3].

$$\dot{x} = y, \dot{y} = z, \dot{z} = -y - \beta z + \mu x(1 - x). \tag{2}$$

that demonstrate spiral chaos with certain values of parameters β and μ. The system has constant divergence $-\beta$, and thus attractor can exist only with $\beta > 0$. Scenario of spiral chaos appearance here is the same as in the Rossler system.

The last model is the system that describes fluctuations of the concentrations of chemical elements [4].

$$\dot{x} = x(\beta x - fy - z + g), \dot{y} = y(x + sz - \alpha), \dot{z} = (x - \alpha z^3 + bz^2 - cz)/\varepsilon, \tag{3}$$

The transition from a stable limit cycle to chaotic dynamics occurs not due to supercritical bifurcation of period doubling, as in the previous two models, but as a result of a subcritical period doubling bifurcation, when a saddle cycle of a double period merges into a stable limit cycle.

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Non-stationary hyperbolic attractors in chaotic driven maps

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In this talk we consider driven maps having the form

$$x(i+1) = F(x(i), u(i)), \tag{1}$$

where $x \in R^n$, F is a n-dimensional vector, $i \in Z$ is a discrete time and the function $u : Z \to R^m$ is a set of driving parameters changing the structure of the map at each time iteration.

Four cases are possible.

The case 1. In the trivial case u =const the map (1) becomes an ordinary autonomous map for which all standard notations of attractors and bifurcations are applicable.

The case 2. In the case when a driving parameter $u(i)$ is a periodic function of discrete time $u(i) = u(i+p)$ with a period $p \in Z^+$ the dynamics of the map (1) is defined by the autonomous map $x(j+1) = \hat{F}(x(j), i_0)$, where $i_0 = 1, 2, \ldots, p$, $j = pi$ is a new discrete time and $\hat{F}$ is the composition of the sequential maps.

The case 3. A driving parameter $u(i)$ is an arbitrary bounded function of discrete time. To study a particular case of an attracting set with hyperbolic properties we use the next definition of non-stationary hyperbolic attractor [1].

Definition 1. *Let $G : (\|x\| \leq x^*, x^* = const)$ be an absorbing domain of the map $F(x(i), u(i))$, $FG \subset G$, $\forall i \in Z^+$. Let at each point $x_0 \in G$ the similar pairs of stable and unstable invariant cones K^s and K^u be defined. Denote the linearization of the map F in the point x_0: $L(x_0, i) = D_x F(x_0, u(i))$, where D_x is a differential with respect to x. Let the next conditions be fulfilled. The operator L (the operator L^{-1}) expands any vector V_0^u (V_0^s, respectively) released from x_0 and lying in the unstable cone K^u (the stable cone K^s, resp.) for any $x_0 \in G$ and $i \in Z^+$. Then the set of points in G on which the map $F(x(i), u(i))$ eventually acts for unboundly increasing i is called a non-stationary hyperbolic attractor.*

In this talk we study the problem of the existence of a non-stationary hyperbolic attractor for the following two-dimensional Lurie-type map [2]

$$F: \quad \begin{array}{l} x(i+1) = x(i) + y(i) + ag(x(i)) \equiv X(x,y), \\ y(i+1) = \lambda u(i)(y(i) + bg(x(i))) \equiv Y(u,x,y). \end{array}$$

where a, b, λ are positive parameters and $g(x)$ is a piecewise-linear function of cubic type

$$g(x) = \begin{cases} 2 + 2x, & x < -\frac{1}{2}, \\ -2x, & |x| \leq \frac{1}{2}, \\ -2 + 2x, & x > \frac{1}{2}. \end{cases}$$

For this map according to Def. 1 we rigorously prove the existence of the non-stationary hyperbolic attractor.

The case 4. The driving parameter $u(i)$ is dynamically defined by the map

$$u(i+1) = f(u(i)). \tag{2}$$

In this case one can join the maps (1) and (2) in one autonomous map defined in the extended phase space and having a master-slave structure where the map (2) serves the master equation.

Any attractor of the obtained autonomous map becomes stationary, and the master-slave structure simplifies the study of the joint map hyperbolic properties.

In our talk we give en example of such map and prove hyperchaotic properties of its attractor.

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Riemannian manifolds and laminations

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A foliated space is a topological space endowed with a partition into connected manifolds, called the *leaves* of the foliated space, that locally looks like a product space in a coherent way. To each such space, we can associate a dynamical system, which can be realized in different ways.

In [3], the author together with Álvarez López and Candel studied the foliated properties of the *smooth Gromov space*, which is the subspace of the Gromov space of pointed proper metric spaces that only consists of pointed, complete and connected Riemannian n-manifolds. Subsequently, this was used to give an answer to a modified version of the *realization problem* in foliation theory[1, 2]. The precise statement of the main results are the following.

Theorem 1. *The smooth Gromov space is a Polish space, and the subspace consisting of locally non-periodic manifolds is a foliated space.*

Theorem 2. *Every connected, complete Riemannian manifold of bounded geometry can be realized in a compact foliated space without holonomy. The foliated space can be chosen so that it is a matchbox manifold.*

An interesting characteristic of the methods used to prove this results is that one could try to modify them in order to study the realization of manifolds in foliated spaces satisfying further dynamical properties. In this talk we will present our previous research as well as the current research regarding the realization of manifolds in foliated spaces with a dense set of compact leaves.

The talk is based on ongoing research with Prof. Álvarez López and Nozawa.

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Dynamics of Kuramoto oscillator networks

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Patterns of phase locked oscillators are observed in many networks, ranging from neuronal populations to power grids. Despite significant interest among physicists and applied mathematicians, the emergence and hysteretic transitions between phase-locked patterns in oscillatory networks, including the celebrated Kuramoto network, have still not been fully understood. In this talk, I will review the state of the art in research on phase-locking in networks of Kuramoto oscillators and discuss new results and research trends.

Emergence of wandering stable components

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In a joint work with Sebastien Biebler, we show the existence of a locally dense set of real polynomial automorphisms of $\mathbb{C}^2$ displaying a wandering Fatou component; in particular this solves the problem of their existence, reported by Bedford and Smillie in 1991. These wandering Fatou components have non-empty real trace and their statistical behavior is historical with high emergence. The proof follows from a real geometrical model which enables us to show the existence of an open and dense set of C^r-families of surface diffeomorphisms in the Newhouse domain, each of which displaying a historical, high emergent, wandering domain at a dense set of parameters, for every $2 \leq r \leq \infty$ and $r = \omega$. Hence, this also complements the recent work of Kiriki and Soma, by proving the last Taken's problem in the C^∞ and C^ω-case.

Dynamics of a Beaver Ball: Topology and Chaos

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This paper addresses the problem of the rolling of a spherical shell with a frame rotating inside, on which rotors are fastened. It is assumed that the center of mass of the entire system is at the geometric center of the shell.

For the rubber rolling model and the classical rolling model it is shown that, if the angular velocities of rotation of the frame and the rotors are constant, then there exists a noninertial coordinate system (attached to the frame) in which the equations of motion do not depend explicitly on time. The resulting equations of motion preserve an analog of the angular momentum vector and are similar in form to the equations for the Chaplygin ball. Thus, the problem reduces to investigating a two-dimensional Poincaré map.

The case of the rubber rolling model is analyzed in detail. Numerical investigation of its Poincaré map shows the existence of chaotic trajectories, including those associated with a strange attractor. In addition, an analysis is made of the case of motion from rest, in which the problem reduces to investigating the vector field on the sphere S^2.

On the boundary between Lorenz attractor and quaisattractor in Shimizu-Morioka system

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We study the boundary between Lorenz attractor and qusiattractors in the Shimizu-Morioka system [1]:

$$\begin{cases} \dot{x} = y, \\ \dot{y} = x - ay - xz, \\ \dot{z} = -bx + x^2. \end{cases}$$

Here x, y, and z are phase variable, a and b are parameters of the system.

Under Lorenz attractor, we mean the stable closed invariant set which satisfies the condition of the geometrical model of Afraimovich, Bykov, Shilnikov [2]. It means that it has a pseudohyperbolic structure and, thus, it is robust with respect to small changing in parameters. The class of quasi-attractors was introduced by Afraimovich and Shilnikov and contains non-robust strange attractors which either possess stable periodic orbits with large periods and narrow absorbing domains or such orbits appear with arbitrarily close perturbation [3, 4].

In the classical Lorenz system the boundary between Lorenz attractor and quasiattractors is formed by the curve $l_{A=0}$ where the separatrix value A of the corresponding Poincaré maps vanishes [5]. On the one side from the curve $l_{A=0}$, when $A > 0$, the attractor is pseudohyperbolic, and it becomes a quaisattractor on the other side, when $A < 0$. The violating of pseudohyperbolicity on the curve $l_{A=0}$ is associated with the destruction of the stable foliations in the corresponding Poincare map [2]. It is important to note, that in the Lorenz system the saddle index ν of the saddle equilibrium $O(0, 0, 0)$ is less than $1/2$ along the part of curve $l_{A=0}$ bounded the region of existence of the Lorenz attractor.

In the Shimizu-Morioka system, the saddle index ν along the curve $l_{A=0}$ belongs to the interval $\nu \in [\nu_1, \nu_2]$, where $\nu_1 \approx 0.31$ and $\nu_2 \approx 0.82$. Thus, the boundary between Lorenz attractor and quasiattractor is much more complicated. For $\nu \leq 1/2$ it coincide with the curve $l_{A=0}$ as in the classical Lorenz model. However, in the case $\nu > 1/2$, stable periodic orbits appear in the system even for positive values of the separatrix value near the curve $l_{A=0}$. Hence the region of existence of the Lorenz attractors in this case is formed by the upper boundary of the corresponding stability windows.

For the detailed analysis of bifurcations in the neighborhood of the curve $l_{A=0}$ we study a one-dimensional factor-map of the corresponding Poincare map:

$$\bar{x} = (-1 + A|x|^{\nu} + B|x|^{2\nu}) \cdot sign(x).$$

We establish that our theoretical investigations are in the full agreement with the numerical study of the boundary of the existence of Lorenz attractor in the Shimizu-Morioka model.

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Twisted states in a system of nonlinearly coupled phase oscillators

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The dynamics of the ring of nonlocally coupled identical phase oscillators is considered [1, 2, 3]. Using the Ott-Antonsen approach, the existence and stability of twisted states is studied [4]. Both fully coherent and partially coherent stable twisted states were found. The existence of partially coherent twisted states in a system of identical phase elements is described for the first time. This regime appears due to the presence of a nonlinear phase shift in coupling term. The analytical results are confirmed in the framework of direct numerical simulation.

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Dynamics of disordered heterogeneous chains of phase oscillators

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Synchronization is a basic concept of rhythms adjustment of self-sustained periodic oscillators due to their weak interaction. This adjustment can be described in terms of phase locking and frequency entrainment. Synchronization phenomena in large ensembles of coupled systems often manifest themselves as collective coherent regimes appearing via non-equilibrium phase transitions. Despite the sufficient success in studies of this phenomenon in wide range of systems [1], there are still several less elaborated problems, such as synchronization in disordered chains and lattices. Recent studies [3, 5] have been restricted to the simplest case of pure sine-coupling of phase oscillators, which is strongly dissipative. Another approach to disordered lattices has been developed in paper [4], which is based on the reformulation of the problem in the basis of linear Anderson (localized) modes with the main focus on weakly nonlinear regimes.

In this study we go beyond these investigations mainly by exploring disordered chains as the simplest case of disordered lattices. Namely, we characterize synchronous states in disordered one dimensional lattices of coupled phase oscillators. We considered disorder in natural frequencies of oscillators, taking phase shift in the coupling, which determine whether interaction is attractive or neutral, as the main parameter.

Previous studies, particularly fundamental paper by Ermentrout and Kopell [2], describe the dynamics and bifurcations of the synchronous state in the case of zero phase shift in details. Our main goal is to extend this study to the case of a non-zero phase shift. In the framework of this study we numerically determined the existence and stability of synchronous states. The strategy was to start with a vanishing disorder, where the synchronous states have just a smooth profile, and to follow numerically these stationary solutions by increasing the disorder level, for different sets of the frequencies, checking for stability of the obtained solutions.

In order to do this we consider the lattice consisting of N phase oscillators with nearest-neighbor coupling. In this case the evolution of the phase φ_n of each unit is given by the following equation

$$\dot{\varphi}_n=\sigma\omega_n+\sin(\varphi_{n+1}-\varphi_n-\alpha)+\sin(\varphi_{n-1}-\varphi_n-\alpha), \tag{1}$$

where normalized natural frequencies ω_n are taken from continuous uniform distribution over a line segment $[0,1]$, the parameter σ defines the level of disorder in natural frequencies, and the phase shift α determines whether the interaction between elements is attractive, repulsive or neutral. Below we assume that the coupling is strongly attractive, i.e. $\alpha \leq 0.3$. It is natural to set boundary conditions in the following way:

$$\sin(\varphi_0-\varphi_1-\alpha)=0, \quad \sin(\varphi_{N+1}-\varphi_N-\alpha)=0, \tag{2}$$

which corresponds to the free boundaries of the chain, i.e. there are no elements with indexes $n=0$ and $n=N+1$. Actually, the system (1) can be interpreted as the Kuramoto – Sakaguchi model, which is relevant to many physical, chemical and biological systems, e.g. lasers, biocircuits, electro-mechanical oscillators [6].

We studied the robustness of stationary solution against phase shift and the level of disorder in the natural frequencies in the chain of phase oscillators. We focused on the case of smooth profile of the stationary solution. Under this assumption we shown that phase shifts of certain values make the regions of existence of stable synchronous regime even wider in comparison to the case of zero phase shift. This result may be counter-intuitive because phase shift is believed to be harmful for synchronization in this case.

Also, in this system we observe different patterns for different disorder realization. Of main interest were stationary synchronous solutions. As we expected to see, these clusters dominate weakly asynchronous states for the phase shifts of the interaction close to zero. Dynamically, the simplest cluster states is periodic or quasi-periodic one, while for stronger disorder and larger number of clusters, chaotic states are expected.

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Lie algebras of heat operators in nonholonomic frame

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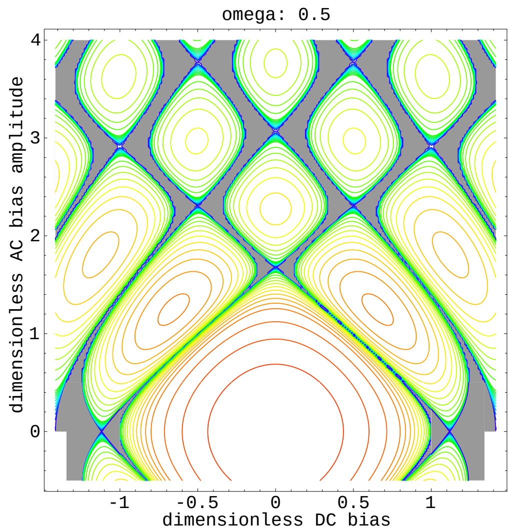
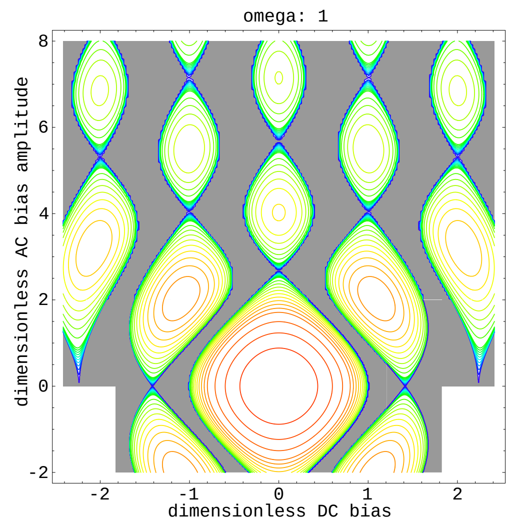
We consider systems of $2g$ heat equations that define sigma functions $\sigma(z,\lambda)$ of the elliptic curve for $g = 1$ and of hyperelliptic curves for $g = 2$ and 3, where $z = (z_1, z_3, \ldots, z_{2g-1})$, and $\lambda = (\lambda_4, \lambda_6, \ldots, \lambda_{4g+2})$ are the parameters of the universal curve. We show that in an infinite-dimensional Lie algebra of linear operators on the ring of smooth functions $\varphi(z,\lambda)$, the operators of this system form a Lie subalgebra $\mathcal{L}_Q$ with $2g$ generators over the ring $\mathbb{Q}[\lambda]$, considered as the set of operators of multiplication by polynomials $p(\lambda) \in \mathbb{Q}[\lambda]$. The Lie algebra $\mathcal{L}_Q$ over $\mathbb{C}$ as a polynomial algebra over $\mathbb{Q}[\lambda]$ turned out to be isomorphic to the polynomial Lie algebra over $\mathbb{Q}[\lambda]$ of vector fields tangent to the discriminant of a hyperelliptic curve in $\mathbb{C}^{2g}$. As a corollary, we find that the system defined by the three operators Q_0, Q_2 and Q_4 is already sufficient to determine the general solution of the original system of $2g$ equations.

A transformation is introduced that maps a system of heat equations in $\varphi(z,\lambda)$ into a system of nonlinear equations in $\nabla \ln \varphi(z,\lambda)$, where ∇ is the gradient of the function in z. This transformation is a multidimensional analogue of the Cole–Hopf transformation, which turns the one-dimensional heat equation into the Burgers equation.

Let $\varphi(z,\lambda)$ be some smooth solution to the system of heat equations. We denote by $\mathcal{R}_\varphi$ the graded commutative ring that is generated over $\mathbb{Q}[\lambda]$ by the logarithmic derivatives of $\varphi(z,\lambda)$ of order at least 2. We have obtained an explicit description of the Lie algebra of derivations of the ring $\mathcal{R}_\varphi$. We will show the close connection of this Lie algebra with our system of nonlinear equations.





The need to obtain effective descriptions of such Lie algebras of differentiations is stimulated by actual problems of describing the dependence on the initial data of solutions to important problems of mathematical physics. In particular, in the case $\varphi(z,\lambda) = \sigma(z,\lambda)$ we obtain the well-known solution to the problem of constructing the Lie algebra of differentiations of hyperelliptic functions of genus $g = 1, 2, 3$.

We give a construction of systems of $2g$ graded heat conduction operators $Q_0, Q_2, \ldots, Q_{4g-2}$. They determine the sigma functions $\sigma(z,\lambda)$ of genus $g = 1, 2$ and 3 hyperelliptic curves. The operator Q_0 is the Euler operator, and each of the operators Q_{2k}, $k > 0$ determines the g-dimensional Schrödinger equation with quadratic potential in z in the nonholonomic frame of vector fields in $\mathbb{C}^{2g}$ with coordinates λ.

For any solution $\varphi(z,\lambda)$ of the system of heat equations, a graded ring $\mathcal{R}_\varphi$ is introduced. It is generated by the logarithmic derivatives of the function $\varphi(z,\lambda)$ of order of at least 2. The Lie algebra of derivations of the ring $\mathcal{R}_\varphi$ is presented explicitly. In the case when $\varphi(z,\lambda) = \sigma(z,\lambda)$, this leads to a known result of constructing the Lie algebras of differentiations of hyperelliptic functions of genus $g = 1, 2, 3$.

Dynamical systems on torus in model of Josephson junction: results, interrelations and conjectures

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In 1973 **B.Josephson** got **Nobel prize** for predicting tunneling effect in superconductivity. The central subject of the talk is one of the known models of dynamics of Josephson junction. It is equivalent to a family of dynamical systems on torus that demonstrates phase-lock effect, see the picture.

The model is described by a family of equations

$$\phi' + \sin\phi = f(\omega t, u), \quad \omega = const, \qquad (1)$$

where $f(\tau, u)$ is a 2π-periodic function of τ; $u \in U$ being the parameter of the family. The variable change $\tau := \omega t$, $g(\tau, u) := f(\omega^{-1}\tau, u)$ transforms (1) to the family of systems on torus:

$$\begin{cases} \dot{\phi} = -\sin\phi + g(\tau, u) \\ \dot{\tau} = \omega, \end{cases} \qquad (\phi(\operatorname{mod} 2\pi\mathbb{Z}), \tau(\operatorname{mod} 2\pi\mathbb{Z})) \in \mathbb{T}^2 = \mathbb{R}^2/2\pi\mathbb{Z}^2. \qquad (2)$$

The classical Poincaré rotation number defines a function $\rho : U \to \mathbb{R}$ for $\omega = const$. Those level sets $L_r = \{u \in U \mid \rho(u) = r\}$ that have non-empty interiors are called the **phase-lock** *areas.* As was shown in [3], the following *quantization effect* holds for system (2): the *phase-lock areas* L_r exist only for *integer* values of r. Note that the quantization effect is **not observed** in the case of famous circle diffeomorphisms family that led to the notion of the **Arnold tongues**. Family (1) with $g(\tau, u) = f(\omega^{-1}\tau, u) = B + A\cos\tau$, $u = (B, A) \in \mathbb{R}^2$ is used in studying the dynamics of Josephson junction and also in different classical and modern problems in physics, mechanics and geometry. In what follows we will denote this family by (1)*. See the **picture above** of the corresponding phase-lock areas for $\omega = 0.7$.

Set $W := \rho^{-1}(\mathbb{R} \setminus \mathbb{Z})$. Let $U = \mathbb{R} \times \mathcal{A}$. Consider the family (1) with the function $f(\tau, u) = B + h(\tau, \alpha)$ being 2π-periodic in τ and analytic in $(\tau, B, \alpha) \in \mathbb{R} \times \mathbb{R} \times \mathcal{A}$. Then it appears that the mapping $\rho : W \to \mathbb{R} \setminus \mathbb{Z}$ is *transversally regular*, and moreover, analytic, has no critical points, and for every $r \in \mathbb{R} \setminus \mathbb{Z}$ the preimage $\rho^{-1}(r)$ is the graph of a function $B = \xi_r(\alpha)$ analytic on the whole manifold $\mathcal{A}$. In the case of system $(1)^*$ we will get functions $\xi_r(A)$, see [2], and each phase-lock area is a **«garland»** of **countable set** of components. Each pair of neighbor components is separated by one point. A separation point is called a **constriction,** if it does not lie in the B-axis. See [7] and the picture above.

Complexification and the variable changes $\Phi = \exp(i\phi)$, $z = \exp(i\omega t)$ transform equations (1) to projectivizations of linear equations on a vector function $(x(z), y(z))$ in form of Riccati equations on the function $\Phi(z) = \frac{x(z)}{y(z)}$. In the case of equation (1)*, setting $\mu = \frac{A}{2\omega}$, $\ell = \frac{B}{\omega}$, the corresponding linear systems can be written as follows:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \left(\frac{1}{z^2} \operatorname{diag}(-\mu, 0) + \frac{1}{z} \begin{pmatrix} -\ell & \frac{1}{2i\omega} \\ \frac{1}{2i\omega} & 0 \end{pmatrix} + \operatorname{diag}(-\mu, 0) \right) \begin{pmatrix} x \\ y \end{pmatrix}. \qquad (3)$$

Systems (3) have only two singular points on the Riemann sphere (at 0 and at ∞). When $\mu \neq 0$, both these singularities are irregular and nonresonant. Variable changes $E(z) := \exp(\mu z)y(z)$, $\mathbb{C}mbda := \frac{1}{4\omega^2} - \mu^2$, transform them to special double confluent Heun equations

$$z^2 E'' + ((\ell + 1)z + \mu(1 - z^2))E' + (\mathbb{C} - \mu(\ell + 1)z)E = 0. \qquad (4)$$

A family of solutions of (1)* that are expressed explicitly via **polynomial solutions** of equation (4) was found in [4]. These solutions correspond to appropriate special points on the intersection of the boundaries of the phase-lock areas with the line $B = \ell\omega$. The manifold MP of parameters (A, B, ω) of such solutions was found. Explicit formulas for the rotation number and the Poincaré mapping of the dynamical system on torus were found for all $(A, B, \omega) \in MP$.

Conditions on the parameters of equation (4) for which its general solution is holomorphic on $\mathbb{C}^* = \overline{\mathbb{C}} \setminus \{0\}$ were obtained in [1, 5]. An explicit function basis in the solution space, one function being holomorphic everywhere except for ∞, the other one being holomorphic everywhere except for the origin, was constructed in [5]. It was shown that in the model of Josephson junction the above situation, when all the solutions of (4) become single-valued on $\mathbb{C}^*$, corresponds to a **constriction** on a phase-lock area.

Together with our colleagues we obtained a series of results on the **structure** of the phase-lock areas for dynamical systems (2), and their applications to the dynamics of Josephson junction in model (1)*, see [2, 6] and references therein. These results are based on the theory of complex differential equations, theory of dynamical systems, theory of double confluent Heun equations and their **isomonodromic** deformation along solutions of Painlevé 3 equation. Some questions on the phase-lock areas are closely related to the generalized Riemann–Hilbert problem for system (3).

We will present a survey of the above-mentioned results and problems that **remain open.**

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Symmetric powers, commuting polynomial Hamiltonians and Hydrodynamic type systems

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With every positive integer N and a polynomial $F(x,y) \in \mathbb{C}[x,y]$, $\frac{\partial}{\partial y}F(x,y) \neq 0$ we associate a family of N polynomial Hamiltonian integrable systems on $\mathbb{C}^{2N}$ with commuting Hamiltonians. The degree of the polynomial $F(x,y)$ does not depend on N. Our construction is based on a canonical transformation of the co-tangent bundle $T^*\mathbb{C}^N$, while the method of integration of the system uses explicit form of the bi-rational equivalence $\mathrm{Sym}^N(\mathbb{C}^2) \to \mathbb{C}^{2N}$ given by this transformation. As a byproduct we obtain integrable hierarchies of Hydrodynamic type systems and a wide class of their explicit solutions. In the talk we present recent developments of our results published in [1,2,3].

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Wasserstein distance for estimating the similarity between an attractor and a repeller for systems that demonstrate the overlapping of these two sets

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Wasserstein distance is a distance function $W(\mu, \nu)$ defined between a pair of probability distributions μ and ν on a given metric space M. If to imagine each distribution as an amount of "dirt" piled on M, Wasserstein distance is the minimum cost of transformation of one pile into the other. It is assumed that the cost of this transformation is proportional to the amount of moved dirt and the distance over which this dirt should be moved.

In this talk, we show how to apply Wasserstein distance for the computation of the similarity between a chaotic attractor and a chaotic repeller for diffeomorphisms demonstrating the overlapping of these two sets. As test examples, we consider three different two-dimensional diffeomorphisms. The first example is a map with a strange nonchaotic attractor given on a torus [1]. The second example is Anosov diffeomorphism perturbed by Mobius transformation. This map is hyperbolic given on a torus. It is important to note, that despite the fact that these two examples demonstrate visual overlapping of numerically obtained attractor and repeller, topologically their dynamics are conservative, since an attractor, as well as a repeller, in these maps coincide with the whole phase space. The last, third, example is the Chirikov map perturbed by Mobius transformation. This map is not hyperbolic, invariant manifolds of its saddle periodic orbit may intersect non-transversally and, therefore, stable, as well as completely unstable, periodic orbits may appear in this case. Thus, in contrast to all previous examples, the perturbed Chirikov map demonstrates the mixed dynamics phenomenon when an attractor of the system intersects with a repeller but does not coincide with it [2]. By computation of Wasserstein distance for all three diffeomorphisms, we show that the proposed method detects perfectly observed in numerical experiments distinguishes between an attractor and a repeller.

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Puiseux series, invariant algebraic curves and integrability of planar polynomial dynamical systems

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Establishing integrability of an ordinary differential equation or a system of ordinary differential equations is an important problem from theoretical and practical points of view. The system of first–order ordinary differential equations

$$x_t = P(x, y), \quad y_t = Q(x, y), \quad P(x, y),\, Q(x, y) \in \mathbb{C}[x, y] \tag{1}$$

with coprime polynomials $P(x, y)$ and $Q(x, y)$ is called integrable with the first integral $I(x, y) \not\equiv \text{const}$ defined in a domain D of full Lebesgue measure in $\mathbb{C}^2$ if the function $I(x, y)$ remains constant along any solution $(x(t), y(t))$ in D. The following equation $\mathcal{X}I = 0$ holds whenever $I(x, y)$ is of class at least C^1 in D. In this relation $\mathcal{X} = P\partial_x + Q\partial_y$ is the vector field associated to system (1). If there exists a function $R(x, y)$ such that the product of the differential form $d\omega = P(x, y)dy - Q(x, y)dx$ and $R(x, y)$ makes the form exact, then this function is called an integrating factor of the differential form and dynamical system (1).

Suppose a differential system under consideration possesses the first integral $I(x, y)$ that is a Liouvillian function; then we shall say that dynamical system (1) is *Liouvillian integrable*. A function is Liouvillian if it can be expressed as a finite superposition of algebraic functions, quadratures and exponential of quadratures over the field of rational functions $\mathbb{C}(x, y)$. It is known [1] that dynamical system (1) is Liouvillian integrable if and only if it has an integrating factor $R(x, y)$ given by

$$R(x, y) = \exp\left\{\frac{g(x, y)}{f(x, y)}\right\} \prod_{j=1}^{r} F_j^{s_j}(x, y), \tag{2}$$

where $g(x, y)$, $f(x, y)$, $F_1(x, y)$, …, $F_r(x, y)$ are bivariate polynomials with coefficients from the field $\mathbb{C}$ and $s_1, \ldots, s_r \in \mathbb{C}$. The algebraic curve $F_j(x, y) = 0$ given by the polynomial $F_j(x, y)$ in expression (2) is an invariant algebraic curve of dynamical system (1). In other words, the polynomial $F_j(x, y)$ satisfies the following partial differential equation $\mathcal{X}F_j = \lambda_j(x, y)F_j$, where $\lambda_j(x, y)$ is a bivariate polynomial called the cofactor of the algebraic curve $F_j(x, y) = 0$. Analogously, the function $E(x, y) = \exp\{g(x, y)/f(x, y)\}$ is an exponential invariant of dynamical system (1). Consequently, the problem of establishing Liouvillian integrability or non–integrability of a dynamical system can be reduced to the problem of constructing all irreducible invariant algebraic curves of $\mathcal{X}$ and all exponential invariants of $\mathcal{X}$. The main difficulty in finding irreducible invariant algebraic curves lies in the fact that bounds on the degrees of $F_j(x, y)$ are as a rule unknown in advance.

The aim of the talk is to present a general method of constructing all irreducible invariant algebraic curves of dynamical system (1) [2, 3]. The main idea of the method is to use the factorization of invariant algebraic curves in the algebraically closed field of Puiseux series. We shall derive the general structure of irreducible invariant algebraic curves and their cofactors for any polynomial dynamical system of the form (1).

As an application of our results we shall solve completely the problem of Liouvillian integrability for a number of physically relevant dynamical systems including the famous Duffing and Duffing–van der Pol oscillators [2, 3]. In addition, we shall demonstrate that a similar method is applicable in the case of time–dependent polynomial dynamical systems in the plane [4].

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Optimal self-similar metrics of expansive homeomorphisms and expanding continuous maps

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It is known since the 1980s that any expansive homeomorphism on a metrizable compactum possesses some Lyapunov or adapted metric. This means that the homeomorphism contracts (resp. expands) local stable (resp. unstable) "manifolds" of a small radius in this metric. Simultaneously, an analogous result was obtained for positively expansive continuous maps on compacta, that is, such a map expands small distances in a suitable metric. The author [1] have sharpened this result in the case of homeomorphism: there exists a Lyapunov metric such that the homeomorphism on local stable (resp. unstable) "manifolds" is approximately representable on a small scale as a contraction (resp. expansion) with constant coefficient λ_s (resp. λ_u^{-1}) in $(0;1)$. Precisely speaking, $\rho(f(x), f(y))/\rho(x, y)$ tends to λ_s (resp. λ_u^{-1}) for two points x, y on one local stable (resp. unstable) "manifold", when $\rho(x, y) \to 0$, where f is the homeomorphism under discussion and ρ is a metric constructed. Also, the homeomorphism together with its inverse are Lipschitz with constants λ_u^{-1} and λ_s^{-1}, respectively, with respect to the metric constructed. Moreover, for homeomorphisms with local product structure the lower bounds for the contraction constants λ_s and expansion constants λ_u are attained simultaneously for some "optimal" metric that satisfies all the conditions described. These results can be immediately transferred to the case of positively expansive continuous maps, though this fact was not pointed out in [1].

Recently, A. Artigue [2] have constructed the so-called self-similar metrics. For the case of homeomorphism f, the metric is called self-similar if $\max\{\rho(f(x), f(y)), \rho(f^{-1}(x), f^{-1}(y))\} = \lambda^{-1}\rho(x, y)$ with some constant $\lambda \in (0;1)$, provided $\rho(x, y)$ is small enough. In particular, on a small scale, the homeomorphism contracts (resp. expands) local stable (resp. unstable) "manifolds" with exact constant λ (resp. λ^{-1}). Respectively, for the case of positively expansive maps, the equality defining self-similar metric takes the form $\rho(f(x), f(y)) = \lambda^{-1}\rho(x, y)$.

Thus, we deal with asymptotic estimates for rate of contraction/expansion, while Artigue states the exact equalities. On the other hand, we consider distinct constants λ_s, λ_u for contraction and expansion, and discuss simultaneous attainability of their lower bounds at some metric, while Artigue simply considers case where these constants are equal.

In the present talk, we adapt Artigue's approach to sharpen our results and to introduce self-similar metrics with distinct contraction and expansion constants λ_s and λ_u. In fact, the basic idea is only to use the sup operation instead of traditional summation in the well-known formula for Lyapunov metric (such an expression was written in [3] and when used in [2]) and in analogous formulas in [1] that were based on the latter!

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Small Perturbations of Smooth Skew Products and Sharkovsky's Theorem

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1. Let $I = I_1 \times I_2$ be a closed rectangle in the plane (I_1, I_2 are closed intervals of the straight line $\mathbf{R^1}$, $I_k = [a_k,\, b_k]$ for $k = 1,\, 2$). We consider a map $F : I \to I$ satisfying the equality

$$F(x,\, y) = (f(x) + \mu(x,\, y),\, g(x,\, y)) \quad \text{for any } (x,\, y) \in I. \tag{1}$$

We suppose that the map (1) is C^1-smooth on the rectangle I, and the map $f : I_1 \to I_1$ satisfies the following conditions:
(i_f) $f(\partial I_1) \subset \partial I_1$, where $\partial(\cdot)$ is the boundary of a set;
(ii_f) f is Ω-stable in the space of C^1-smooth maps of the interval I_1 into itself with the invariant boundary.

We suppose also that for a C^1-smooth function μ (of variables x and y) the following property holds:
(i_μ) the equalities $\mu(x,\, a_2) = \mu(x,\, b_2) = 0$ are correct for every $x \in I_1$; and the equalities $\mu(a_1,\, y) = \mu(b_1,\, y) = 0$ are correct for every $y \in I_2$.

Moreover, we consider functions $\mu = \mu(x,\, y)$ so small that the following inequality is valid:
(ii_μ) $||\mu||_{1,\,(1,\,1)} < \varepsilon$, where ε is found for an arbitrary $\delta > 0$ by the condition of f- Ω-stability in the space of C^1-smooth self-maps of the interval I_1 with the invariant boundary, and $||\cdot||_{1,\,(1,\,1)}$ is the standard C^1-norm of the linear normalized space of C^1-smooth maps of the rectangle I into the straight line $\mathbf{R^1}$ (that contains the interval I_1).

Let $C^1_\omega(I)$ be the space of C^1-smooth maps (1) such that f satisfies conditions (i_f) – (ii_f), and μ satisfies conditions (i_μ) – (ii_μ).

This talk is the presentation of results of the paper [1]. We prove here that the Sharkovsky's order is reserved for maps from the space $C^1_\omega(I)$. But the proof of this claim requires a large preliminary work. Therefore, first of all, we prove existence of the invariant set (under the map $F \in C^1_\omega(I)$) of continuous pairwise disjoint curvelinear fibers over the points of the nonwandering set $\Omega(f)$ of the map f. Then we rectify these fibers, deduce the map under consideration on the above set to the skew product of interval maps on the set $\Omega(f) \times I_2$, and apply Kloeden's result on the preservation of the Sharkovsky's order for continuous skew products of interval maps (see [2]).

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About optimal harvesting of renewable resource at a finite period of time

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The problem of rational use of renewable resources is one of the important tasks in mathematical biology. The optimal harvesting of the excess portion of individuals from the population contributes to more intensive reproduction of resources. In recent years a large number of works devoted to analytical and numerical study of dynamics regimes of two-age population have been published, for example [1, 2]. In more detail, the current state of research in the field of optimal resource extraction for different models of exploited populations is described in [3].

This paper considers the structured population at which individuals are divided into age or typical groups, given a normal autonomous system of difference equations. For such a population, the task of optimal collection of a renewable resource at a finite period of time.

Define $x_i(k)$, $i = 1, \dots, n$ the number of resources of each of the $n \geq 2$ of the species or classes at the moment $k = 0, 1, 2, \dots$. We will consider the model of the exploited population in the form

$$x(j+1) = F\big((1-u(j))x(j)\big), \quad j = 0, 1, 2, \dots, k-1,$$

where $x(j) = \big(x_1(j), \dots, x_n(j)\big) \in R^n_+$, $R^n_+ \doteq \big\{x \in R^n : x_1 \geq 0, \dots, x_n \geq 0\big\}$,
$u(j) = \big(u_1(j), \dots, u_n(j)\big) \in [0,1]^n$ — control that can be varied to achieve the best collection result, $(1-u_i(j))x_i(j)$ — number of remaining resource of the i-th species at the moment k after harvesting, $F(x) = \big(f_1(x), \dots, f_n(x)\big)$, $f_i(x)$ — real non-negative functions defined for all $x \in R^n_+$ of them are $f_i(0) = 0$, $f_i \in C^2(R^n_+)$, and Jacobi matrix $\left(\frac{\partial f_i}{\partial x_j}\right)_{i,j=1,\dots,n}$ is nondegenerate for all $x \in R^n_+$.

Let $C_i \geq 0$, $i = 1, \dots, n$ — the cost of the conditional unit of each class, then the cost of all all extracted products at the moment k is equal $z(k) = \sum\limits_{i=1}^{n} C_i x_i(k) u_i(k)$.

Define $\overline{u}(k) \doteq (u(0), \dots, u(k-1))$, where $u(j) = (u_1(j), \dots, u_n(j)) \in [0,1]^n$, $j = 0, 1, \dots, k-1$. For any $k = 1, 2, \dots$ consider the function

$$h\big(\overline{u}(k), x(0)\big) \doteq \sum_{j=0}^{k-1} z(j) = \sum_{j=0}^{k-1} \sum_{i=1}^{n} C_i x_i(j) u_i(j),$$

this is equal to the value of a resource extracted for k seizures.

Theorem 1. *Let the function $D(x) \doteq \sum\limits_{i=1}^{n} C_i\big(f_i(x) - x_i\big)$ reaches the maximum value in the only one point $x^* \in R_+^n$ и $x_i^* \le f_i(x^*) \neq 0$ for any $i = 1, \dots, n$. Then for any $x(0) \in R_+^n$ such that $x_i(0) \ge x_i^*$, $i = 1, \dots, n$, function $h\big(\overline{u}(k), x(0)\big)$ reached the highest value*

$$h\big(\overline{u}^*(k), x(0)\big) = (k-1) \cdot D(x^*) + \sum_{i=1}^{n} C_i x_i(0)$$

on multiple $[0,1]^{kn}$ at the following exploitation mode: (1) if $k = 1$, then $u^*(0) = (1, \dots, 1)$;

(2) if $k = 2$, then $\overline{u}^*(2) = \big(u^*(0), u^*(1)\big)$, $u^*(0) = \Big(1 - \frac{x_1^*}{x_1(0)}, \dots, 1 - \frac{x_n^*}{x_n(0)}\Big)$, $u^*(1) = (1, \dots, 1)$;

(3) if $k \ge 3$, then $\overline{u}^*(k) = \big(u^*(0), \dots, u^*(k-1)\big)$, where $u^*(0) = \Big(1 - \frac{x_1^*}{x_1(0)}, \dots, 1 - \frac{x_n^*}{x_n(0)}\Big)$; $u^*(j) = \Big(1 - \frac{x_1^*}{f_1(x^*)}, \dots, 1 - \frac{x_n^*}{f_n(x^*)}\Big)$ при $j = 1, \dots, k-2$; $u^*(k-1) = (1, \dots, 1)$.

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Algorithms for computing Intersection Numbers and Bases of Cohomology Groups for Triangulated Closed Three-Dimensional Manifolds

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We solve some computational problems for triangulated closed three-dimensional manifolds P using groups of simplicial homology and cohomology modulo 2. One of the interesting and important tasks of the computational topology is the development of algorithms for calculating the intersection numbers of cycles of a closed manifold. For two-dimensional manifolds, the first versions of its solution were proposed in [1] and [2]. In [3], this problem is solved for a simple $(n-1)$-dimensional cycle x and one-dimensional cycle y of a manifold of arbitrary dimension n. But the proposed algorithm was not applicable for the case of a non-simple cycle x.

In this work two efficient algorithms for computing intersection numbers of 1- and 2-dimensional cycles are developed. We show that computational complexity of described algorithms are $O(|P|)$ and $O(|P|\ log\ |P|)$ respectively, where $|P|$ is the size of the polyhedron, i.e. the number of simplices in the model. Thus, the described algorithms have similar or better efficiency than the previously developed algorithms, but can be applied to the cases not covered by existing algorithms.

Using these algorithms it is possible to obtain a basis of the cohomology group $H^1(P)$ $(H^2(P))$ from a given basis of the homology group $H_2(P)$ $(H_1(P))$ of complementary dimension. This cohomology group basis can be used for constructing covering polyhedron $\hat{P}$ of a specific form. This covering polyhedron, in turn, can be used to reduce the problem on finding conditional minimum in original polyhedron P, like finding shortest path or cycle inside particular homology class, to finding absolute minimum in covering polyhedron $\hat{P}$.

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Solution-giving formula to Cauchy problem for multidimensional parabolic equation with variable coefficients

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This talk is based on two papers [1,2]. We consider for integer dimension $d \geq 1$, $x = (x_1, \cdots, x_d) \in \mathbb{R}^d$, $t \geq 0$ and $u : [0,\infty) \times \mathbb{R}^d \longrightarrow \mathbb{R}$ and set the Cauchy problem for a second-order parabolic partial differential equation

$$\left\{ \begin{array}{lcl} u'_t(t,x) & = & \sum_{j=1}^{d}(a_j(x))^2 u''_{x_j x_j}(t,x) + \langle b(x), \nabla u(t,x)\rangle + c(x)u(t,x) = Hu(t,x) \\ u(0,x) & = & u_0(x) \end{array} \right. \qquad (1)$$

where a_j and c are $\mathbb{R}$-valued functions for each $j = 1, \cdots, d$, and b is an $\mathbb{R}^d$-valued function, and all of the coefficients are bounded and uniformly continuous.

The objective is to express the solution of (1) in terms of a_j, b_j, c and u_0 assuming that the closure of the operator H is an infinitesimal generator of the C_0-Semigroup $\left(e^{tH}\right)_{t\geq 0}$.

The solution to (1) is obtained by means of theory of C_0-Semigroups [3], which states that the solution is given by $u(t,x) = e^{tH}u_0(x)$, we then apply the Chernoff theorem [4] to a specially constructed family of operators $(S(t))_{t\geq 0}$ explicitly defined in terms of a_j, b_j and c by

$$\begin{aligned}(S(t)f)(x) &= \frac{1}{4d}\sum_{j=1}^{d}\Big(f(x+2\sqrt{d}a_j(x)\sqrt{t}e_j)+f(x-2\sqrt{d}a_j(x)\sqrt{t}e_j)\Big) \\ &+ \frac{1}{2}f(x+2tb(x))+tc(x)f(x) \qquad (2)\\ (H\varphi)(x) &= \sum_{j=1}^{d}(a_j(x))^2\varphi''(x)_{x_jx_j}+\langle b(x),\nabla\varphi(x)\rangle + c(x)\varphi(x) \qquad (3)\end{aligned}$$

Where $e_j \in \mathbb{R}^d$ is a constant d-dimensional vector with 1 at position j and 0 at the other $d-1$ positions. And the solution to (1) is obtained by

$$u(t,x) = \left(e^{tH}u_0\right)(x) = \lim_{n\longrightarrow\infty}\left(S(t/n)^n u_0\right)(x)$$

Where $S(t/n)$ is obtained via substitution of t by t/n in (2) and $S(t/n)^n$ is a composition of n copies of linear bounded operator $S(t/n)$.

We also state the conditions that guarantee that the closure of H generates a C_0-Semigroup, see [2] for details.

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Formulas that Represent Cauchy Problem Solution for Momentum and Position Schrödinger Equation

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This talk is based on paper [1]: the purpose is to derive two formulas representing solutions of Cauchy problem for two SchrÖdinger equations:

1. **One-dimensional momentum space equation with polynomial potential**

   We consider for $K \in \mathbb{N}$ the following Cauchy problem for $x \in \mathbb{R}$, $t \geq 0$

$$\left\{\begin{array}{lcl} i\frac{\partial}{\partial t}\psi(t,x) & = & \sum_{k=1}^{K}\frac{\partial^k}{\partial x^k}\left(a_k(x)\frac{\partial^k}{\partial x^k}\psi(t,x)\right) = \mathcal{H}\psi(t,x) \\ \psi(0,x) & = & \psi_0(x) \end{array}\right. \quad (1)$$

   where $a_k : \mathbb{R} \longrightarrow \mathbb{R}$ are bounded smooth functions with bounded derivatives up to $(2k)$-th order for each $k = 1, \cdots, K$, while $a_0 : \mathbb{R} \longrightarrow \mathbb{R}$ is measurable but may be unbounded. The operator $\mathcal{H}$ is self-adjoint and defined on some dense linear subspace of $L_2(\mathbb{R})$.The initial condition $\psi_0 : \mathbb{R} \longrightarrow \mathbb{C} \in L_2(\mathbb{R})$ which is a Hilbert space over $\mathbb{C}$.

   The objective is to find a formula for the solution ψ in terms of the coefficients such that for each $t \geq 0$ we have $\psi(t,\cdot) \in L_2(\mathbb{R})$ and equation (1) is satisfied in sense of $L_2(\mathbb{R})$.

   This solution is known to exist for each $\psi_0 \in L_2(\mathbb{R})$ and is provided by resolving C_0-Semigroup for the equation considered because the operator is self-adjoint. We employ general approach to find an explicit formula for the resolving C_0-Semigroup and thus reaching the proposed goal.

2. **Multidimensional position space equation with polynomial potential**

   For arbitrary fixed $d \in \mathbb{N}$ we obtain the solution of the Cauchy problem for a d-dimensional SchrÖdinger equation. In the space $L_2(\mathbb{R})$ over $\mathbb{C}$ we study the problem for $t \in \mathbb{R}$, $x \in \mathbb{R}^d$.

$$\left\{\begin{array}{lcl} \psi'_t(t,x) & = & \frac{1}{2}i\left(\sum_{m=1}^{d}\psi''_{x_m x_m}(t,x)\right) - iV(x)\psi(t,x) \\ \psi(0,x) & = & \psi_0(x) \end{array}\right. \quad (2)$$

   We assume $V : \mathbb{R}^d \longrightarrow \mathbb{R}$ is measurable and has a locally summable second power, $V \in L_2^{loc}(\mathbb{R}^d)$.

Let us consider the Cauchy Problem for the Schrödinger equation

$$\left\{\begin{array}{lcl} i\psi'_t(t) & = & \mathcal{H}\psi(t) \\ \psi(0) & = & \psi_0 \end{array}\right. \quad (3)$$

In general case, the Hamiltonian $\mathcal{H}$ is a self-adjoint operator in $L_2(Q)$ with dense domain $Dom(\mathcal{H}) \subset L_2(Q)$, this guarantees due to the Stone Theorem that for each $t \in \mathbb{R}$, the operator $e^{-it\mathcal{H}}$ exists and can be shown to be unitary. Moreover, the family $(e^{-it\mathcal{H}})_{t\in\mathbb{R}}$ is a one-parameter C_0-Semigroup of unitary linear bounded operators with infinitesimal generator $i\mathcal{H}$. The Cauchy problem (3) then has unique solution provided by the formula $\psi(t) = e^{-it\mathcal{H}}\psi_0$.

If we want to determine the evolution of this system and succeed finding a strongly continuous family of bounded self-adjoint operators that are Chernoff-tangent to the operator $\mathcal{H}$, then we can apply the following theorem which allows us to obtain the solution for Cauchy Problem (3). We obtain such an expression for (3) when is representable in the form (1) or (2).

Theorem(Remizov 2016). Let $\mathcal{F}$ be a complex Hilbert space and let $Dom(\mathcal{H}) \subset \mathcal{F}$ be its dense linear subspace. Suppose that the operator $\mathcal{H} : Dom(\mathcal{H}) \longrightarrow \mathcal{F}$ is linear and self-adjoint and real number a is non zero. Suppose that we have such family $(W(t))_{t\geq 0}$ of bounded linear operators in $\mathbb{F}$ that $(W(t))^* = W(t)$ $\forall t \geq 0$, and denoting $S(t) = I + W(t)$, the family $(S(t))_{t\geq 0}$ is Chernoff-tangent to $\mathcal{H}$. Set $R(t) = \exp[ia(S(t) - I)] = \exp[iaW(t)]$. Then there exists a C_0-Semigroup $(e^{iat\mathcal{H}})_{t\geq 0}$, family $(R(t))_{t\geq 0}$ is Chernoff-equivalent to this semigroup, and for each $f \in \mathcal{F}$ and $t_0 \geq 0$ the following equalities hold with respect to the norm in $\mathcal{F}$:

$$e^{iat\mathcal{H}}f = \lim_{n\longrightarrow\infty} R(t/n)^n f = \lim_{n\longrightarrow\infty} \exp[ianW(t/n)]f \quad 0 \le t \le t_0$$

and

$$\lim_{n\longrightarrow\infty} \sup_{t\in[0,t_0]} \| e^{iat\mathcal{H}}f - \lim_{j\longrightarrow\infty} \sum_{k=0}^{j} \frac{(ian)^k}{k!} W(t/n)^k f \| = 0 \quad (4)$$

We express the solution of (1) in terms of its coefficients and provide a family of translation operators that is Chernoff-tangent to the self-adjoint operator from (1) and then apply this theorem. The same is done for finding the solution of (2).

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On some parabolic equations for measures

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Let $\mathcal{H}$ be a separable Hilbert space, $B_{\mathcal{H}}$ is σ–algebra of its Borel subsets, $\mathcal{M}(\mathcal{H})$ is the set of Borel measures on $B_{\mathcal{H}}$, $\mathfrak{U}_{\mathcal{H}}$ is the algebra of cylindrical subsets in $\mathcal{H}$, $\mathcal{M}_c(\mathcal{H})$ is the family of cylindrical measures on $\mathfrak{U}_{\mathcal{H}}$, B and D are linear bounded operators on $\mathcal{H}$, B is symmetric and nonnegative, and B is nuclear.

Definition 1. We will call the function $\widetilde{\mu}(y) = \int_{\mathcal{H}} e^{i(x,y)} d\mu(x)$, where $y \in \mathcal{H}$, Fourier transform of the measure $\mu \in \mathcal{M}_c(\mathcal{H})$.

Definition 2. If $\mu \in \mathcal{M}(\mathcal{H})$, then let us denote through $\mathrm{tr}(B\mu'')$ such a measure $\lambda \in \mathcal{M}_c(\mathcal{H})$, that $\widetilde{\lambda}(\varphi) = -(B\varphi,\varphi)\widetilde{\mu}(\varphi)$ (of course, if it exists), and through $[(D\mu',\cdot)+\mathrm{tr}\, D\cdot\mu]$ — such $\nu \in \mathcal{M}_c(\mathcal{H})$, that $\widetilde{\nu}(\varphi) = -\widetilde{\mu}'_{D\varphi}(\varphi)$ (here the derivative is understood in the sense of Gato).

Definition 3. Let $\mathcal{L} : \mathcal{M}(\mathcal{H}) \to \mathcal{M}_c(\mathcal{H})$ be some linear operator with the domain $D_{\mathcal{L}}$. The family of measures $\mu(t)$, $t > 0$, lying in $D_{\mathcal{L}}$, is called *the weak solution of the equation* $\mu'_t = \mathcal{L}\mu$ *with initial conditions* $\mu_0 \in \mathcal{M}(\mathcal{H})$, if the next two conditions are met:

1) $\dfrac{d}{dt} \int_{\mathcal{H}} f_c\, d\mu(t) = \int_{\mathcal{H}} f_c\, d(\mathcal{L}\mu(t))$ for any continuous cylindrical bounded function f_c on $\mathcal{H}$;

2) $\lim_{t\to+0} \int_{\mathcal{H}} f\, d\mu(t) = \int_{\mathcal{H}} f\, d\mu_0$ for any continuous bounded function f on $\mathcal{H}$.

We call *the fundamental solution* of such an equation, the family of its solutions is $[G(t)](x)$, $t \geqslant 0$, depending on the parameter $x \in \mathcal{H}$, with initial conditions $[G(0)](x) = \delta_x$.

Let's denote by $[G_0(t)](x)$ the fundamental solution of the equation

$$\mu_t' = \frac{1}{2}\operatorname{tr}(B\mu'') - \Big[(D\mu', \cdot) + \operatorname{tr} D \cdot \mu\Big].$$

Let the symbol $\boldsymbol{C}^T_{x,A}$ means, for fixed $T > 0$, $x \in \mathcal{H}$ and $A \in B_{\mathcal{H}}$, the space of all continuous functions f on the segment $[0, T]$, taking values in $\mathcal{H}$, such that $f(0) = x$ and $f(T) \in A$. For any kits of points $0 < t_1 < \ldots < t_n < T$ and sets $A_1, \ldots, A_n \in B_{\mathcal{H}}$, the subsets $I^{t_1,\cdots,t_n}_{A_1,\cdots,A_n} = \Big\{ f \mid f(t_i) \in A_i,\ i = 1, \ldots, n\Big\}$ form a semiring K in $\boldsymbol{C}^T_{x,A}$. Let's set the measure $U^T_{x,A}$ on it by the equality

$$U^T_{x,A}(I^{t_1,\cdots,t_n}_{A_1,\cdots,A_n}) = \int_{A_1} [G_0(t_1)](x)(dy_1) \int_{A_2} [G_0(t_2 - t_1)](y_1)(dy_2) \cdot \ldots \cdot$$
$$\cdot \int_{A_n} [G_0(t_n - t_{n-1})](y_{n-1})(dy_n)[G_0(T - t_n)](y_n)(A).$$

Theorem 1. *The measure $U^T_{x,A}$ on $K \subset \boldsymbol{C}^T_{x,A}$ has unique countably-additive continuation to Borel measure on $\boldsymbol{C}^T_{x,A}$.*

Definition 4. We will call *a conditional generalized Ornstein — Uhlenbeck measure* the continuation of the measure $U^T_{x,A}$, described in theorem 1.

Theorem 2. *Let $V : \mathbb{R}_+ \times \mathcal{H} \to \mathbb{C}$ be a function, continuous by the totality of variables, and there are functions $C_1, C_2 \in L_{1,loc}(\mathbb{R}_+)$ and number $0 \leqslant r \leqslant 2$, such that for any $t \in \mathbb{R}$, $x \in \mathcal{H}$, the next inequalities are satisfied:*

$$|V(t, x)| \leqslant C_1(t) \exp\Big\{\overline{o}(\|x\| r)\Big\} \quad \text{and} \quad \operatorname{Re} V(t, x) \leqslant C_2(t)\overline{o}(\|x\|^r).$$

Then the equation $\nu_t' = \frac{1}{2}\operatorname{tr}(B\nu'') - \big[(D\nu', \cdot) + \operatorname{tr} D\,\nu\big] + V\nu$ has a weak fundamental solution $[G_V(t)](x)$ of the form

$$[G_V(t)](x)(A) = \int_{\boldsymbol{C}^T_{x,A}} \exp\Big\{\int_0^t V(s, q(s))\, ds\Big\} U^t_{x,A}(dq),$$

where $A \in B_{\mathcal{H}}$. Also, if $\nu_0 \in \mathcal{M}(\mathcal{H})$ and $\int_{\mathcal{H}} \exp\{\|x\|^r\}\nu_0(dx) < \infty$, then the formula $\nu(t) = \int_{\mathcal{H}} [G_V(t)](x)\nu_0(dx)$ sets the weak solution of this equation with the initial condition ν_0.

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Symmetry breaking in a system of two coupled microbubble contrast agents

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In this work we study a dynamical system that describes behavior of two different interacting microbubble contrast agents. Contrast agents are micro-meter size gas-filled bubbles, which are encapsulated into a visco-elastic shell [1, 2]. Such bubbles can be used for various biomedical applications, for example, for enhancing ultrasound visualization of blood flow. It is known that contrast agents can demonstrate complex dynamics and its type is important for applications [1, 3].

The dynamics of two interacting bubbles is described by a non-autonomous system of four differential equations (or an equivalent autonomous system of five equations). If the equilibrium radii of both bubbles are the same then the system describing their dynamics is invariant with respect to the symmetry: $R_1 \leftrightarrow R_2, \dot{R}_1 \leftrightarrow \dot{R}_2$, where $R_1(t)$ and $R_2(t)$ denote the first and second bubbles' radii respectively and dot is the derivative with respect to time. This symmetry leads to the appearance of the three-dimensional invariant manifold $R_1 = R_2, \dot{R}_1 = \dot{R}_2$, the orbits lying in which can only be periodic or chaotic with one positive Lyapunov exponent. Solutions embedded in this manifold are characterized by completely in-phase (synchronous) oscillations of both bubbles. Some of these solutions can be asymptotically stable (attractive). Various synchronous (periodic, chaotic) and asynchronous (periodic, quasiperiodic, chaotic and hyperchaotic) states were studied recently in work [4].

The main aim of this talk is to study the influence of destruction of the synchronization manifold on various dynamical regimes in the system. We introduce a perturbation of the equilibrium radius of one of the bubbles which leads to the symmetry breaking. Since synchronous attractors are essentially defined by presence of the symmetry, it is natural to assume that they are in general more sensitive to the symmetry breaking. We show that multistable states consisting of synchronous and asynchronous attractors often transit to monostable states via crisis of a previously synchronous state. Asynchronous states, especially hyperchaotic ones, are in general more stable with respect to symmetry breaking perturbations. However we also demonstrate that in some cases symmetry breaking in a monostable synchronous state does not lead to qualitative changes in the dynamics. Further we consider different transition scenarios of symmetry breaking that can lead to both death of old multistable states or to the birth of new multistable ones.

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Instability induced by prey-taxis in a prey-predator model

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The possibility of spatial structure formation in the activator-inhibitor systems, if diffusion coefficient for inhibitor is considerably greater than that for activator, was shown by Turing. Later, a number of papers established that this condition is unnecessary for the Turing instability. Conditions for the emergence of spatial and spatiotemporal patterns after flow-induced instabilities [1] of spatially uniform populations were derived by Malchow [2, 3] and illustrated by patterns in a minimal phytoplankton-zooplankton model. Instabilities in the uniform distribution can arise, if phytoplankton and zooplankton move with different velocities, regardless of which one is faster. This mechanism of generating patchiness is more general than the Turing mechanism, which depends on strong conditions on the diffusion coefficients. Present study deal with a prey-predator model for spatiotemporal dynamics of phytoplankton, zooplankton and nutrients. The system is described by reaction-diffusion-advection equations in a one-dimensional vertical column of water in the surface layer. Advective term of the predator equation represents the vertical movements of zooplankton with velocity, which is assumed to be proportional to the gradient of phytoplankton density. This study aimed to determine the conditions under which these movements (taxis) lead to the spatially heterogeneous structures generated by the system in the case of equal diffusion coefficients of all model components.

Necessary conditions for the flow-induced instability were obtained through linear stability analysis. Depending on the parameters of the model local kinetics, increasing the taxis rate leads to Turing or wave instability. This fact is in good agreement with conditions for the emergence of spatial and spatiotemporal patterns derived by other authors. While the taxis exceeding a certain critical value, the wave number corresponding to the fastest growing mode remains unchanged. This value determines the type of spatial structure.

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Cluster Synchronization in Fully Coupled Genetic Networks

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In order to describe the mathematical model of an oscillatory genetic network, we consider an isolated self-repressor gene. By [1,2], the change of the concentration $u = u(t)$ of the corresponding protein in the course of time has the form $\dot{u} = -u + \alpha(1 + u^\gamma(t-h))^{-1}$, where α, γ, and h are positive constants. Next, we suppose that there are m genes of this kind, $m \ge 2$, which are related as "each to all." As a result, we obtain the system

$$\dot{u}_j = -u_j + \frac{\alpha}{1 + u_j^\gamma(t-h)} + \sum_{s=1, s\neq j}^{m} \frac{\beta}{1 + u_s^\gamma(t-h)}, \quad j = \overline{1,m}, \tag{1}$$

where α, β, γ, and h are positive parameters. In the present paper, we study the problem on attractors of system (1) for the case in which the parameter $\gamma = 1/\varepsilon$ is large $(0 < \varepsilon \ll 1)$ and the remaining parameters have the order of unity $(\beta > \alpha > 1)$. Next, in system (1), we make the change of variables $u_j = \exp x_j$, $j = \overline{1,m}$. As a result it acquires the form

$$\dot{x}_j = -1 + \exp(-x_j)\Bigg(\frac{\alpha}{\Omega(x_j(t-h),\varepsilon)} + \sum_{s=1, s\neq j}^{m} \frac{\beta}{\Omega(x_s(t-h),\varepsilon)}\Bigg), \tag{2}$$

where $j = \overline{1,m}$, $\Omega(y,\varepsilon) = 1 + \exp(y/\varepsilon)$. System (2) is complicated; thus, we do not try to perform a complete analysis but restrict our considerations to its special periodic solutions, known as two-cluster synchronization modes. To describe the above-mentioned modes, we fix an arbitrary positive integer k, $1 \le k \le m-1$, and suppose that the set of indices $1 \le j \le m$ splits into two disjoint sets $\mathcal{A}$ and $\mathcal{B}$ that consist of k and $m-k$ elements, respectively; i.e., $\{1, 2, \ldots, m\} = \mathcal{A} \cup \mathcal{B}$. Then, obviously, system (2) admits solutions with the components $x_j = v(t)$ for $j \in \mathcal{A}$, $x_j = w(t)$ for $j \in \mathcal{B}$, where the variables v and w satisfy the auxiliary system

$$\begin{aligned} \dot{v} &= -1 + \exp(-v)\left(\frac{\alpha_k}{\Omega(v(t-h),\varepsilon)} + \frac{\beta_{m-k}}{\Omega(w(t-h),\varepsilon)}\right), \\ \dot{w} &= -1 + \exp(-w)\left(\frac{\alpha_{m-k}}{\Omega(w(t-h),\varepsilon)} + \frac{\beta_k}{\Omega(v(t-h),\varepsilon)}\right), \end{aligned} \tag{3}$$

where $\alpha_s = \alpha + (s-1)\beta$ and $\beta_s = s\beta$, $s = \overline{1,m}$. If system (3) has an inhomogeneous periodic solution (such that $v(t) \not\equiv w(t)$), then the corresponding solution of the original system (2) is referred to as a periodic two-cluster synchronization mode (see [3]). Thus, the problem of the existence of two-cluster synchronization modes can be reduced to finding inhomogeneous periodic solutions of system (3) (see also [2,3]).

Theorem. *Let k, $1 \le k \le m-1$, be an arbitrarily fixed positive integer, let the parameters α, β satisfy and the delay h satisfy the estimates $\beta > \alpha > 1$, $h < \ln(\beta_k/\alpha_k)$. Then there exists a sufficiently small $\varepsilon_k > 0$ such that, for all $0 < \varepsilon \le \varepsilon_k$ system (3) admits an exponentially orbitally stable inhomogeneous cycle $C_k : (v,w) = (v_k(t,\varepsilon), w_k(t,\varepsilon))$, whose component $v_k(t,\varepsilon)$ is alternating and the component $w_k(t,\varepsilon)$ is strictly positive.*

The cycle C_k generates an family of periodic two-cluster synchronization modes given by relations $x_j = v_k(t,\varepsilon)$ for $j \in \mathcal{A}$, $x_j = w_k(t,\varepsilon)$ for $j \in \mathcal{B}$. Note that the number of such modes coincides with the number of k-combinations in the set of n elements.

In addition, note that we have found only the simplest periodic two-cluster synchronization modes, whose component v is alternating, and the other component w is strictly positive. However, as numerical analysis shows, system (2) with $m \ge 9$ can additionally have stable two-cluster synchronization modes whose both components v and w are alternating.

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The 1:3 resonance under reversible perturbations

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We study how reversible non-conservative perturbations influence on the 1:3 resonance, i.e. bifurcations of fixed points with eigenvalues $e^{\pm i2\pi/3}$, in the conservative cubic Hénon maps with positive and negative cubic term. We pay special attention to local symmetry-breaking bifurcations of 3-periodic orbits which lead to the emergence of nonsymmetric non-conservative orbits. Such bifurcations turn out to be the so-called reversible pitchfork bifurcations. Due to these bifurcations, for the perturbed map with positive cubic term, a symmetric elliptic orbit becomes a symmetric saddle orbit and in its neighborhood a pair of nonsymmetric asymptotically stable and completely unstable orbits appears. For perturbations of the cubic Hénon map with the negative cubic term, under reversible pitchfork bifurcations a symmetric saddle orbit breaks into a symmetric elliptic orbit surrounded by two nonsymmetric saddles. Moreover, the Jacobian in one saddle is greater than 1 and the Jacobian in the other saddle is less than 1. The presence of these nonsymmetric orbits may indicate mixed dynamics. In addition, we present various methods to construct reversible non-conservative perturbations which break down the conservative dynamics in the two-dimensional Hénon-like maps. This is a joint work with A. O. Kazakov, E. A. Samylina, A. I. Shyhmamedov.

On discrete Lorenz-like attractors

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We give an overview of some results on recently discovered strange attractors of new types, the so-called discrete Lorenz attractors. These attractors can exist in three-dimensional maps – diffeomorphisms, and they belong to the class of homoclinic attractors, that is, strange attractors containing only one saddle fixed point and, hence, entirely its the unstable invariant manifold. We discuss the most important features of these attractors, such as their geometric and homoclinic structures, phenomenological scenarios of their appearance, pseudohyperbolic properties, etc. We also observe various types of such discrete Lorenz-like attractors.

On discrete Lorenz attractors in a Celtic stone model

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We consider the problem on the existence of discrete Lorenz attractors in a nonholonomic Celtic stone model. To this end, in two-parameter families of such models of certain types, the main local and global bifurcations leading to both the appearance and destruction of the attractors are studied. In the plane of governing parameters (one of them is the angle of dynamical asymmetry of the stone, and the other is the total energy), we construct the corresponding bifurcation diagram, where the region of existence of the discrete Lorenz attractor is constructed and its boundaries are explained. We point out the similarities and differences in the scenarios of the emergence of the discrete Lorenz attractor in the nonholonomic model of Celtic stone and the attractor from the classical Lorenz model.

Absorbing domain and Smale horseshoe in multidimensional Henon map

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In this talk we consider a multidimensional Henon map of a general form [1, 2]

$$\begin{cases} \bar{x} = f(x) + \sum\limits_{i=1}^{n} a_i v_i, \\ \bar{v_1} = x, \\ \bar{v_i} = v_{i-1}, i = \overline{2, n}, \end{cases} \tag{1}$$

where $a_i \in \mathbb{R}^1, f(x)$ is the quadratic function of the form

$$f(x) = \mu - x^2, \mu \in \mathbb{R}^1. \tag{2}$$

This map written in reverse numbering with the help of the variables change $v_j = u_j + x, j = \overline{1, n}, u_0 \equiv 0$ takes the next form

$$\begin{cases} \bar{x} = x + \sum\limits_{j=1}^{n} a_j u_j + F(x), \\ \bar{u_j} = -a_1 u_1 - a_2 u_2 - ... - (a_{j-1} - 1) u_{j-1} - ... - a_n u_n - F(x), \end{cases} \tag{3}$$

where $u_0 \equiv 0, F(x) = \big(\sum\limits_{i=1}^{n} a_i - 1\big) + f(x)$.

This map is the Lurie-type map with one nonlinearity and admits the comparison principle [3]. Using this approach for some region of parameters we prove the existence of absorbing domain G containing an attractor of the Henon map (3). We find another region of parameters for which G is no longer absorbing domain. We prove that this domain G and its image form the multidimensional

Smale horseshoe. Therefore in this region of parameters the chaotic component of non-wandering set exists.

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On topological classification and realization of Morse-Smale cascades on the sphere

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A Morse-Smale cascade $f : S^n \to S^n$ (or a flow f^t) on the sphere S^n is a structurally stable diffeomorphism (flow) whose non-wandering set belongs to finite number of periodic hyperbolic orbits, including fixed points.

A wide set of Morse-Smale flows admit combinatorial description of topological equivalence classes, while similar cascades do not (see, for example, [1] for references). We discuss the difference and define a class of Morse-Smale cascades to which it is possible to borrow the topological invariants from the flows.

Describe a combinatorial invariant for a class G of Morse-Smale diffeomorphisms on the sphere S^n of dimension $n \geq 4$ without heteroclinic intersection of invariant manifolds of saddle periodic points introduced in [2] and called a colored graph.

Let Ω_f be a non-wandering set of the diffeomorphism $f \in G$ and $\Omega^i_f = \{p \in \Omega_f |\ dim\ W^u_p = i\}$, $i \in \{0, 1, n-1, n\}$.

For any saddle point σ of the diffeomorphism $f \in G$ the closure $cl\ W^\delta_\sigma$ of its invariant manifold W^δ_σ, $\delta \in \{s, u\}$ of dimension $(n-1)$ consists of the union of W^δ_σ and exactly one periodic point (a sink if $\delta = u$ and a source otherwise). A union $\mathcal{L}_f = (\bigcup\limits_{p\in\Omega^1_f} cl\ W^s_p) \cup (\bigcup\limits_{q\in\Omega^{n-1}_f} cl\ W^u_q)$ cuts the sphere S^n in $k = |\Omega^1_f \cup \Omega^{n-1}_f| + 1$ connected components (where $|P|$ is a cardinality of the set P). Denote these components by $D_1, \dots, D_k$ and put $\mathcal{D}_f = \bigcup\limits_{i\in 1}^{k} D_i$.

A colored graph of the diffeomorphism $f \in G$ is the graph Γ_f with the following properties:

1) a set Γ^0_f of vertices of the graph Γ_f is isomorphic to the set $\mathcal{D}_f$, a set Γ^1_f of edges is isomorphic to the set $\mathcal{L}_f$ by an isomorphism $\xi : \Gamma^0_f \cup \Gamma^1_f \to \mathcal{D}_f \cup \mathcal{L}_f$;

2) vertices v_i, v_j are joined by an edge $e_{i,j}$ if and only if the correspondent domains $D_i = \xi(v_i), D_j = \xi(v_j)$ have the common boundary;

3) an edge $e_{i,j}$ have a color s (u) if $\xi(e_{i,j}) = W^s_p$ $(\xi(e_{i,j}) = W^u_q)$ for some points $p, q \in \Omega_f$.

One can show that the colored graph Γ_f of any $f \in G$ is a tree.

Endow the graph Γ_f by an automorphism $P_f : \Gamma_f \to \Gamma_f$ such that $\xi P_f = f\xi$.

Theorem 1. *Diffeomorphisms $f, f' \in G$ are topologically conjugated if and only if there exists an isomorphism $\zeta : \Gamma_f \to \Gamma_{f'}$ preserving color of edges such that $P_{f'} = \zeta P_f \zeta^{-1}$.*

Theorem 2. *For any 2-colored tree Γ enriched by an automorphism P there is a diffeomorphism $f \in G$ such that Γ_f is isomorphic to Γ by means of isomorphism $\zeta : \Gamma_f \to \Gamma$ preserving color of edges such that $P = \zeta P_f \zeta^{-1}$.*

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Three-dimensional Poincare cross-sections in the model of oscillatory interaction of different-scaled structures in solids

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As is well known the properties of solids are the result of the joint influence of structures of various scales. The report presents the model of the oscillatory interaction of different-scaled structures in solids.

In the model exchange of energy between structural levels is described as evolution of the dynamic system consisting of subsystems that interact defined by laws:

$$\Phi(x, y, z, w) = \begin{cases} x_{n+1} = x_n - k_{xy} p x_n^2 + k_{yx} q y_n^2 + x_{in} \\ y_{n+1} = y_n + k_{xy} p x_n^2 - (k_{yx} + k_{yz}) q y_n^2 + k_{zy} r z_n^2 \\ z_{n+1} = z_n + k_{yz} q y_n^2 - (k_{zy} + k_{zw}) r z_n^2 + k_{wz} s w_n^2 \\ w_{n+1} = w_n + k_{zw} r z_n^2 - (k_{wz} + k_{out}) s w_n^2 \end{cases},$$

where x, y, z and w are dynamic variables characterizing the energy of scale structural levels, and k_{ij}, p, q, r and s are distributing coefficients, that have a clear interpretation depending on a physical nature of the system.

Solution of the system was obtained numerically. Stability of phase trajectories was computed by methods of Lagrange and Lyapunov; it was shown that the region of existence of stable trajectories is limited.

It obvious, evolution of this dynamical system is described by various types of attractors in the four-dimensional phase space. For determined this types in the computer realization of the model, visualization of attractors is provided. For this, the trajectory of the system in phase space was dissecting by a three-dimensional analogue of the Poincare cross-sections.

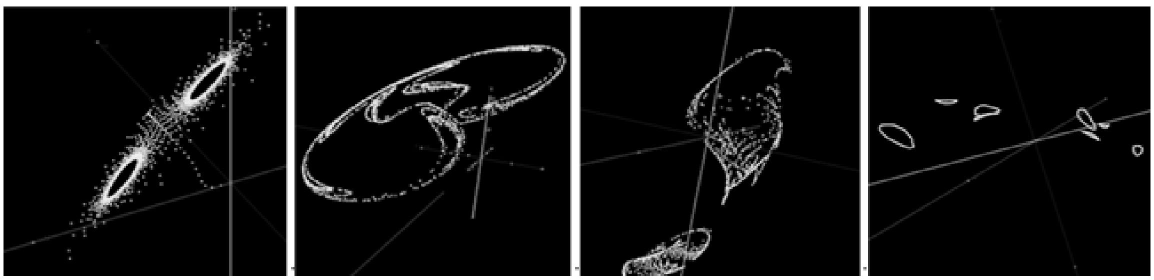

Figure 1: Attractors in three-dimensional Poincare cross-sections for various values of control parameters. Segments of lines are axes of coordinate.

Trajectories obtained in that cross-sections, as obviously, have dimension on one less then initial attractors, and can be visualize (Fig.1). It is by these visualizations in the model is determined character of the evolution of the system for different values of the control parameters.

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Morse theory and rigidity for transversely affine foliations

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This is a work in progress, joint work with Gilbert Hector (University of Lyon I).

Codimension one taut foliations on 3-manifolds have been studied for many years. Their classification on hyperbolic 3-manifolds has a mysterious finite aspect, but still very few is known. In this talk, we consider classification of taut foliations on surface bundles with pseudo-Anosov monodromy, which is a typical example of hyperbolic 3-manifolds.

On a torus bundle M over S^1 with linear Anosov monodromy map f, taut foliations are essentially classified by Ghys-Sergiescu [GS]: Any foliation without compact leaves on M is isotopic to either of the stable or unstable foliation of f. On a surface bundle M over S^1 whose fiber is an orientable closed surface Σ of genus > 1, the classification becomes much more complicated. For example, Cooper-Long-Reid [CLR] produced uncountably many minimal foliations which are close to $\mathcal{E}$ and mutually non-isotopic by some surgery on the bundle foliation. On the other hand, Nakayama [Na] generalized Ghys-Sergiescu's theorem in the context of transversely affine foliations under some conditions. Here, a transversely orientable codimension one foliation is transversely affine if it is defined by a 1-form ω such that $d\omega = \eta \wedge \omega$ for some closed 1-form η. In Nakayama's result, (un)stable foliations of linear Anosov maps are replaced with Meigniez's example [Me], so-called suspension foliation of pseudo-Anosov map.

Our main results are generalizations of Meigniez's example and Nakayama's theorem: Given $f \in \mathrm{Diff}_+(\Sigma)$ and $\sigma \in H^1(\Sigma)$ with $f^*\sigma = \lambda\sigma$ for some $\lambda\ (\neq 1) > 0$, we construct a foliation $\mathcal{F}_\sigma$ on the surface bundle with monodromy f by modifying Meigniez's construction with Moser's technique. These foliations $\mathcal{F}_\sigma$ are transversely affine, and share good properties with Meigniez's examples. The following generalizes Nakayama's theorem.

Theorem 1. *Let $f \in \mathrm{Diff}_+(\Sigma)$ be a pseudo-Anosov map and let M_f be the Σ-bundle whose monodromy is f. Let $\mathcal{E}$ be the bundle foliation on M_f. If $b_1(M_f) = 1$, then any orientable transversely affine foliation without compact leaves whose tangent plane field is homotopic to $T\mathcal{E}$ is isotopic to $\mathcal{F}_\sigma$ for some real eigenvector $\sigma \in H^1(\Sigma)$ of f^*.*

Since $\mathcal{F}_\sigma$ can be isotoped to a foliation which is almost tangent to $\mathcal{E}$, we have the following consequence.

Theorem 2. *Let f, M_f and $\mathcal{E}$ be as in Theorem 1. If $b_1(M_f) = 1$, then, for any $\varepsilon > 0$, any orientable transversely affine foliation without compact leaves whose tangent plane field is homotopic to $T\mathcal{E}$ is ε-coarse isotopic to $\mathcal{E}$ in the sense of Gabai [Ga].*

The main step of the proof of Theorem 1 is to isotope given foliation $\mathcal{F}$ to a foliation transverse to given 1-dimensional flow transverse to the surface fibers. This is done by eliminating certain tangent points of $\mathcal{F}$ to $\mathcal{E}$ by isotopies, based on a one-parameter family version of the Morse theory and Cerf's theorem. The argument is related to the argument of Roussarie-Quê [QR] and Blank-Laudenbach [BL] based on Morse theory for foliations without holonomy.

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Cascade of period-doubling bifurcations in the "generalized" FitzHugh-Nagumo system

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We study a FitzHugh-Nagumo-like system of three ODEs with one fast variable corresponding to the membrane potential and two slow gating variables:

$$\begin{aligned} \varepsilon\dot{x} &= x - x^3/3 - y - z \\ \dot{y} &= a + x \\ \dot{z} &= a + x - z, \end{aligned}$$

where ε is a small parameter and the parameter a is assumed to be slightly less than one. The slow manifold of the system is described by the eqution $x - x^3/3 - y - z = 0$ and possesses folds at $x = \pm 1, y + z = \pm 2/3$.

One may observe that the system has a unique equilibrium, which is stable for sufficiently large a. However, decrease of a leads to the supercritical Andronov-Hopf bifurcation at a value $a_H = 1 - \frac{1}{4}\varepsilon + O(\varepsilon^2)$ (see e.g. [1]) . Immediately after the bifurcation the amplitude of the newborn stable periodic orbit is small and lies below the threshold of spiking. In contrast, for $a \ll a_H$ the system exhibits large-scale periodic oscillations: continuous spiking known as "canards".

In [1] the author found numerically that dynamics near the slow surface can effectively become three-dimensional. As a result, the initial periodic state may lose stability already before the canard transition via a sequence of period-doubling bifurcations. Studying numerically the period-doubling cascades for small but fixed values of the parameter ε, M. Zaks observed that the cascade follows the Feigenbaum law with the Feigenbaum constant $4.67\ldots$, which is typical for dissipative systems. On the other hand for smaller values of ε the process switches to the Feigenbaum constant of a conservative map as, in the limit $\varepsilon \to 0$, two-dimensional Poincaré map nearly preserves the area.

The reason for such phenomenon lies in the closeness of the equilibrium to a fold of the slow manifold. Varying a the position of the equilibrium moves and reaches the fold at $a = a_H$.

In this paper we study the system in a vicinity of the pair "equilibrium-fold" and derive the asymptotic formula for the Poincaré return map. We calculate the parameter values for the first period-doubling bifurcation and also discuss more general $3d$ model with a similar bifurcation scenario.

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On the classification of homoclinic attractors of three-dimensional flows

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This report covers some issues of classification of strange homoclinic attractors of three-dimensional dynamical systems with continuous-time (flows). Strange attractors are called homoclinic if they contain a specific saddle equilibrium together with its unstable manifold. The classification of homoclinic attractors is based on the properties of the equilibrium state belonging to the attractor. These properties are determined by eigenvalues of the equilibrium. Depending on the signs of real parts of the eigenvalues, the saddle equilibrium states of three-dimensional flows are of only two types: (2,1) – with two-dimensional stable and one-dimensional unstable invariant manifolds, and (1,2) – with one-dimensional stable and two-dimensional unstable manifolds. If a saddle equilibrium has a pair of complex-conjugated eigenvalues, then it is called a saddle-focus. Topologically, the saddle-focus equilibrium topologically is not distinguished from the saddle (with real eigenvalues). However, concerning the dynamics, the saddle focus is fundamentally different from the saddle [1]. Another important characteristic of saddle equilibrium states is the sum of the real parts of the eigenvalues closest to the imaginary axis, but lying on it on opposite sides. Depending on the described characteristics, strange homoclinic attractors can be of six different types: Shilnikov attractor containing a saddle focus (1,2); figure-eight spiral attractor containing a saddle focus (2,1) with the Shilnikov homoclinic loop of the saddle focus when the saddle value is positive; a figure-eight spiral attractor with a saddle-focus loop (2,1), whose saddle value is negative; attractor of the Lorenz type containing a saddle (2,1) with a positive saddle value; Lyubimov-Zaks-Rovella attractor containing a saddle (2,1) with a negative saddle value; and the Shilnikov saddle attractor containing the saddle equilibrium (1,2).

The idea of classifying homoclinic attractors according to the type of equilibrium state was proposed in [2]. It was framed as a saddle charts method and applied to the class of three-dimensional Henon maps in [3]. In this report the classification of homoclinic attractors based on the saddle charts method is applied to the three-dimensional flows of the following form:

$$\begin{cases} \dot{x} = y + g_1(x,y,z), \\ \dot{y} = z + g_2(x,y,z), \\ \dot{z} = Ax + By + Cz + g_3(x,y,z), \end{cases} \tag{1}$$

where A, B и C – parameters of the system and $g_i, i = 1,2,3$ – non-linear functions satisfying to

$$g_i(0,0,0) = \frac{\partial g_i}{\partial x}(0,0,0) = \frac{\partial g_i}{\partial y}(0,0,0) = \frac{\partial g_i}{\partial z}(0,0,0) = 0, \ i = 1,2,3.$$

whose linearization matrix is represented in the Frobenius form, and the eigenvalues are determined by the coefficients A, B and C. In the parameters space A, B and C, a saddle chart (extended bifurcation diagram) is constructed, where 7 regions corresponding to attractors of various types are distinguished. It is noted that a wide class of three-dimensional flows can be reduced to the class of systems under consideration.

The report also discusses problems related to the pseudohyperbolicity of homoclinic attractors of three-dimensional flows. According to the theory of Turaev and Shilnikov chaotic attractors are called pseudohyperbolic if any its trajectory has a positive Lyapunov exponent, and this property persists after small perturbations of the system [4]. It is proved that in three-dimensional flows only two types of homoclinic attractors can be pseudohyperbolic: Lorenz-like attractors containing a saddle equilibrium state with a two-dimensional stable manifold whose saddle value is positive; as well as Shilnikov saddle attractors containing a saddle equilibrium with a two-dimensional unstable manifold. The remaining attractors inevitably belong to a class of quasiatractors by Afraimovich-Shilnikov (such attractors either contain stable periodic orbit with narrow absorbing domains or such orbits appear after arbitrarily small perturbations).

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Bifurcations in the singular perturbed second-order system with delay

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Consider second-order delay dynamical system

$$\begin{aligned} &\gamma^{-1}\dot{x} + x = x(t-T)(a + d_1 y + d_2 y^2), \\ &\dot{y} = by + cx^2. \end{aligned} \tag{1}$$

Study the dynamics of (1) in the neighborhood of zero equilibrium.

Main assumption is γT is sufficiently large, i.e. $0 < \varepsilon = (\gamma T)^{-1} \ll 1$. Thus, system (1) is singular perturbed. The critical cases (points of bifurcation) has infinite dimension. The quasinormal forms, nonlinear evolutional equations, are constructed in each critical case. Solutions of these equations determine the behavior and main terms of asymptotics of the solutions of (1).

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Asymptotics of self-oscillations in chains of coupled oscillators

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Consider the local dynamics of system of coupled oscillators. Under condition of sufficiently large number of oscillators a spatially distributed model is obtained. Critical cases in the problem of the stability of its solutions have infinite dimension. Special nonlinear systems of partial differential equations are constructed whose nonlocal dynamics describes the behavior of all solutions of the original system in a small neighborhood of its equilibrium state.

Routes to spiral chaos and hyperchaos in a three-dimensional Henon map.

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In this work, we study scenarios of the appearance of chaotic and hyperchaotic attractors in three-dimensional Hénon map of the form

$$\begin{cases} \bar{x} = y, \\ \bar{y} = z, \\ \bar{z} = Bx + Cy + Az - y^2. \end{cases}$$

where x, y, z are map variables and A, B, C are parameters. Note, that the determinant of the Jacoby matrix of this map is equal to B. From the papers [1] and [2], it is known that this map demonstrates discrete Shilnikov attractors. These attractors appear in accordance with the scenario presented in [3, 1]. The main stage of this scenario is the absorption of the saddle-focus fixed point with two-dimensional unstable manifold which appears after supercritical Neimark-Sacker bifurcation. In [4] this scenario was extended to the systems demonstrating secondary Neimark-Sacker bifurcation with stable periodic orbits emerging inside Arnold's tongues. In accordance with this scenario, a chaotic attractor absorbs the periodic saddle-focus orbit which appears via secondary Neimark-Sacker bifurcations and the discrete Shilnikov-like attractor containing this periodic orbit appears. Also, it was shown in this paper that this scenario can lead to the birth of hyperchaotic attractors.

In this work, we show that the above scenario leads to the emergence of spiral chaos and hyperchaos in the three-dimensional Hénon map under consideration. Various types of discrete Shilnikov-like attractors containing different period orbits are found. We also discuss that depending on the measure of saddle-focus periodic orbits belonging to the attractor comparing with the saddles with one-dimensional unstable manifold the resulting Shilnikov-like attractors may be chaotic or hyperchaotic.

In the second part of this work, we show that for some values of parameters, e.g. for small Jacobian B, the map under consideration demonstrates hyperchaotic attractors without saddle-focus periodic orbit. We propose for this case a new scenario. The key part of this scenario is the cascades of period-doubling bifurcations with periodic saddle orbits belonging to the Hénon-like chaotic attractor which, in its turn, appears from the stable fixed point due to the Feigenbaum's cascade followed by the cascade of heteroclinic bifurcations of band merging. Note, that after the period-doubling cascades with the saddle orbits almost all periodic orbits belonging to the attractor are saddles with two-dimensional unstable manifold and, thus, the created attractor becomes hyperchaotic one.

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Structures of a parabolic problem with spatial variable transformation

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A mixed boundary value problem for a nonlinear parabolic equation in a circle

$$v_t + v - D\triangle v + K\gamma \sin w\, Qv = K\gamma((\cos w(\cos Qv - 1) - \sin w(\sin Qv - Qv)),$$

is considered with Neumann conditions on the boundary for $r = r_1$

$$\frac{\partial v(r_1, \varphi, t)}{\partial r} = 0,$$

periodicity conditions

$$v(r, \varphi, t) = v(r, 2\pi + \varphi, t),$$

boundedness conditions at the origin

$$|v(0, \varphi, t)| \leq c < \infty$$

and initial condition

$$v(r, \varphi, 0) = v_0(r, \varphi).$$

Lemma [1]. The linear operator L has eigenfunctions

$$X_{km}(r, \varphi) = \{J_k(\lambda_{km}^c r) \cos k\varphi, J_k(\lambda_{km}^s r) \sin k\varphi\},$$

which correspond to eigenvalues

$$\lambda_{km}^c = D\left(\frac{\mu_{km}}{r_1}\right)^2 + (-1)^k K\gamma \sin\omega + 1,$$

$$\lambda^s_{km} = D\left(\frac{\mu_{km}}{r_1}\right)^2 + (-1)^{k+1}K\gamma\sin\omega + 1, k = 0,1,2,\ldots, \; m = 1,2,\ldots,$$

where $J_k(x)$ — Bessel function, μ_{km} — solutions of the equation

$$J'_k(\mu_{km}) = 0, \; k = 0,1,2,\ldots, \; m = 1,2,\ldots.$$

To analyze the structure of the solution, depending on the parameter D, it is necessary to evaluate the eigenvalues.

Denote by $\Lambda = -K\gamma\sin w$. Choose $\Lambda = \Lambda(K,\gamma) < -1$.

At $D_1 = \frac{-1-\Lambda}{\left(\frac{\lambda^c_{11}}{r_1}\right)^2}$, λ^c_{11} can change sign when decreasing D. As a result of bifurcation, a pair of spatially inhomogeneous stationary solutions branches off from the zero solution.

Theorem [2]. There is $\delta_0 > 0$ such that if $0 < D - D_1 < \delta_0$, then the equation has two asymptotically stable solutions:

$$v^{\pm}(r,\varphi,D) \approx \pm\left(\frac{D-D_1}{c_1(D)}\right)^{1/2} J_1(\lambda^c_{11}r)\cos\varphi +$$

$$+\frac{1}{2!}\left(\frac{D-D_1}{c_1(D)}\right)\frac{\Lambda}{2}ctg\,\omega((\lambda^c_{10}-2\lambda^c_{11})^{-1} + (\lambda^c_{12}-2\lambda^c_{11})^{-1}\cos 2\varphi)J_1^2(\lambda^c_{11}r)\pm$$

$$\pm\frac{1}{3!}\left(\frac{D-D_1}{c_1(D)}\right)^{3/2}(\lambda^c_{13}-3\lambda^c_{11})^{-1}\left(\frac{\Lambda}{4}-\frac{3}{4}\Lambda^2 ctg^2\,\omega(\lambda^c_{12}-2\lambda^c_{11})^{-1}\right)\cdot$$

$$\cdot J_1^2(\lambda^c_{11}r)J_3(\lambda^c_{11}r)\cos 3\varphi,$$

where

$$c_1(D) = \left[\frac{\Lambda}{8} - \frac{1}{4}(\Lambda ctg\,\omega)^2\left((\lambda^c_{10}-2\lambda^c_{11})^{-1} + \frac{1}{2}(\lambda^c_{12}-2\lambda^c_{11})^{-1}\right)\right]J_1^2(\lambda^c_{11}r) < 0.$$

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Symmetry broken states in a chain of pendulums

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We consider the rotational dynamics in a chain of coupled pendulums [1], where the wide variety of in-phase and out-of-phase regimes exists. Many of these states appear due to instability of in-phase rotational regime [2, 3]. Our theoretical analysis allows to find the boundaries of the in-phase regime instability domain for chains with arbitrary number of pendulums in the case of small dissipation. The modes responsible for the instability development are matched with arising out-of-phase regimes in the chain. The analytical results are confirmed with direct numerical simulations. The work was supported by RSF grant No. 19-12-00367.

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Superradiant mode transition in a heterolaser via the formation of a self-consistent population-inversion grating

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A talk is devoted to an intriguing physical phenomenon observed in our numerical modeling of a superradiant lasing in a low-Q symmetric cavity [1, 2], namely, a spontaneous breaking of the mirror symmetry of counter-propagating waves which results in the asymmetric profiles of the field, polarization and population inversion of an active medium. The phenomenon is owing to the nonlinear population inversion grating which is generated self-consistently by the inhomogeneous counter-propagating waves under CW pumping. Such coherent dynamical effects take place in the case of very dense active medium and low-Q cavities when a photon (cavity) lifetime is much shorter than a polarization (optical dipole) lifetime of an active center.

Here, on the basis of numerical solution to the Maxwell-Bloch equations, we describe in detail the steady and dynamic spontaneous symmetry breaking of the structure of the field, polarization and population inversion of an active medium with almost homogeneously broadened spectral line placed into a symmetric combined distributed feedback (DFB) Fabry-Perot cavity. Under the conditions of this breaking in the steady-state or weakly modulated regimes of superradiant lasing, the spatial profiles of the counter-propagating waves become strongly asymmetric and differ essentially from the symmetric profiles of known so-called cold and hot modes, calculated at zero or quasistationary homogeneous population inversion, respectively.

It is shown that the asymmetric (with respect to the cavity center $\zeta = 0$) superradiant lasing is typical below or near the non-stationary lasing threshold. The symmetry breaking occurs during the long transient stage of moving to steady or slow self-modulated lasing and exists even without DFB. We discuss a range of the laser parameters where the phenomenon is present and suggest possible designs of the semiconductor heterolasers of this kind.

In general, in any non-stationary superradiance regime, a dynamic spontaneous symmetry breaking can take place for the profiles of mode fields and the consistent population inversion profile averaged over a long enough time interval containing several characteristic sets of pulses of all lasing modes. The appearing asymmetry can be metastable and the regions of the maximum inversion of the medium and the minimum intensity of the mode field can displace alternately to one side or another from the cavity center. Such spontaneous switchings of metastable laser states can cause temporal changes in the average emission intensity and in the correlation properties of laser pulses, these averages being considerably different for opposite ends of the laser. We attract attention to same mathematical problems of the theory of spontaneous symmetry breaking in both cases of the steady-state and non-stationary superradiant lasing.

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Evolution of the spatial spectra of the quasi-magnetostatic Weibel turbulence in an anisotropic collisionless plasma and the relayed particle magnetization

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On the basis of numerical modeling of the nonlinear stage of the Weibel instability in a homogeneous nonrelativistic electron-ion plasma with a strong initial temperature anisotropy and comparable initial energies of electrons and ions, the evolution of the spatial spectrum of the developing quasimagnetostatic turbulence is investigated. Until very recently, analytical and numerical studies of the dynamics of this instability (see, e.g., [1, 2]) were mainly limited to the analysis of a single spatial harmonic of magnetic field or electric current, the one with the maximum growth rate at the linear stage. The dynamics of various harmonics of the field and current and their interaction at the nonlinear stage remain essentially unexplored. We started such an analysis in our recent work [3], where the Weibel instability in nonrelativistic electron-ion bimaxwellian plasma was investigated. This report discusses peculiarities and difficulties of the analytical description of the considered evolution of the Weibel turbulence.

The calculations presented in the report were carried out with the DARWIN code, which implements the particle-in-cell method in a 5-dimensional (2D3V) phase space and is based on the non-radiative Vlasov – Darwin model for electromagnetic field dynamics. Initially, we set the Maxwellian distributions of particles by each of the velocity components, but with different temperatures parallel and orthogonal to the z-axis of Cartesian coordinates. The longitudinal temperature was the greatest, and the simulation was carried out in the xy-plane.

It is shown that after the growth of the total magnetic field RMS value stops, the exponential growth of the electron current harmonics at a certain stage before their saturation changes to a power-law one, and that the long-wave harmonics saturate later than the short-wave ones. On the whole, the dynamics of the spatial spectra of the magnetic field is largely determined by the relay processes of the trapped electrons release from decaying short-wave current filaments and subsequent trapping into growing longer-wavelength ones. This leads to a universal power law of the decay of the magnetic field (or current) spatial spectrum components that decrease in time with an exponent close to 5/2. Meanwhile, the wave number corresponding to the maximum of the magnetic field and current spectrum decreases with time approximately according to the root law. Finally, the RMS value of the inductive electric field decreases as a power-law with an exponent close to 5/3. The spectral indices of the spatial spectra of the Weibel turbulence are also established.

The question of the universality of the discovered spectral indices and the laws of the temporal evolution of the Weibel instability in a plasma with various types of anisotropy remains open.

The considered scenario can be realized in the solar (stellar) wind, coronal mass ejections on the late spectral class stars, or in laboratory conditions, e.g., in laser experiments on the ablation of solid targets, where anisotropic heating of electrons is possible.

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Studying of the spatial distribution of the chlorophyll "a" in the Bering Sea on the basis of satellite data

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The concentration of chlorophyll "a" is the one of the parameters allowing to estimate a condition of ecosystems of the ocean. Chlorophyll "a" is the main pigment of cages of phytoplankton providing photosynthesis process. The amount of photosynthetic primary production which is the speed of producing organic substance in the course of photosynthesis which defines the general overall bio-productivity of the ocean depends on quantity and intensity of functioning chlorophyll "a".

To observe the phytoplankton (more precisely, "chlorophyll-a") and its spatial distribution are developed special space sensors, scanners of color of the sea such as the SeaWiFS (Sea-View Wide Field-of-View Sensor) on the Seastar satellite and also the MERIS spectroradiometers (Medium Spectrometer with image resolution) on the Envisat and MODIS (moderate resolution spectrometer) satellites on the Aqua and Terra satellites.

The regularity of collection of data over the entire area of the World Ocean allows to distinguish features of dynamics of a chlorophyll "a" on various water areas, compare them, to reveal long-term tendencies of change. The results of satellite monitoring do not contain direct information about phytoplankton, but give an opportunity to judge his state on the basis of indicators of content of a chlorophyll in the upper water layer of the ocean. Monitoring of distribution of concentration of a chlorophyll has important practical value for fishery because phytoplankton is the food base of zooplankton and fishes.

For a research the region of the Bering Sea limited to coordinates 45° -75° NL, 160° EL-155° WL was chosen. The Bering Sea is rich in nutrients for phytoplankton, the is rather biologically various, certain areas of the sea are abundant of different types of fishes. The open area allowing to allowing to analyze the patterns of formation of the lower trophic levels of the marine ecosystem has been chosen.

From satellite data are used concentration of a chlorophyll, temperature and illumination on a surface. Data of May 2014 are processed. The size of the spatial cell (points) is 4 x 4 km, the time interval is 1 day.

The averaging of satellite characteristics on time and space (the surface of the sea) are constructed. If averaging over space has a small variability in time, the averaging on time is highly dynamic depending on the spatial coordinates. The results are compared to similar researches in the Okhotsk and Japanese seas.

Transition to estimates of a bio-productivity of the Bering Sea by satellite information with application of mathematical models of dynamics of plankton is supposed.

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Simple foliated flows

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Let $\mathcal{F}$ be a smooth transversely oriented codimension one foliation on a closed manifold M and let $\phi = \{\phi^t : M \to M : t \in \mathbf{R}\}$ be a foliated flow on M (that is, each ϕ^t maps leaves to leaves). Denote by $(\Sigma, \mathcal{H})$ the holonomy pseudogroup of $\mathcal{F}$. The foliated flow ϕ induces an $\mathcal{H}$-equivariant local flow $\bar{\phi}$ on the transversal manifold Σ. A leaf L preserved by ϕ corresponds to a fixed point $\bar{p} \in \Sigma$ of $\bar{\phi}$. It is called *transversely simple* if $\bar{p}$ is a simple fixed point of $\bar{\phi}$. In this case, $\bar{\phi}^t_* = e^{\kappa t}$ on $T_{\bar{p}}\Sigma \equiv \mathbf{R}$ for some $\kappa \in \mathbf{R}\backslash\{0\}$, which depends only on L. If all leaves preserved by ϕ are transversely simple, then ϕ is called *transversely simple*. The goal is to describe codimension one foliations on closed manifolds that admit transversely simple foliated flows.

Let M^0 be the union of leaves preserved by ϕ. The set M^0 is ϕ-invariant and closed in M. If ϕ is transversely simple, then M^0 is a finite union of compact leaves. Moreover, the holonomy group $Hol(L)$ of any leaf $L \subset M^0$ consists of germs at 0 of homotheties on $\mathbf{R}$.

Let L be a compact leaf whose holonomy group $Hol(L)$ is described by germs at 0 of homotheties on $\mathbf{R}$ in a foliated chart $(U, (x, y))$ around any point of L, where x is the transverse coordinate. Then $Hol(L)$ is also described by germs at 0 of homotheties on $\mathbf{R}$ in the foliated chart $(U, (u, y))$, with $u = x\,|x|^{\alpha-1}$ $(0 < \alpha \neq 1)$, which is not smooth at $U \cap L$. A transverse power change of the differentiable structure of M around L is defined by requiring all of these new charts to be smooth.

We claim that $\mathcal{F}$ admits a transversely simple foliated flow if and only if:

1. $\mathcal{F}$ is a fiber bundle over S^1 with connected fibers.
2. $\mathcal{F}$ is a minimal $\mathbf{R}$-Lie foliation.
3. $\mathcal{F}$ is an elementary transversely affine foliation whose developing map is surjective over $\mathbf{R}$, and whose global holonomy group is a non-trivial group of homotheties.
4. $\mathcal{F}$ is a transversely projective foliation whose developing map is surjective over the real projective line $S^1_\infty = \mathbf{R} \cup \{\infty\}$, and whose global holonomy group consists of the identity and hyperbolic elements with a common fixed point set.
5. $\mathcal{F}$ is obtained from a foliation as in (3) or (4) using transverse power changes of the differentiable structure of M around the leaves in M^0.

In all these cases, $\mathcal{F}$ is almost without holonomy.
This is joint work with J.A. Álvarez López and E. Leichtnam.

Time-reversal dynamics in ensemble of two excitatory coupled elements

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Model. The studied model consists of two elements. Elements are modeled by Adler's equation:

$$\dot{\phi} = \gamma - \sin\phi,$$

where ϕ is phase of element.

The model of ensemble of two excitatory coupled elements is

$$\begin{cases} \dot{\phi}_1 = \gamma - \sin\phi_1 + d \cdot I(\phi_2) \\ \dot{\phi}_2 = \gamma - \sin\phi_2 + d \cdot I(\phi_1) \end{cases}, \tag{1}$$

where $I(\phi)$ – the function which define the coupling between elements:

$$I(\phi) = \frac{1}{1 + e^{k(\cos(\delta/2) - \cos(\phi - \alpha - \delta/2))}}. \tag{2}$$

Parameters γ, d and k are fixed: $\gamma = 0.7$, $d = 1$, $k = 50$. So there are two free parameters: α and δ.

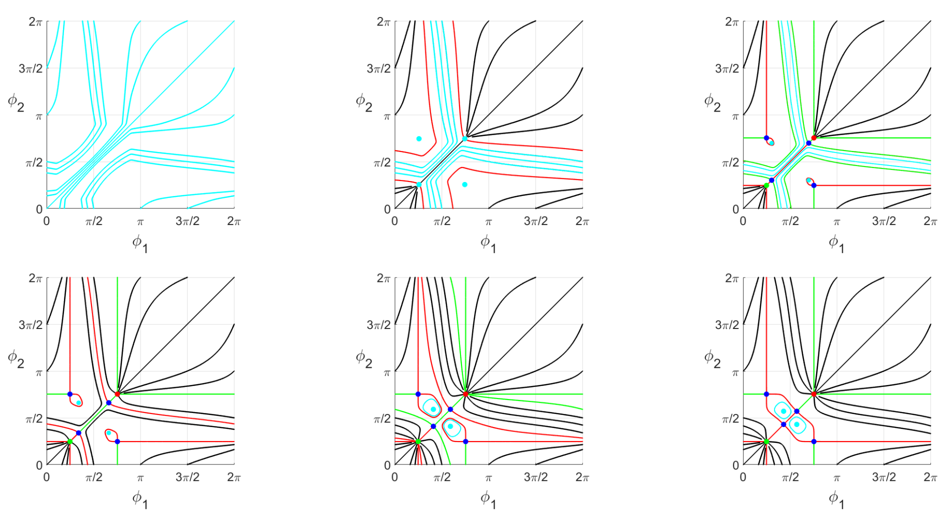

Figure 1: Phase portraits of system (3). Green and red points are stable and unstable equilibrium. Blue points are saddle. Cyan points are centers. Red curves are unstable separatrices of saddles. Green curves are stable separatrices of saddles. Cyan curves are closed trajectories.

Time-reversal symmetry in model. Dynamical system $\dot{x} = F(x)$ $(x \in \Omega)$ is reversible if it satisfies condition $\frac{dR(x)}{dt} = -F\left(R(x)\right)$ $(R : \Omega \to \Omega)$.

Case $\delta = \pi - 2\alpha$**.** If $\delta = \pi - 2\alpha$ then system (1) becomes

$$\begin{cases} \dot{\phi}_1 = \gamma - \sin\phi_1 + \frac{d}{1+e^{k(\sin\alpha - \sin\phi_2)}} \\ \dot{\phi}_2 = \gamma - \sin\phi_2 + \frac{d}{1+e^{k(\sin\alpha - \sin\phi_1)}} \end{cases}. \tag{3}$$

System (3) is reversible. For system (3) reversing symmetry function is $R : (x, y) \mapsto (\pi - y, \pi - x)$. Figure 1 shows all types of dynamics of system (3) when parameter α is varying.

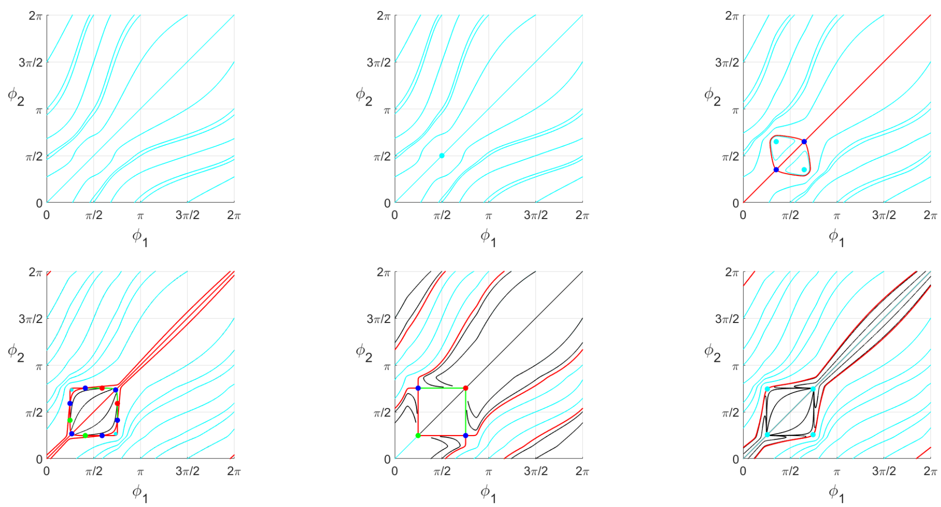

Figure 2: Phase portraits of system (4). Denotation is the same as for figure 1.

Case $\delta = 3\pi - 2\alpha$**.** If $\delta = 3\pi - 2\alpha$ then system (1) becomes

$$\begin{cases} \dot{\phi}_1 = \gamma - \sin\phi_1 + \frac{d}{1+e^{k(\sin\phi_2 - \sin\alpha)}} \\ \dot{\phi}_2 = \gamma - \sin\phi_2 + \frac{d}{1+e^{k(\sin\phi_1 - \sin\alpha)}} \end{cases}. \tag{4}$$

For system (4) reversing symmetry function is the same as for system (3). Figure 2 shows all types of dynamics of system (4) when parameter α is varying.

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On quasi-periodic perturbations of systems with a double limit cycle

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The effect of quasi-periodic perturbations on systems close to two-dimensional Hamiltonian ones is studied in the case where the perturbed autonomous systems have a double limit cycle. The structure of the resonance zone of the original non-autonomous systems in this case is investigated. The results obtained are illustrated by the example of a pendulum-type equation.

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Hamiltonian generalization of Topaj – Pikovsky lattice

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We discuss the Hamiltonian model of oscillator lattice with nonlinear local coupling:

$$\begin{aligned}\dot{I}_j = -\frac{\partial \mathcal{H}}{\partial \phi_j} &= -2\varepsilon\sqrt{I_{j+1}I_j}\left(I_{j+1}-I_j\right)\cos\left(\phi_{j+1}-\phi_j\right) - \\ &\quad - 2\varepsilon\sqrt{I_{j-1}I_j}\left(I_{j-1}-I_j\right)\cos\left(\phi_{j-1}-\phi_j\right), \\ \dot{\phi}_j = \frac{\partial \mathcal{H}}{\partial I_j} &= \omega_j + \beta I_j + \varepsilon\left\{3\sqrt{I_{j+1}I_j} - I_{j+1}\sqrt{\frac{I_{j+1}}{I_j}}\right\}\sin\left(\phi_{j+1}-\phi_j\right) + \\ &\quad + \varepsilon\left\{3\sqrt{I_{j-1}I_j} - I_{j-1}\sqrt{\frac{I_{j-1}}{I_j}}\right\}\sin\left(\phi_{j-1}-\phi_j\right),\end{aligned} \tag{1}$$

with free boundary conditions $\phi_0 = \phi_1$, $\phi_{N+1} = \phi_N$, $I_0 = I_1$, $I_{N+1} = I_N$. Equations (1) are generated by Hamiltonian function

$$\begin{aligned}\mathcal{H}\left(\ldots, I_j, \phi_j, \ldots\right) = &\sum_{j=1}^{N}\omega_j I_j + \frac{1}{2}\beta\sum_{j=1}^{N} I_j^2 - \\ &- 2\varepsilon\sum_{j=1}^{N}\sqrt{I_{j+1}I_j}\left(I_{j+1}-I_j\right)\sin\left(\phi_{j+1}-\phi_j\right).\end{aligned} \tag{2}$$

Equations (1) describe in the classical limit the dynamics of quantum bosonic gas in a tilted periodic lattice [1]. I_j are intensities of oscillations in potential wells, ϕ_j are phases of oscillations. Frequencies are distributed linearly: $\omega_{j+1} - \omega_j = 1$. Since there are only differences of phases on right-hand-sides of equations (1), we write equations for phase shifts $\psi_j = \phi_{j+1} - \phi_j$:

$$\begin{aligned}\dot{\psi}_j = &1 + \beta\left(I_{j+1}-I_j\right) + \varepsilon\left\{3\sqrt{I_{j+2}I_{j+1}} - I_{j+2}\sqrt{\frac{I_{j+2}}{I_{j+1}}}\right\}\sin\psi_{j+1} + \\ &+ \varepsilon\left\{3\sqrt{I_{j-1}I_j} - I_{j-1}\sqrt{\frac{I_{j-1}}{I_j}}\right\}\sin\psi_{j-1} - \\ &- \varepsilon\left\{6\sqrt{I_{j+1}I_j} - I_{j+2}\sqrt{\frac{I_{j+2}}{I_{j+1}}} - I_{j+1}\sqrt{\frac{I_{j+1}}{I_j}}\right\}\sin\psi_j.\end{aligned} \tag{3}$$

System (1)-(3) has an invariant manifold $I_j = I = const\ \forall j$, on which the dynamics of phases is described by Topaj – Pikovsky lattice [2, 3] of locally coupled phase oscillators. Furthermore, the system (1)-(3) has an involution, that reduces to the Topaj – Pikovsky involution on the invariant manifold:

$$\mathbf{R} : I_j \mapsto I_{N-j+1},\ \psi_j \mapsto \pi - \psi_{N-j}. \tag{4}$$

The dynamics on the invariant manifold is not conservative, but that does not contradict to the Hamiltonicity of the system (1). We show by the numerical simulation, that asymptotic trajectories on the invariant manifold are saddle ones with sum of Lyapunov exponents equal to zero.

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Smale – Williams solenoids in autonomous model of coupled oscillators with "figure-eight" homoclinic bifurcations

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We continue the research of autonomous model [1] with uniformly hyperbolic chaotic attractor of Smale – Williams type. It is composed of two self-oscillators with saddle equilibrium at the origin. Each oscillator is governed by equations:

$$\begin{aligned} \dot{x} &= u, \\ \dot{u} &= \left(1-x^2\right)x + \left[L - \left(1-x^2\right)^2\right]u, \end{aligned} \tag{1}$$

and demonstrates "gluing" of the limit cycles into the asymptotic trajectories of the saddle equilibrium at $L \approx 0.3197$. At the same parameter value a "figure-eight" pair of bi-asymptotic trajectories of the saddle exists. We investigate two coupled subsystems (1) with coordinates (x, y) and velocities (u, v). Equations in complex variables $z = x + iy$ and $w = u + iv$ are:

$$\begin{aligned} \dot{z} &= w, \\ \dot{w} &= \left(1-|z|^2\right)z + \left[L - \left(1-|z|^2\right)^2\right]w + \varepsilon w^M. \end{aligned} \tag{2}$$

The term εw^M describes auxiliary coupling. We discuss examples of system (2) with $M = 2$ and $M = 3$. Let us introduce an angular variable θ as an argument of complex variable $z \propto \exp i\theta$. If a typical trajectory is close to the saddle equilibrium at $z = 0$, $w = 0$ so that the amplitude $|z|$ is small, then the angular variable θ expands M times due to auxiliary coupling εw^M. If one constructs a proper Poincaré cross-section of the flow (2), for example by the surface $|z|^2 = 1$ (trajectories crossing outwards), which is far from saddle equilibrium, the angular variable θ undergoes an expanding circle map $\theta_{n+1} = M\theta_n + \text{const} \pmod{2\pi}$ after each iteration of Poincaré return map. If there is strong contraction of phase space in all other directions in Poincaré map, the Smale – Williams solenoid emerges with factor of angular expansion $M = 2$ or 3.

We investigate system (2) by means of numerical simulation. We obtain atlases of dynamical regimes for Poincaré maps of (2) with expansion factors $M = 2$ and $M = 3$. We qualify parameters at which Smale – Williams attractors exist with recently developed numerical technique [2]. We check numerically that average expansion of angular variable is close to M. Namely we accumulate values of angular variable θ_n in small intervals $\left[\frac{2\pi}{N}k,\, \frac{2\pi}{N}(k+1)\right]$ of the circle $[0,\, 2\pi)$ until smooth distribution, find averages $\langle \exp i\theta \rangle_k$ for every interval and calculate the sum $\sum_{k=0}^{N-1} \arg \frac{\langle \exp i\theta \rangle_{k+1}}{\langle \exp i\theta \rangle_k}$. If obtained sum is close to $2\pi M$, while there are no empty intervals and negative values of $\arg \frac{\langle \exp i\theta \rangle_{k+1}}{\langle \exp i\theta \rangle_k}$, we confirm that trajectory belongs to Smale – Williams solenoid. We show that the domain of Smale – Williams attractors is large and continuous on the parameter plane $(L,\, \varepsilon)$ for models (2) with $M = 2$ or 3. We spot other regimes by calculation of Lyapunov exponents. We report that birth of Smale – Williams solenoid is preceded by Feigenbaum transition to non-hyperbolic chaos. The details of the transition between non-hyperbolic and hyperbolic attractors will be discussed elsewhere.

In addition we check the hyperbolicity of attractor at typical values of parameters by numerical test of the angles between stable and unstable subspaces with technique developed in [3]. Absence of zero angles gives us reason to consider the attractor uniformly hyperbolic.

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Topological conjugacy of flows with two limit cycles

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Two flows are called topologically equivalent if there exists a homeomorphism sending trajectories of one flow into trajectories of another one preserving directions of trajectories. Two flows are called topologically conjugate if there exists a homeomorphism sending trajectories of one flow into trajectories of another one preserving time of moving along the trajectories.

Here we study flows with two hyperbolic limit cycles without singular points on a torus. A domain restricted by two limit cycles has only two topological equivalence classes. But in case of topological conjugacy the situation is a lot more complicated.

In 1978 J. Palis [1] invented continuum topologically non-conjugate systems in a neighbourhood of a system with a heteroclinic contact (moduli). We tried to find some similar moduli for our class of flows to describe a class of topological conjugacy. Even existence of a finite number of moduli leads to infinite number of conjugacy classes for a non-singular flow on a torus. But we found that even the number of moduli is infinite. More precisely, the condition of conjugacy is coincidence of two special multivalued functions constructing by the flow up to a composition with a monotonic function.

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Bursting activity in system of two predator–prey communities coupled by migration

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The report is discusses the mechanism of bursting activity for a group of biological predator–prey communities. The simplest situation is considered, when two non-identical communities live in isolated patches and are weakly coupled by migration. Coupling is predator migration at constant rate. The non-identity is the difference in the prey growth rates or predator mortalities in each patch. It should be noted here, the completely identical communities demonstrate only full synchronization with an arbitrarily small nonzero coupling and any initial conditions. By changing of the variables and characteristic time, it is easy to show the model of communities that differ in the prey birth rate or the predators mortality rates are equivalent. In the latter case, the equations of dynamics after all simplifications and replacements of the parameters have the form:

$$\begin{cases} \frac{dx_1}{dt} = x_1\left(1 - ax_1\right) - \frac{x_1 y_1}{1+hx_1}, \\ \frac{dy_1}{dt} = -c_1 y_1 + \frac{c_1 x_1 y_1}{1+hx_1} + c_1 m\left(\frac{c_1}{c_2} y_2 - y_1\right), \\ \frac{dx_2}{dt} = x_2\left(1 - ax_2\right) - \frac{x_2 y_2}{1+hx_2}, \\ \frac{dy_2}{dt} = -c_2 y_2 + \frac{c_2 x_2 y_2}{1+hx_2} + c_2 m\left(\frac{c_2}{c_1} y_1 - y_2\right), \end{cases} \tag{1}$$

where x_i and y_i are numbers or density of prey and predator, $1/a$ is prey habitat carrying capacity, c_i is rate of decline in predator numbers or mortality, h is handling time, mc_i is predator migration rate ($i = 1, 2$). Without coupling ($m = 0$) the model is as well known Rosenzweig–MacArthur equations with logistic growth of prey and Hollings predation functional response of II type.

The analysis of local stability for all equilibrium points and limit cycles of system (1) was performed as well as the qualitative analysis of global bifurcations of periodic solutions. As result it was shown with increasing difference between predator mortality ($c_2 - c_1$) there are changes in the types of dynamics differing by a period of oscillation in each patch, the ratio of numbers and the degree of synchronization. Typically in the first patch there is a fast-slow limit cycle (canard) with a large period and amplitude which modulates the limit cycle with a small period in the second patch. Here, the coupling, in fact, is unidirectional so that the fast cycle of second community does not qualitatively affect the first. As a result there exists a hysteresis loop of bursting connecting the fast spiking oscillations and slow orbit. Consequently the typically phase trajectory lies on the surface of a Klein bottle or torus.

Using a singular perturbation analysis, it was found that the fast cycle emerges and disappears at certain values of the numbers of predator and prey in the first patch (according to the Andronov–Hopf bifurcation and saddle-node bifurcation). As a result, if the differences between mortality parameters are significant ($0 < c_1 \ll c_2 < 1$), then the dynamics of the system (1) contains segments of slowly resting dynamic and fast bursts of spikes. Moreover, in the resting part the dynamics of the second community, as a rule, follow the slow changes in the first community, i.e. the dynamics in different patches are synchronous. But in the fast part there is only phase synchronization between the fast-slow cycle in first patch and bursts in second. However, depending on the system parameters, spiking manifold can be differently lies relative to canard. For example, the start of bursting activity (divergent fast oscillations) coincides with the minimum numbers of prey in the first territory ($x_1 \approx 0$). After a rapid increase in the number of prey in the first patch, diverging fluctuations give way to damped in the second patch. Such dynamics correspond to the rhombus shape of spikes cluster. Another case is interesting, when the bursting activity is possible only after the full recovery of prey in the first patch ($x_1 > 0$). In this case, the spikes cluster has the shape of a triangle or a truncated rhombus.

This work was performed in the framework of the State targets of the Institute for Complex Analysis of Regional Problem FEB RAS and partially supported jointly by the Russian Foundation for Basic Research (18-04-00073 a), and Program of fundamental research of the Russian academy of Sciences "Priority research in the interests of the integrated development of the Far Eastern Branch of Russian academy of sciences" (no. 18-5-013).

Attractors of nonlocal Ginzburg-Landau equation

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The integro-differential equation

$$u_t = u - (1+ic)u\Big[\frac{1}{2\pi}\int\limits_0^{2\pi} |u|^2 dx\Big] + (a+ib)u_{xx}, \tag{1}$$

where $u = u(t,x)$ is a complex-valued function, $a \geq 0, b, c \in R$ and
$a^2 + b^2 \neq 0$, is usually called the non-local Ginzburg-Landau equation (see, for example, [1,2,3], and also the references therein). It appeared in the study of ferromagnetism. Usually, equation (1) is considered together with periodic boundary conditions

$$u(t, x + 2\pi) = u(t, x). \tag{2}$$

In [3], the dimension of the global attractor was estimated. These results can be substantially supplemented.

We set $a_k = 1 - ak^2$ and consider only those k, for which $a_k > 0$, i.e. $k^2 \leq m_0^2, m_0 = \Big[\Big(\frac{1}{a}\Big)^{1/2}\Big]$ or $m_0 = \Big[\Big(\frac{1}{a}\Big)^{1/2}\Big] - 1$, if $\Big(\frac{1}{a}\Big)^{1/2} \in N$.

Theorem 1. *The boundary value problem (1), (2) has a homogeneous cycle V_0 :*

$$u(t,x) = u_0(t) = \exp(ict + i\varphi_0),\ \varphi_0 \in R,$$

as well as m_0 *invariant varieties* V_k $(k^2 \leq m_0^2)$ *of dimension 3, which are formed by periodic solutions of the form*

$$u_k(t,x) = \eta_k \exp(i\sigma_k t + ikx + i\varphi_k) + \eta_{-k} \exp(i\sigma_k t - ikx + i\varphi_{-k}),$$

where $\varphi_k, \varphi_{-k} \in R$ *and are arbitrary,* $\sigma_k = -bk^2 - ca_k, a_k = 1 - ak^2 > 0$,
$k = 1, 2, \ldots, m_0, \eta_k, \eta_{-k} \geq 0$ *and they satisfy the following equality*

$$\eta_k^2 + \eta_{-k}^2 = a_k,\ k = 1, \ldots, m_0.$$

Theorem 2. *All solutions of the boundary value problem (1), (2) tend to one of the manifolds over time* V_0 *or* $V_k (k = 1, \ldots, m_0)$*. Moreover, the one-dimensional manifold* V_0 *is a local attractor, and the remaining invariant manifolds* V_k *of dimension 3 are saddle.*

In other words, the global attractor of a dynamical system generated by a nonlinear boundary value problem (1), (2)

$$M = \bigcup_{k=0}^{m_0} V_k.$$

A special case arises if $a = 0$. For such case, the statement holds.

Theorem 3. *The global attractor* M_∞ *can be selected by the condition*

$$\int_0^{2\pi} |u(t,x)|^2 dx = 2\pi.$$

All the solutions of the boundary value problem (1), (2) belonging to M_∞*, in the general case, are quasiperiodic functions of the evolution variable* t.

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Birth of hyperbolic hyperchaos in a time-delay system with periodic forcing

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We consider a non-autonomous time delay system whose excitation parameter is periodically modulated so that the system produces a sequence of oscillation pulses. Due to specially tuned nonlinear mechanism phase of the oscillations is doubled after each modulation period. As a result a stroboscopic map for this system demonstrates hyperbolic chaos [1, 2]. Varying relation between the delay time and the excitation period one can observe a transition to a regime when this map operates as two weakly coupled chaotic subsystems excited alternately. The overall dynamics in this case still being hyperbolic becomes hyperchaotic with two positive Lyapunov exponents [3].

We analyze transition to this hyperbolic hyperchaos and reveal the following scenario. After regular oscillations the hyperchaos appears almost immediately: An area with a single positive exponent is very narrow. Then, the following hyperchaotic regimes take place sequentially: (a) intermittency as an alternation of staying near a fixed point and chaotic bursts; (b) competition between the fixed point and chaotic subset which appears near it; (c) plain hyperchaos without hyperbolicity after termination visiting neighborhoods of the fixed point; (d) transformation of chaos to hyperbolic form.

The competition in the regime (b) results in a non-Gaussian distribution of large time finite time Lyapunov exponents with power law tails and power law growth of Lyapunov sums. This type of behavior related with wandering of trajectories near subsets with different numbers of expanding directions is called unstable dimension variability (UDV). Usually it is observed as a part of scenario of destruction of chaotic synchronization of two subsystems [4]. In our case we also can talk about two chaotic subsystems with rather non-trivial interaction. The UDV effect is observed for them as their effective coupling strength is decreased.

The transition to hyperbolic hyperchaos is accompanied by vanishing of a non-hyperbolic chaotic subset embedded into attractor. We detect it using covariant Lyapunov vectors. The hyperbolic hyperchaos is found to be of two types. The difference is due to the presence or absence of a low dimensional embedded hyperbolic chaotic subset also detected with the help of covariant Lyapunov vectors. When the subset exists it is visited by trajectories and as a result the attractor has more complicated structure with higher Kaplan-Yorke dimension. After its vanish the system operates merely as two weakly coupled identical hyperbolic chaotic subsystems.

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Theory of hidden oscillations and stability of control systems

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The development of the theory of absolute stability, the theory of bifurcations, the theory of chaos, and new computing technologies made it possible to take a fresh look at a number of well-known theoretical and practical problems in the analysis of multidimensional control systems, which led to the emergence of *the theory of hidden oscillations* [1, 2] which represents the genesis of the modern era of Andronov's theory of oscillations.

The theory of hidden oscillation is based on a new classification of attractors as self-excited or hidden [3]. While trivial attractors (stable equilibrium points) can be easily found analytically or numerically, the search of periodic and chaotic attractors can turn out to be a challenging problem (see, e.g. famous 16th Hilbert problem on the number of coexisting periodic attractors in two-dimensional polynomial systems, which was formulated in 1900 and is still unsolved). *Self-excited attractors*, even coexisting in the case of multistability, can be revealed numerically by the integration of trajectories, started in small neighborhoods of unstable equilibria, while *hidden attractors* have the basins of attraction, which are not connected with equilibria and are "hidden somewhere" in the phase space. Thus, the search and visualization of hidden attractors in the phase space require the development of special analytical and numerical methods.

The suggested classification of attractors as being self-exited or hidden not only demonstrated difficulties of fundamental problems and applied systems analysis, but also triggered the discovery of new hidden attractors in well-known engineering and physical models [4]. For the engineering dynamical models the importance of identifying hidden attractors is related with the classical problems of determining the exact boundaries of global stability and identifying classes of models for which the necessary and sufficient conditions for global stability coincide.

This lecture is devoted to well-known theoretical and engineering problems in which hidden oscillations (their absence or presence and location) play an important role.

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Chaos with positive and zero Lyapunov exponents in a three-dimensional map: discrete Lorenz-84 system

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Chaotic behavior is one of the fundamental properties of nonlinear maps [1-3]. Chaos can be most easily and reliably diagnosed using the largest Lyapunov exponent, which will be positive for the chaotic regime. Chaotic dynamics can occur in diffeomorphisms of dimension two or higher, or even in one-dimensional endomorphisms. For maps, the absence of a zero exponent in the spectrum of Lyapunov exponents is characteristic, since they are discrete. A zero exponent in the spectrum will indicate the possibility of embedding such a map in the flow.

In the frame of this work, the possibility of the appearance of chaotic attractors will be shown, the spectrum of Lyapunov exponents of which contains one positive, one close to zero, and one negative exponents. As objects of study, three-dimensional discrete oscillator will be used: a discrete Lorentz-84 oscillator [4]. The paper will present charts of Lyapunov exponents, on which areas with chaotic dynamics with zero Lyapunov exponent are localized, and characteristic phase portraits are shown. A mechanism of occurrence of chaos with a close to zero Lyapunov exponent via cascade of period-doubling bifurcations of an invariant curve will be discussed.

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Homoclinic orbits for conservative surface diffeomorphisms

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Let S be a closed surface furnished with an area form ω. For $1 \le r \le \infty$, denote $\mathrm{Diff}^r_\omega(S)$ the set of C^r diffeomorphisms of S preserving ω, endowed with the C^r-topology. In a join work with Martín Sambarino (Universidad de la República, Montevideo) we prove that there exists a residual set $\mathcal{R} \subset \mathrm{Diff}^r_\omega(S)$ such that if $f \in \mathcal{R}$, there exist hyperbolic periodic points, and every such point has a transverse homoclinic intersection.

Bifurcations in integrable Hamiltonian systems

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One of contemporary problems in the theory of integrable Hamiltonian systems (IHS) is to classify Liouville fibrations generated by low dimensional integrable systems, i.e. in 2, 3, and 4 degrees of freedom. This includes semi-local description of the Liouville fibrations for nondegenerate singularities (rank 0), semi-local nondegenerate singularities of rank 1, 2, and 3 [1]. But such classification unavoidably requires studying bifurcations [4, 5]. This can be seen in many integrable systems depending on parameters, for instance, in integrable systems of mechanics (see [2, 3]).

Bifurcations in IHS are related with the fact that in such systems for the related Poisson action all its singular orbits (of dimension lesser than the half of the manifold dimension) are met in families. For instance, periodic orbits being 1-dimensional orbits of the induced Poisson action belong to a 1-parameter families, 2-dimensional Lagrangian tori belong to 2-parameter families, etc. This implies the following phenomenon: if one moves along the family, an orbit being more degenerate (in transverse direction) than neighboring orbits can be met and hence one may expect branching the family. Also bifurcations are met at the study of parametrized families of integrable systems, then parameters of the family play the similar role. It is important to stress that common tool to study integrable systems uses some assumptions on the linearized system on the related Poisson orbits (like to be a Cartan algebra for the related set of commuting integrals, etc). Such properties are usually violated at the bifurcation and one needs to use another tool to study the related orbit structure.

In the talk I intend to discuss these themes for 3 degree of freedom integrable Hamiltonian systems. In this case (if no outer parameter exist) the related integrable system can contain 1-parameter families of periodic orbits and 2-parameter families of Lagrangian 2-tori. The reduction procedure allows one to reduce locally near a point on the degenerate orbit to studying an integrable family of IHS of lesser dimension depending on related number of parameters. We investigate such bifurcations and after that globalize this study to get a semi-local description.

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Effects of memristor-based coupling in the ensemble of FitzHugh-Nagumo elements

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The aim of our work is to study the impact of electrical and memristor-based couplings (connections through a common field) on the dynamics of a minimal ensemble of neuron-like systems with chemical (synaptic) excitatory couplings. Previously, the authors has proposed in [1] a new method for modelling of chemical synaptic couplings using a rectangular function that describes both the strength of the coupling and the start time and the duration of its impact.This model is quite simple from a computational point of view and allows to organize a phenomenological modelling the actions of chemical synapses that is in a good agreement with the biological principles of their functioning [2]. Described model was tested in the studies [3, 1], in which an ensemble of FitzHugh-Nagumo coupled systems was studied in the case of chemical (synaptic) excitatory connections alone. Depending on the relations between coupling parameters various coexisting regimes were observed: regular spiking in-phase and anti-phase regimes, regimes of sequential activity with different sequences of activations, as well as chaotic 'anti-phase' regimes.

In order to study and describe the effects arising due to taking into account electrical and memristor-based couplings to the ensemble of neurons with chemical synaptic couplings alone, we need to build a correct mathematical model. This model should be both biologically relevant and not very complicated. In this section we introduce such a model. This model can be considered as an extension of the model proposed in [1], which describes interaction of two identical FitzHugh-Nagumo neurons:

$$\begin{aligned} &\epsilon \dot{x_1} = x_1 - {x_1}^3/3 - y_1 + I(\phi_2), \\ &\dot{y_1} = x_1 - a, \\ &\epsilon \dot{x_2} = x_2 - {x_2}^3/3 - y_2 + I(\phi_1), \\ &\dot{y_2} = x_2 - a. \end{aligned} \tag{1}$$

Here the variables x_i and y_i ($i = 1, 2$) are one-dimensional variables: x_i describes the dynamics of membrane potential of i-th element, and y_i, the so-called recovery variable, sets a slow negative feedback for i-th element; also ϵ is a small parameter, $0 < \epsilon << 1$. In further studies we will assume that each of coupled elements is initially (before we set all couplings) in an excitable regime ($a = -1.01$). We also fix $\epsilon = 0.01$.

The chemical synaptic couplings are given by the following formula

$$I(\phi) = \frac{g}{1 + e^{k(\cos(\delta/2) - \cos(\phi - \alpha - \delta/2))}}, \tag{2}$$

where $\phi = \arctan \frac{y}{x}$ is measured in degrees, $0 \leq \phi < 360°$, the parameter g describes the strength of chemical synaptic couplings between elements. For suitable sufficiently large values of k, the coupling function $I(\phi)$ is a smooth function that approximates very well the rectangular wave-pulses. In further studies we will take the following values of chemical coupling parameters: $k = 50$, $g = 0.1$, $\delta = 50°$.

The extended model, in addition to the described above synaptic coupling, takes into account electrical and memristor-based couplings [4]. In this case neurons can exchange signals by setting different electromagnetic field and, thus, additional couplings through magnetic field arise here [5, 6]. In the framework of described approach we will use a flux-controlled memristor [7] with the memductance

$$\rho(\phi) = \frac{dq(\phi)}{d\phi} = k_1 + k_2\phi^2, \tag{3}$$

depending on parameters k_1 and k_2 to simulate such type of coupling. In the case of $k_2 = 0$ this coupling can be viewed as electrical one, while in the case of $k_2 \neq 0$ it is the typical memristor-based coupling.

Finally, the model of two FitzHugh-Nagumo elements interacting via chemical, electrical or memristor-based couplings takes the following form

$$\begin{aligned} &\epsilon \dot{x_1} = x_1 - {x_1}^3/3 - y_1 + I(\phi_2) + \rho(z) \cdot (x_2 - x_1), \\ &\dot{y_1} = x_1 - a, \\ &\epsilon \dot{x_2} = x_2 - {x_2}^3/3 - y_2 + I(\phi_1) + \rho(z) \cdot (x_1 - x_2), \\ &\dot{y_2} = x_2 - a, \\ &\dot{z} = x_1 - x_2. \end{aligned} \tag{4}$$

It is important to note, that when $k_2 = 0$ term $\rho(z)(x_2 - x_1)$ could be rewritten in the form $k_1 \cdot (x_2 - x_1)$ that corresponds to the most common way of description of electrical couplings.

In this study we propose an extended and more precise model of an ensemble of coupled neurons and examine how the additional couplings (both electrical and via the common field) allow us to more effectively manage the dynamics of the ensemble. To confirm this, we carry out one-parameter bifurcation analysis of the model in order to reveal the nature of the effects of additional couplings on previously detected regimes of activity. Using this approach we study how the regimes of in-phase, anti-phase and sequential activity can transform into other regimes of neuron-like activity in the presence of additional couplings. We also show that the presence of electrical and/or memristor-based coupling can lead to the emergence of extreme events related to the appearance of the spiral attractors containing a saddle-focus equilibrium with its homoclinic orbit. In this case interspike intervals can become arbitrarily large when orbits (corresponding to this regime) pass near the saddle-focus.

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About the Bifurcations of the Logistic Equation with Diffusion and Non-linear Multiplier of Delay

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We consider bifurcations in the problem

$$\frac{\partial u}{\partial t} = d\frac{\partial^2 u}{\partial x^2} + ru(t,x)(1 - a(x)u(t-1,x)) \tag{1}$$

with boundare conditions

$$\left.\frac{\partial u}{\partial x}\right|_{x=0} = \left.\frac{\partial u}{\partial x}\right|_{x=1} = 0. \tag{2}$$

For consideration, the function $a(x)$ is chosen

$$a(x) = Cx^{-\alpha}, \quad 0 < \alpha < 1, \quad C > 0. \tag{3}$$

The problem (1), (2) under the condition (3) has a clear biological meaning and models the dynamics of the development of a population of animals living in mountainous areas. For convenience, the function $a(x)$ is normalized so that

$$\int_0^1 a(x)dx = 1 \tag{4}$$

and the parameter C was chosen

$$C = 1 - \alpha. \tag{5}$$

In the problem (1), (2) for $r > 4.05265$, a stable cycle is born. At the left border of the habitat, a small change in the number of individuals is observed, whereas, when approaching the right border, these indicators increase markedly. It is shown that for $x \to 0$ the solution is $u(t, x) \to 0$. A decrease in the degree of in the term with delay increases the amplitude of the oscillations.

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On the classes of stable isotopic connectivity of polar cascades on a torus.

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The problem of the existence of an arc with no more than a countable (finite) number of bifurcations connecting structurally stable systems (Morse-Smale systems) on manifolds is included in the list of fifty Palis-Pugh problems [6] under number 33. The report will present a solution this problem for polar gradient-like diffeomorphisms of a torus.

In 1976, S. Newhouse, J. Palis, F. Takens [3] introduced the concept of a stable arc connecting two structurally stable systems on a manifold. Such an arc does not change its quality properties with little movement. In the same year, S. Newhouse and M. Peixoto [4] proved the existence of a simple arc (containing only elementary bifurcations) between any two Morse-Smale flows. From the result of the work of J. Fleitas [1] it follows that a simple arc constructed by Newhouse and Peixoto can always be replaced by a stable one. For Morse-Smale diffeomorphisms given on manifolds of any dimension, examples of systems that cannot be connected by a stable arc are known. In this connection, the question naturally arises of finding an invariant that uniquely determines the equivalence class of the Morse-Smale diffeomorphism with respect to the connection relation by a stable arc (*is a component of stable connection*).

Consider the class G of polar gradient-like diffeomorphisms on a torus with a fixed nonwandering set. The diffeomorphisms $f_0, f_1 \in G$ are smoothly isotopic to the identity map of the 2-sphere and, therefore, can be connected by some arc $\{f_t : S^2 \to S^2, t \in [0,1]\}$. However, *stability* of such an arc with a finite number of bifurcation values $0 < b_1 < \dots < b_k < 1$ is characterized by the fact that all of its points are structurally stable diffeomorphisms, with the exception of a finite number of bifurcation points, which typically pass through saddle-node bifurcations or flip.

In this report, a stable arc will be constructed connecting any two cascades from the class in question. Note that it was shown in [5] that polar cascades on a two-dimensional sphere are always connected by an arc without bifurcations. For a two-dimensional torus, the situation is different due to the fact that the closures of the invariant manifolds of saddle points of the polar cascade are circles belonging to any previously defined homotopy class. It follows directly from this that in the general case there is no arc without bifurcations between the systems under consideration. Nevertheless, the authors of this paper established the following result.

Theorem 1. *Theorem Any diffeomorphisms $f_0, f_1 \in G$ belong to the same class of stable isotopy connection. Moreover, there exists a stable arc connecting them, all the bifurcation points of which are saddle-nodes.*

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The Depth of the Centre for Continuous Maps on Dendrites

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By *continuum* we mean a compact connected metric space. *Dendrite* is a locally connected continuum without subsets homeomorphic to the circle. Dendrite with a finite set of end points is called *a finite tree*. *A graph* is a continuum if it can be written as the finite union of arcs such that every two of them meet at their end points.

The depth of the centre for continuous maps on one-dimensional continua is studied in [1] – [8]. The depth of the centre of a continuous map on a closed interval and the circle is at most 2 (see, e.g., [1], [2]). In [6] J. Mai, T. Sun proved that the depth of the centre of a continuous map on a finite tree and a graph is at most 2. In [7] H. Kato showed that for any countable ordinal number λ there are a dendrite X and a continuous map $f : X \to X$ such that the depth of the centre of f is λ. In [8] T. Sun and H. Xi proved that the depth of a centre for a continuous map $f : X \to X$ on dendrite X with finite branch points is at most 3. Moreover they constructed a dendrite X with finite branch points and a continuous map $f : X \to X$ such that the depth of the centre of f is 3.

Let X be a dendrite, $f : X \to X$ be a continuous map.

Denote by $E^{(0)}(X)$ the set of end points of a dendrite X. For any ordinal number $\lambda \geq 1$ we define $E^{(\lambda)}(X)$ as follows:
if $\lambda = \alpha + 1$, then we denote by $E^{(\lambda)}(X)$ the set of limit points of the set $E^{(\alpha)}(X)$;
if λ is a limit ordinal number, then we set $E^{(\lambda)}(X) = \bigcap\limits_{\alpha<\lambda} E^{(\alpha)}(X)$.
There is a countable ordinal number γ such that $E^{(\gamma)}(X) = E^{(\gamma+1)}(X)$. The minimal such γ is called *the rank of a dendrite X*.

In the report the relationship between the depth of the centre of a continuous map on a dendrite and a rank of a dendrite is studied.

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Deformations of functions on surfaces

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Let M be a compact surface, $f \in C^\infty M\mathbb{R}$ be a Morse function and Γ_f its Kronrod-Reeb graph. Denote by $\mathcal{O}(f) = \{f \circ h \mid h \in \mathcal{D}(M)\}$ the orbit of f with respect to the natural right action of the group of diffeomorphisms $\mathcal{D}(M)$ on $C^\infty M\mathbb{R}$, and by $\mathcal{S}(f) = \{h \in \mathcal{D}(M) \mid f \circ h = f\}$ the stabilizer of this function.

S. Maksymenko [1], proved that

- if f has at least on saddle critical point, then the connected components of $\mathcal{S}(f)$ are contractible;
- otherwise, every path component of $\mathcal{S}(f)$ is homotopy equivalent to the circle.

In that paper is was also shown that for **generic** Morse function f connected components of its orbit $\mathcal{O}(f)$ is homotopy equivalent to

- $(S^1)^k \times SO(3)$ for some k if M is either a 2-sphere or a projective plane;
- and to $(S^1)^k$ for some k in all other cases;

Further E. Kudryavtseva [2] extended that result proving that for **arbitrary** Morse function f there exists a free action of a certain finite group G on the torus $(S^1)^k$ such that the connected components of orbits $\mathcal{O}(f)$ are homotopy equivalent to the spaces of the form $(S^1)^k/G \times SO(3)$ if $M = S^2$ or $\mathbb{R}P^2$ and $(S^1)^k/G$ in all other cases.

The aim of the talk is to describe recent progress in the computations of homotopy types of the fundamental group $\pi_1\mathcal{O}(f)$ and the groups G.

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Generalization of Milnor-Thurston's theorem on continuity of topological entropy to discontinuous maps

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The problem on continuity and monotonicity of topological entropy for certain families of chaotic dynamical systems is still important and attracting because this quantity is responsible for complexity of the limit behavior of orbits. In this talk, we consider this problem for the class of piecewise monotone, piecewise smooth maps on the interval endowed with C^1-topology. It is well-known that the topological entropy is lower semi-continuous on the space of piecewise monotone maps with C^0-topology. However, there are many examples of dynamical systems for which the jumps of topological entropy take place under arbitrarily small perturbations of the initial system (which lead to instant complication of the orbit behavior). Hence one cannot expect topological entropy to be a continuous function without any additional assumptions. For instance, as shown in Milnor and Thurston' paper [1], the topological entropy depends continuously on the map in the class of smooth piecewise monotone maps with C^1-topology. In [2], M. Malkin proved that in the class of discontinuous Lorenz maps, the topological entropy also has no jumps under the assumption that the set of the preimages of the discontinuity point is dense (or, alternatively, if the initial map has positive entropy).

Our aim is to generalize Milnor-Thurston's theorem to the class of piecewise-monotone maps with finitely many points of discontinuity. We show that for piecewise monotone piecewise smooth maps f, the topological entropy $h_{top}(f)$ is continuous as a function of f under the next (natural) assumption. Namely, the following holds:

Theorem. *In the space of piecewise monotone, piecewise C^1-smooth maps with C^1-topology and with zero one-sided derivatives at the discontinuity points c_i, i.e., with*

$$\lim_{x \to c_i+0} f'(x) = \lim_{x \to c_i-0} f'(x) = 0,$$

the topological entropy depends continuously on f.

We also give counterexamples in the cases when the assumption of the above theorem is violated (along with providing exact estimates of the jumps of topological entropy) and discuss the relationship with one-dimensional and high-dimensional perturbations.

A family of difference equations $\Phi_\lambda(y_n, y_{n+1}, \ldots, y_{n+m}) = 0$, $n \in \mathbf{Z}$, of order m with parameters λ (multidimensional, in general) is considered near nonperturbed value λ_0 at which the function Φ is in two variables only :

$$\Phi_{\lambda_0}(x_0, \ldots, x_m) = \xi(x_N, x_{N+L}),$$

where $0 \leq N, N+L \leq m$. It is also assumed that the implicit function induced by the equation $\xi(x, y) = 0$ has a piecewise monotone, piecewise smooth branch $y = \varphi(x)$ with positive topological entropy: $h_{top}(\varphi) > 0$. Under above assumptions it was proved in [3] that for perturbed difference equations with parameters λ near λ_0, there is a closed (in the product topology) shift-invariant set of solutions such that the shift map restricted to this set has positive topological entropy which approximates the value $h_{top}(\varphi)/|L|$.

Now in the talk we show that if the branch φ is a Lorenz map (nonsymmetric, in general) then it has at most two ergodic measures of maximal entropy $h_{top}(\varphi)$, and for perturbed difference equations these measures can be continued in order to construct invariant measures with positive entropy (sufficiently close to $h_{top}(\varphi)/|L|$)) when restricted to correspondent invariant sets of solutions.

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Fermi-like acceleration growth in nonholonomic systems

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The equations of motion in the Suslov problem [1] with two rotors, which govern the evolution of the angular velocity of a rigid body are represented as

$$\dot{u} = -vu - K(t)v - \dot{\Lambda}(t), \quad \dot{v} = u^2 + K(t)u, \tag{1}$$

where $K(t)$ and $\Lambda(t)$ are periodic functions with the same period T, which define the angular velocities of the rotors.

Of particular interest is the question of whether the reduced system has trajectories unbounded on the plane (v, u) (i.e., trajectories which leave any bounded region on the plane). In this case the angular velocity of the carrying body and hence the kinetic energy must increase indefinitely (in absolute values) with time.

The case $K(t) = 0$. Then the reduced system coincides with the reduced system describing another nonholonomic system: a Chaplygin sleigh with gyrostatic momentum. For this system the following theorem holds [2]:

Theorem 1. *Let $\Lambda(t)$ be a periodic function. Let us calculate the following average value:*

$$\langle \dot{\Lambda}^2 \rangle = \frac{1}{T} \int\limits_0^T \dot{\Lambda}^2(t)dt.$$

If at the initial time $v > 0$, then the function $v(t)$ increases indefinitely and $u(t)$ tends to zero:

$$v(t) = Ct^{\frac{1}{3}} + o\big(t^{\frac{1}{3}}\big), \quad u(t) = -C\dot{\Lambda}(t)t^{-\frac{1}{3}} + o\big(t^{-\frac{1}{3}}\big), \; C = \big(3\langle \dot{\Lambda} \rangle^2\big)^{\frac{1}{3}}. \tag{2}$$

If $K(t) \neq 0$, then for the reduced system the following theorem holds:

Theorem 2. *If the average*

$$\langle G \rangle = \frac{2}{T} \int\limits_0^T K(t)\dot{\Lambda}(t)dt > 0, \tag{3}$$

then the reduced system has trajectories unbounded in v, which have the following asymptotics:

$$v(t) = Ct^{\frac{1}{2}} + o(t^{\frac{1}{2}}), \; u(t) = -K(t) + o(t^{-\frac{1}{2}}), \; C = \sqrt{\langle G \rangle}. \tag{4}$$

If $\langle G \rangle < 0$, then there are no unbounded trajectories. The case $\langle G \rangle = 0$ requires a separate analysis.

Numerical experiments for the one-parameter family of functions:

$$\Lambda(t) = \alpha \cos t - \frac{1}{2}\sin t, \quad K(t) = \sin t, \quad \alpha = \mathrm{const}$$

show that, depending on α, the reduced system exhibits the following qualitatively different dynamical regimes.

— As $t \to +\infty$, all trajectories tend to one or several periodic solutions of the system.

— Chaotic oscillations: the system exhibits a strange attractor.

— An intermediate situation: noncompact and bounded chaotic trajectories are observed.

— Speedup: except for the unstable fixed points of the map, all trajectories are noncompact, and $v \to +\infty$ as $t \to +\infty$. Numerical experiments show that their asymptotics is described by relations.

The work of I.S. Mamaev and I.A. Bizyaev was carried out within the framework of the state assignment of the Ministry of Education and Science of Russia (1.2405.2017/4.6 and 1.2404.2017/4.6 respectively).

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The Generalized Integral Minkowski Inequality and L_p -norms of Some Special Functions

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In this report we prove two theorems concerning L_p -norms of some functions $f(x)$:

$$\|f(x)\|_{L_p([A,B])} \equiv \left[\int_A^B |f(x)|^p\,dx\right]^{\frac{1}{p}}, \qquad p > 1\,,$$

namely,

Theorem 1. Let $I_0(z)$ is the modified Bessel function of the first kind and zero order [1] then the following inequality is true:

$$\|I_0(\cos x)\|_{L_p([0,2\pi])} \leq \int_0^{2\pi} I_0^{\frac{1}{p}}(p\cos x)\,dx\,.$$

Theorem 2. Let $\beta > \alpha > 0$ and

$$M(\alpha,\beta,x) = \frac{1}{B(\alpha,\beta-\alpha)}\int_0^1 e^{xy}y^{\alpha-1}(1-y)^{\beta-\alpha-1}\,dy$$

is a confluent hypergeometric function [1] where

$$B(\xi,\eta) = \int_0^1 x^{\xi-1}(1-x)^{\eta-1}\,dx$$

is the beta function [1] then the following inequality is true:

$$\|M(\alpha,\beta,x)\|_{L_p([-1,1])} \le \int_0^1 \frac{x^{\alpha-1}(1-x)^{\beta-\alpha-1}}{B(\alpha,\beta-\alpha)} \left(2\,\frac{\sinh p\,x}{p\,x}\right)^{\frac{1}{p}} dx$$

The corner stone for proof of these theorems is the generalized integral Minkowski inequality for arbitrary continuous function $f(x,y)$ [2]:

$$\left\{\int_a^b \left[\int_c^d |f(x,y)|\,dy\right]^p dx\right\}^{\frac{1}{p}} \le \int_c^d \left[\int_a^b |f(x,y)|^p\,dx\right]^{\frac{1}{p}} dy\,.$$

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On 2-dimensional expanding attractors of A-flows on 3-manifolds

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We consider closed 3-manifolds supporting A-flows with 2-dimensional expanding attractors. The natural question is what closed 3-manifolds admits A-flows with 2-dimensional expanding attractors. The main results are the following statements.

Theorem 1. Given any closed 3-manifold M^3, there is an A-flow f^t on M^3 such that the non-wandering set $NW(f^t)$ consists of a non-orientable two-dimensional expanding attractor and trivial basic sets.

Theorem 2. There is a nonsingular A-flow f^t on a 3-sphere S^3 such that the non-wandering set $NW(f^t)$ contains an orientable two-dimensional expanding attractor.

The study was implemented in the framework of the Basic Research Program at the National Research University Higher School of Economics (HSE) in 2019.

Concentration of Haar measure and estimate of medians of matrix elements of real linear irreducible representations of classical compact Lie groups

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The main question addressed in this report is: “what kind real numbers appear as the median of matrix elements representations of classical matrix compact Lie groups?”

First of all recall important P. Levy result on concentration of the invariant measure on the Euclidean sphere S^{n-1} [1, 2].

Theorem (P. Levy). *Let $f\colon S^{n-1} \to \mathbb{R}$ be Lipschitz function with Lipschitz constant L and let X be a uniform measure on S^{n-1}. Then for m_f denoting the median of function f with respect to measure on S^{n-1},*

$$|Ef(X) - m_f| \leq L\sqrt{\pi/(n-2)}$$

and

$$P[|f(X) - Ef(X)| > Lt] \leq \exp(\pi - nt^2/4).$$

That is, a Lipschitz function on the sphere S^{n-1} essentially constant.

Above $Ef(X)$ is the mean of random variable $f(X)$ and we say that real number m_f is a median of function f if $P(\{f \leq m_f\}) \geq 1/2$ and $P(\{f \geq m_f\}) \geq 1/2$. It is clear that the set of median of f is a closed and bounded interval on real line $\mathbb{R}$.

The Levy mean $lm(f,\mu)$ of f with respect to measure μ is defined to be $lm(f,\mu) = (\underline{m} + \overline{m})/2$, where $\underline{m}$ is the minimum of medians of f and $\overline{m}$ the maximum of medians of f.

Our goal is to obtain the estimate of medians probability distribution of matrix elements of the real irreducible representations of the classical compact Lie groups $G = SO(n)$, $SU(n)$ and $Sp(n)$ under probabilistic Haar measure on G. More precisely, let X be distributed according to Haar measure on G and let A be fixed $n \times n$ matrix over real field $\mathbb{R}$, where $n = \dim\rho$. Assume also that $W = \mathrm{Tr}(A\rho(X))$ be matrix element of ρ considered as a random variable on G.

Our main result is

Theorem. *Medians of the matrix elements $W = \mathrm{Tr}(A\rho(X))$ of real irreducible representation ρ classical compact Lie group G with respect probability Haar measure satisfy following inequality:*

$$|m_f| \leq \sqrt{2}\,\mathrm{Tr}(AA^*)/\sqrt{\dim\rho}.$$

Here an A is linear operator on representation space and A^* is its conjugate operator.

Proof of theorem based on the orthogonality relations for matrix elements representations and the Chebyshev inequality [3].

From the geometric point of view it is impotent to find more structured subsets on which functions are concentrated. Some preliminary results of such kind see in [4, 5].

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Self-reliance of attractors

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It is known that there is the locally residual set of C^3 diffeomorphisms with maximal attractors of positive Lebesgue measure [1]. We can improve this result without understanding the proof. So called self-reliance of attractors allows us to conclude that there is an open set of C^1-systems with thick attractors.

The idea of self-reliance is basically about standard topological tricks with G_δ sets, and will be clear after few examples.

This technique also can be used for the research of Milnor attractors of C^1-Anosov diffeomorphisms [2], and to establish connection between unstable Milnor attractors and so-called Takens Last Problem, introduced in [3].

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Infinitesimal symmetries of special multi-flags

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Special m-flags, $m \geq 2$, constitute a natural follow-up to Goursat flags. The latter compactify (in certain precise sense) the contact Cartan distributions on the jet spaces $J^r(1,1)$, while the former do the same with respect to the jet spaces $J^r(1,m)$.

Sequences of Cartan prolongations of rank-$(m+1)$ distributions are the key players in producing (only locally) virtually all rank-$(m+1)$ distributions generating special m-flags. Immediately there emerges an immense tree of singularities of positive codimensions, all of them adjoining the unique open dense Cartan-like strata.

The local classification problem is well advanced for the Goursat flags, most notably after the work [2]. However, it is much less advanced for special multi-flags. The complete classification of them was given in [4]: in length $r = 3$ for all $m \geq 2$, and in length 4 for $m = 2$ (the number of equivalence classes 34). After the year 2010 researchers were aiming at defining various invariant stratifications in the spaces of germs of special multi-flags. The actual state of the art is reflected in [1].

A new approach [5] to the classification starts with the effective (recursive) computation of all *infinitesimal symmetries* of special multi-flags. (They are computed explicitly only for $m = 2$, but implicitly for all $m > 1$. The recursive patterns depend uniquely on the so-called singularity classes of special m-flags defined still in [3]. Those classes are coarser (hence fewer) than the RVT classes recapitulated in [1].

Polynomial visualizations of objects in the singularity classes are called EKR's (Extended Kumpera-Ruiz). They 'only' feature finite families of real parameters. Then the classification problem is rephrased as a search for ultimate normalizations among such parameter families. Having an explicit hold of the infinitesimal symmetries at each prolongation step, the freedom in varying those parameters is being reduced to solvability questions of (huge) systems of linear equations.

In fact, that linear algebra involves only partial derivatives, at the reference point, of the first $m+1$ components of a given infinitesimal symmetry (which initially, by the classical Backlund theorem, are completely free functions of $m+1$ variables). Keeping the preceding part of a [germ of a] flag in question frozen imposes a sizable set of linear conditions upon those derivatives up to certain order. Then some other linear combinations of them appear, or not, to be free – just in function of the local geometry of the prolonged distribution. This, in short, determines the scope of possible normalizations in the new (emerging from prolongation) part of EKR's.

When algorithmized to the software level, the present new approach will give an answer filling in the gap in knowledge as of 2010: on one side the local finite classification of special 2-flags known in lengths not exceeding four ([4]), on the other side the existence of a continuous numerical modulus of that classification in length seven.

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On limit cycles and resonances in systems close to nonlinear Hamiltonian

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The purpose of this report is to give a short review of my basic results, concerning systems close to nonlinear two-dimensional Hamiltonian.

- Limit cycles [1], [2]
- Resonances in periodic case [1], [2], [3]:
  - bifurcations in non-degenerate zones

– bifurcations in degenerate zones [4], [5]

- Resonances in quasi-periodic case [6]

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On parametric quasi-periodic perturbations of two-dimensional Hamiltonian systems

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We consider non-conservative quasi-periodic perturbations of 2-dim Hamiltonian systems that include terms depending on both t and the phase variables:

$$\begin{aligned} \dot{x} &= \frac{\partial H(x,y)}{\partial y} + \varepsilon[g_0(x,y) + a(\omega_1 t, ..., \omega_m t) g_1(x,y)]; \\ \dot{y} &= -\frac{\partial H(x,y)}{\partial x} + \varepsilon[f_0(x,y) + a(\omega_1 t, ..., \omega_m t) f_1(x,y)]. \end{aligned} \tag{1}$$

Here H, g_0, f_0, g_1, f_1 are nonlinear and assumed to be smooth while a and b are continuous and quasi-periodic; ε is a small parameter. We examine a certain domain $D = \{(x,y)|h_- \leq H(x,y) \leq h_+\} \subset \mathbf{R}$ filled with closed phase curves of the unperturbed system. Resonance levels of energy are determined by the condition $n\omega(h_{n,\mathbf{k}}) = k_1\omega_1 + ... + k_m\omega_m, n, k_1, ..., k_m \in \mathbf{Z}$, where $\omega(h)$ is the natural frequency of the unperturbed system. In the $\sqrt{\varepsilon}$-neighborhood of a fixed resonance level $h = h_{n\mathbf{k}}$ we obtain the following 2-dim averaged system

$$\begin{aligned} \dot{u} &= A(v) + \sqrt{\varepsilon}\sigma(v)u; \\ \dot{v} &= b_1 u + \sqrt{\varepsilon} b_2 u^2. \end{aligned} \tag{2}$$

$A(v), \sigma(v)$ are $2\pi/n$–periodic functions. In contrast to the previous studies [1,2], $\sigma(v)$ is signalternative for the perturbation under consideration. This may result in the emergence of limit cycles in (2) that correspond to quasi-periodic solutions in the initial system with $m+1$ frequencies. The conditions of such tori existence are established. We use the equation

$$\ddot{x} + x + x^3 = \varepsilon((p_1 - x^2)\dot{x} + (1 + x\dot{x}) \sin t \sin \sqrt{5}t) \tag{3}$$

to illustrate the study.

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Analytical approach to synchronous states of globally coupled noisy rotators

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The Kuramoto model is a paradigmatic model to study synchronization phenomena. In the thermodynamic limit, stationary synchronized states can be found analytically for an arbitrary distribution of natural frequencies [1]. Another analytical approaches also have restrictions since they apply only for identical oscillators like the Watanabe–Strogatz theory, or for a population with a Lorentzian distribution of frequencies like the Ott–Antonsen ansatz.

An important generalization of the Kuramoto model considers an ensemble of globally coupled rotators. It is in particular relevant for modeling power grid networks [2]. Synchronization features of deterministic and noisy globally coupled rotators have been widely studied. However, for noisy coupled rotators, so far the stationary distributions were found only numerically.

We present a fully analytical approach to the problem of globally coupled noisy rotators for small inertia. Based on this analysis, we derive the asymptotic form with respect to the small order parameter and use it to classify the transition to synchrony as a supercritical or a subcritical one.

We consider an ensemble of N globally coupled rotators characterized by their angles φ_n and velocities $\dot{\varphi}_n$ $(n = 1, 2, \ldots, N)$. The rotators are coupled via the complex mean field and obey equations of motion

$$R \equiv re^{i\psi} = \frac{1}{N}\sum_{n=1}^{N} e^{i\varphi_n}, \quad \mu\ddot{\varphi}_n + \dot{\varphi}_n = \omega_n + \varepsilon r \sin(\psi - \varphi_n) + \sigma\xi_n(t). \tag{1}$$

Parameter μ describes the mass of rotators. Parameter ε is the coupling strength. Parameters ω_n describe torques acting on rotators; we assume them to be distributed with a density $g(\omega)$.

The rotators are acted upon by the independent white Gaussian noise forcing $\sigma\xi_n(t)$ with equal amplitudes σ, zero means $\langle\xi_n(t)\rangle = 0$, and auto-correlations $\langle\xi_n(t_1)\,\xi_{n'}(t_2)\rangle = 2\delta_{nn'}\delta(t_1 - t_2)$.

A method of matrix continuous fractions was employed to solve Eqs. (1) in the thermodynamic limit ($N \to \infty$). The main result is the closed-form formula to the first order in the mass parameter μ for the subgroup order parameter $a_{0,1}(\nu, A)$ for the case of frequency distributions $g(\omega)$ symmetric with respect to ω_0,

$$a_{0,1}(\nu, A) = \frac{1}{\sqrt{2\pi}} \frac{I_{1+i\frac{\nu}{\sigma^2}}\left(\frac{A}{\sigma^2}\right)}{I_{i\frac{\nu}{\sigma^2}}\left(\frac{A}{\sigma^2}\right)} \left(1 - \mu\frac{\sigma^2}{\pi} \frac{\sin\left(i\pi\frac{\nu}{\sigma^2}\right)}{I_{-i\frac{\nu}{\sigma^2}}\left(\frac{A}{\sigma^2}\right) I_{i\frac{\nu}{\sigma^2}}\left(\frac{A}{\sigma^2}\right)}\right) + o(\mu). \tag{2}$$

The general parametric representation of the order parameter as a function of the coupling constant,

$$r = \sqrt{2\pi}\int \mathrm{d}\nu g(\omega_0 + \nu)\, a^*_{0,1}(\nu, A), \quad \varepsilon = \frac{A}{r}. \tag{3}$$

The nontrivial branch of solutions $r(\epsilon)$ starts at

$$\varepsilon_c^{(1)} = \left(\frac{1}{2}\int \mathrm{d}y \frac{g\left(\omega_0 + \sigma^2 y\right)}{1 + y^2}\left(1 - \mu\sigma^2 y^2\right)\right)^{-1}. \tag{4}$$

The character of the synchronization transition depends on the sign of the coefficient C_1,

$$C_1 = -\frac{1}{8\sigma^4}\int \mathrm{d}y \frac{g\left(\omega_0 + \sigma^2 y\right)}{\left(1 + y^2\right)^2\left(4 + y^2\right)}\left(2\left(1 - 2y^2\right) - \mu\sigma^2 y^2\left(13 + y^2\right)\right). \tag{5}$$

Supercritical transition occurs for $C_1 < 0$. Here one observes a continuous (second-order) transition with the solution branch existing for $\varepsilon > \varepsilon_c^{(1)}$

$$r = C_0^2\sqrt{\left(\varepsilon_c^{(1)} - \varepsilon\right)/C_1}. \tag{6}$$

Subcritical transition occurs for $C_1 > 0$. Here, the branch of solutions exists for $\varepsilon < \varepsilon_c^{(1)}$.

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Stability by linear approximation of time scale dynamical systems

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We study systems on time scales that are generalizations of classical differential or difference equations and appear in numerical methods. We consider linear systems and their small nonlinear perturbations. In terms of time scales and of eigenvalues of matrices we formulate conditions, sufficient for stability/instability by linear approximation. We use techniques of central upper Lyapunov exponents (a common tool of the theory of linear ODEs) to study stability of solutions. We develop a new technique to demonstrate that methods of non-autonomous linear ODE theory may work for time-scale dynamics.

The talk is based on joint paper with Sergey Kryzhevich [1].

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Phase change for perturbations of Hamiltonian systems

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We consider a small perturbation of a Hamiltonian system with one degree of freedom that has a separatrix loop. We also assume that the perturbation is such that the solutions starting outside the separatrix loop approach it and eventually cross it. For study of such systems see, e.g., [1] and references therein.

We are interested in the change of phase while approaching the separatrix. A parameter called the pseudo-phase ([2]) describes the phase at the moment of separatrix crossing. In [2] a formula for the dependence of the pseudo-phase on the initial conditions was obtained for slow-fast Hamiltonian systems. We show that a similar formula also holds for our case. The main tool we use is the averaging method.

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On universality of discrete Lorenz attractors in three-dimensional maps

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In papers [1, 2, 3, 4] a class of maps with constant Jacobian was studied, called three-dimensional Henon maps:

$$\bar{x} = y, \;\; \bar{y} = z, \;\; \bar{z} = Bx + f(y, z). \tag{1}$$

It was shown that when map (1) possesses a fixed point x_0 with eigenvalues $(-1, -1, +1)$, and condition

$$G = (c - a)(a - b + c) > 0 \tag{2}$$

is fulfilled, where $a = 1/2 f''_{yy}(x_0)$, $b = f''_{yz}(x_0)$, $c = 1/2 f''_{zz}(x_0)$, then a discrete Lorenz attractor is born near x_0 in arbitrary small perturbations. The proof is based on the fact that the second iterate of map (1) can be approximated by a flow of a system of differential equations, which, in turn, can be brought to the form of Shimizu-Morioka system by rescaling coordinates, parameters and time. Existence of the Lorenz attractor in the latter implies the existence of the discrete Lorenz attractor in map (1), because all the operation performed preserve the properties of attractivity and chain-transitivity as well as the pseudo-hyperbolic structure.

It can be easily seen by [3, 4] that the violation of condition (2) implies immediately the existence of a discrete Lorenz repeller in map (1). It follows from the fact that when $G < 0$, the map also can be approximated by the Shimizu-Morioka system, but in this case the scaling factor of time should be negative. Also it can be checked that if a map has $G > 0$, then its inverse has $G < 0$, and the attractor becomes a repeller.

In the present work we aim to establish the existence of discrete Lorenz attractors in systems with $G < 0$, namely we take the following map with $G = -1$ (this is the inverse to the 3D Henon map studied in [1, 2]):

$$\bar{x} = y, \;\; \bar{y} = z, \;\; \bar{z} = Bx + M + Az - y^2. \tag{3}$$

In this system we find a period-4 orbit with multipliers $(-1, -1, +1)$ and establish that condition (2) is fulfilled at the points of this orbit for the fourth iterate of map (3), this means that the attractor exists near it.

This result, in particular, implies that discrete Lorenz attractors are typical for bifurcations of homoclinic and heteroclinic cycles in the case when the effective dimension of the problem is at least three.

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Integrable and Non-Integrable Equations of the Korteweg-de Vries Hierarchy

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Here we consider the generalized Korteweg-de Vries (gKdV) equation in the form

$$\partial u/\partial t + su^n \partial u/\partial x + \partial^3 u/\partial x^3 = 0$$

or

$$\partial u/\partial t + s|u|^p \partial u/\partial x + \partial^3 u/\partial x^3 = 0,$$

where n is integer, $p > 0$ – is an arbitrary constant, and $s = \pm 1$. The Korteweg-de Vries (KdV, $s = 1, n = 1$) equation and modified Korteweg-de Vries (mKdV, $s = \pm 1, n = 2$) equation are famous members of this series of equations. They are integrable and thoroughly investigated.

The gKdV equations with higher order of the nonlinearity, $n > 2$, may appear in the application to the hydrodynamics of stratified fluid [1]. Some versions of the gKdV equation contain non-integer values of p; for instance, the power $p = 1/3$ is present in the Schamel equation applicable to ion-acoustic waves which interact with resonant electrons [2]. The log-KdV equation for solitary waves in FPU lattices can be mentioned in addition [3]. In all papers cited above the main attention was paid to the soliton dynamics, their stability and interactions. Dynamics of periodic and modulated wave packets in KdV-like systems is less studied. Two problems are discussed here.

1. Dispersionless limit of the generalized KdV equation is a generalized Hopf equation. The general explicit procedure to find the Fourier spectrum is described. In the case of a sinusoidal initial condition all expressions are found in closed form. These results are presented in [4]. The asymptotic shape of the spectrum of the breaking Riemann wave is found [5].

2. If the wave amplitude is small (wave dispersion prevails), the standard approach to investigate the stability of weakly modulated wave trains is to derive the nonlinear Schrödinger equation (NLS) and to determine its type. For the classic KdV equation the modulations are described by the defocusing NLS equation, and therefore waves packet are stable [6]. In the case of the mKdV equation with $s = +1$ a wave train is modulationally unstable, what leads to the generation of rogue waves [7]. In this work such analysis is extended to the gKdV equation. It is discussed in the context of nonlinear mechanisms of rogue wave formation. These results are summarized in the recent publications [8].

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On 4-dimensional flows with wildly embedded invariant manifolds of a periodic orbit

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Qualitative study of dynamical systems reveals various topological constructions naturally emerged in the modern theory. For example, Cantor set with cardinality of continuum and Lebesgue measure zero as expanding attractor or contracting repeller. Also, a curve in 2-torus with irrational winding number, which is not a topological submanifold but is injectively immersed subset, can be found being invariant manifold of Anosov toral diffeomorphism's fixed point.

Another example of intersection of topology and dynamics is the Artin-Fox arc [1] appeared in work by D. Pixton [2] as the closure of a saddle separatrix of a Morse-Smale diffeomorphism on the 3-sphere. A wild behavior of the Artin-Fox arc complicates the classification of dynamical systems, it does not admit already a combinatorial description like to Peixoto's graph [3] for 2-dimensional Morse-Smale flows.

It is well known that there are no wild arcs in dimension 2. In dimension 3 they exist and can be realize as invariant set for a discrete dynamics, in different from regular 3-dimensional flows, which do not possess wild invariant sets. The dimension 4 is very rich. Here wild objects appear both for discrete and continuous dynamics. Despite the fact that there are no wild arcs in this dimension, there are wild objects of co-dimension 2. So the closure of 2-dimensional saddle separatrix can be wild for 4-dimensional Morse-Smale system (diffeomorphism or flow). Such examples recently were constructed by V. Medvedev and E. Zhuzoma [4]. T. Medvedev and O. Pochinka [5] shown as wild Artin-Fox 2-dimension sphere appears as closure of heteroclinic intersection of Morse-Smale 4-diffeomorphism.

In the present paper we prove that the suspension under a non-trivial Pixton's diffeomorphism provides a 4-flow with wildly embedded 2-dimensional invariant manifold of a periodic orbit. Moreover, we show that there are countable many different wild suspensions.

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Combinatorial Ricci flow for degenerate circle packing metrics

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Chow and Luo [1] in 2003 had shown that the combinatorial analogue of the Hamilton Ricci on surfaces under certain conditions converges to Thruston's circle packing metric of constant curvature. The combinatorial setting includes weights defined for edges of a triangulation. Crucial assumption in the paper [1] was that the weights are nonnegative. Recently we have shown that same statement on convergence can be proved under weaker condition, see [3]

On the other hand, for weights not satisfying conditions of Chow-Luo's theorem we observed in numerical simulation a degeneration of the metric with certain regular behaviour patterns, [2].

In the talk, based on joint papers with Ruslan Pepa, we introduce degenerate circle packing metrics, and under weakened conditions on weights, prove that under certain assumptions an analogue of the combinatorial Ricci flow for any initial metric has a unique limit metric with a constant curvature outside of singularities. [4]

Assume T is a triangulation of a closed suface X. We assume that a lift of a closed face or an edge to the universal cover $\tilde{X}$ is an embedding. Denote the sets of vertices, edges and faces of T by V, E, F correspondingly. Divide the set of vertices into a disjoint union $V = V_n \sqcup V_d$, such that there is no edge connecting two vertices from V_d. Without loss of generality we can assume $V_n = \{A_1, \ldots, A_M\}$ and $V_d = \{A_{M+1}, \ldots, A_N\}$. Vertices from V_n are called *nondegenerate* and vertices from V_d are called *degenerate*. Call a cell of T (that is edge or face) *nondegenerate* iff all its vertices are nondegenerate, and *degenerate* otherwise. Denote the set of (non)degenerate edges and faces by E_d (E_n) and F_d (F_n), correspondingly. Clearly, $E = E_n \sqcup E_d$ and $F = F_n \sqcup F_d$.

A weight is a function $w : E_n \to (-1, 1]$. Fix a triple (X, T, w). A (degenerate) circle packing metric is defined by a collection of numbers $r = (r_1, r_2, \ldots, r_N)$, where $r_j > 0$ for $1 \le j \le M$ and $r_j = 0$ for $M + 1 \le j \le N$. This definition differs from the classical circle packing metric where all r_j are positive, see [1]. For the Euclidean background define the length of an edge connecting two vertices A_i and A_j by the formula $l_{ij}^2 = r_i^2 + r_j^2 + 2r_i r_j w_{ij}$. For a degenerate edge one of the numbers r_i or r_j is zero, therefore the last summand is assumed to be zero although the weight w_{ij} is not defined. Moreover, if $r_i = 0$ then $l_{ij} = r_j$. The curvature K_i at the vertex A_i is defined in a usual way:

$$K_i = 2\pi - \sum_{\triangle A_i A_j A_k \in F} \angle A_k A_i A_j.$$

The curvature at a degenerate vertex $A_i \in V_d$ does not depend on r and can be expressed in terms of the weight w:

$$K_i = 2\pi - \sum_{\triangle A_i A_j A_k \in F} (\pi - \arccos(w_{jk})).$$

The combinatorial Ricci flow is the system of ODE

$$\frac{dr_i}{dt} = -K_i r_i,\ i = 1, \dots, M.$$

For $i = M+1, M+2, \dots, N$ we have $r_i = \frac{dr_i}{dt} = 0$, hence in the previous formula we can assume $1 \le i \le N$.

For a degenerate metric define the *averaged curvature* K^{av}:

$$K^{av} = \frac{1}{M}\left(2\pi\chi(X) - \sum_{j=M+1}^{N} K_j\right).$$

The *normalized combinatorial Ricci flow* is the system of ODE

$$\frac{dr_i}{dt} = -(K_i - K^{av})r_i,\ i = 1, \dots, M.$$

The normalized and non-normalized Ricci flows are in a certain sence equivalent. Namely, functions $r_i(t), i = 1, \dots, M$ are a solution of the non-normalized flow iff functions $e^{K^{av}t} r_i(t)$ are solution for normalized flow.

For the hyperbolic background geometry the length of the edge e_{ij} joining vertices A_i and A_j is defined by the equation $\cosh l_{ij} = \cosh r_i \cosh r_j + \sinh r_i \sinh r_j w_{ij}$. As in the Euclidean case for degenerate edge one of the radii r_i or r_j is zero so the last summand is assumed to be zero though the weight w_{ij} is undefined. Clearly for $r_i = 0$ one has $l_{ij} = r_j$. The curvature K_i at the vertex A_i is defined by the same formula as in Euclidean case.

Hyperbolic combinatorial Ricci flow — is the system of ODE

$$\frac{dr_i}{dt} = -K_i \sinh r_i,\ i = 1, \dots, M.$$

For $i = M+1, M+2, \dots, N$ one has $r_i = \frac{dr_i}{dt} = 0$, hence in this equations one can assume $1 \le i \le N$.

We say that a weight function satisfies condition (W), any face of the triangulation satisfies one of the following conditions:

(a) the face is nondegenerate and all the weights of its edges are nonnegative;

(b) the face is nondegenerate, exactly one weight α of its edges is negative, two others weights β, γ are positive, and $\alpha + \beta\gamma \ge 0$;

(c) the face is degenerate and the weight of the nondegenerate edge of the face is not equal to 1.

Theorem 1. *Suppose X is a closed surface with a triangulation T and a weight w, satisfying the condition (W).*

The solution to the normalized Ricci flow converges for any initial metric iff for any proper subset $I \in V_n$,

$$|I|K^{av} + \sum_{j \in D_I} K_j > -\sum_{(e,v) \in Lk(I \cup D_I)} (\pi - \arccos w(e)) + 2\pi\chi(F_{I \cup D_I}). \tag{1}$$

Furthermore, if the solution converges, then it converges exponentially fast to the metric with $K_i = K^{av}$, $i = 1, \dots, M$.

Theorem 2. *Suppose X is a closed surface of negative Euler characteristic with a triangulation T and a weight w, satisfying the condition (W).*

The solution to the hyperbolic Ricci (1) flow converges for any initial metric iff for any subset $I \in V_n$,

$$\sum_{j \in D_I} K_j > - \sum_{(e,v) \in Lk(I \cup D_I)} (\pi - \arccos w(e)) + 2\pi \chi(F_{I \cup D_I}). \tag{2}$$

Furthermore, if the solution converges, then it converges exponentially fast to the metric with $K_i = 0$, $i = 1, \ldots, M$.

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On Interactions Between Flexural and Torsion Vibrations of a Bar

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In the paper we consider a boundary-value problem for equations of flexural-torsional vibrations of a bar, described by the system of two differential equations in the domain $Q = \{0 < x < l,\, 0 < t < T\}$:

$$E\,I\,\frac{\partial^4 y}{\partial x^4} + \rho\,A\,\frac{\partial^2 y}{\partial t^2} - \varepsilon\,\rho\,A\,\frac{\partial^2 \theta}{\partial t^2} = 0\,, \tag{1}$$

$$-G\,C\,\frac{\partial^2 \theta}{\partial x^2} - \varepsilon\,\rho\,A\,\frac{\partial^2 y}{\partial t^2} + \rho\,(I + A\,\varepsilon^2)\,\frac{\partial^2 \theta}{\partial t^2} = 0\,, \tag{2}$$

where $y(x,t)$ is the lateral displacement of the bar, $\theta(x,t)$ is the turning angle of the bar cross-section, E is the Young modulus, I is a polar inertia moment of the cross section with respect to its gravity center, ρ is a density of the bar material, A is the area cross section, G a shear modulus, C is geometrical rigidity of free torsion and ε is the distance from the gravity center to the center of torsion of the bar (see [1] and references there in).

We underline that in equation (2) we neglect by the sectional moment of inertia of the bar's cross-section.

These equations ought to be provided by initial conditions:

$$y(x,0) = y_0(x)\,, \quad \frac{\partial y}{\partial t}(x,0) = y_1(x)\,, \quad 0 \le x \le l\,, \tag{3}$$

$$\theta(x,0) = \theta_0(x)\,, \quad \frac{\partial \theta}{\partial t}(x,0) = \theta_1(x)\,, \quad 0 \leq x \leq l\,. \tag{4}$$

Further let us suppose that ε is small parameter: $0 < \varepsilon \ll 1$. In this case system (1)-(2) is closely connected with the problem of influence of small distinction between the gravity center and the center of torsion of the cantilever of scanning probe microscope on precision of measurement of the local properties of the solid body surface.

In order to answer on this question first of all one ought to solve system (1)-(2) with homogeneous boundary conditions:

$$y|_{x=0} = 0\,, \quad \frac{\partial y}{\partial x}|_{x=0} = 0\,, \quad \frac{\partial^2 y}{\partial x^2}|_{x=l} = 0\,, \quad \frac{\partial^3 y}{\partial x^3}|_{x=l} = 0\,, \quad 0 \leq t \leq T\,, \tag{5}$$

$$\theta|_{x=0} = 0\,, \quad \frac{\partial \theta}{\partial x}|_{x=0} = 0\,, \quad 0 \leq t \leq T\,. \tag{6}$$

In this preliminary investigation we construct solution of this problem in the framework of the perturbation theory:

$$y(x,t) = \sum_{k=0}^{\infty} \varepsilon^k\, y^{(k)}(x,t)\,, \quad \theta(x,t) = \sum_{k=0}^{\infty} \varepsilon^k\, \theta^{(k)}(x,t)\,. \tag{7}$$

Under $\varepsilon = 0$ input system (1)-(2) is split on two independent equations namely initial approximation $y^{(0)}(x,t)$ for the lateral displacement of the bar obeys to the next equation:

$$E\,I\,\frac{\partial^4 y^{(0)}}{\partial x^4} + \rho\,A\,\frac{\partial^2 y^{(0)}}{\partial t^2} = 0\,. \tag{8}$$

Equation (8) we solve exactly with initial conditions (3) and boundary conditions (5).

Initial approximation $\theta^{(0)}(x,t)$ for the turning angle of the bar cross-section obeys to the following equation:

$$-G\,C\,\frac{\partial^2 \theta^{(0)}}{\partial x^2} + \rho\,I\,\frac{\partial^2 \theta^{(0)}}{\partial t^2} = 0\,. \tag{9}$$

Equation (9) we solve exactly with initial conditions (4) and boundary conditions (6).

Equations for higher order approximations $y^{(k)}(x,t)$ and $\theta^{(k)}(x,t)$ we obtain substituting asymptotic series (7) into input system (1)-(2). After that we solve these equations with the same boundary conditions (5)-(6) but with zero initial conditions.

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C_0-group generated by Fourier transform and C_0-group that includes Fourier transform

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We consider well-known Fourier transform in the space of all square integrable functions on the real line. Using spectral representation of the unitary Fourier transform operator F we construct two one-parameter groups of linear bounded operators: A(t)=exp(tF) for which F is the infinitesimal generator, and B(t) satisfying B(1)=F. In other words, we compute the exponent and the logarithm of F. It seems natural that those groups may be helpful in mathematical physics and studies of evolution equations, however the research in this direction is in the very beginning now.

Scenarios of transition to chaos in the discrete–time predator–prey model

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The study of biological communities, such as predator–prey or host–parasite systems, is one of the most important environmental problems. Predator–prey interactions are crucial to formation of the species composition in a community and their dynamics. In particular, predator–prey interactions can cause fluctuations in the numbers of both interacting species and can amplify such fluctuations if they exist due to other causes. In this work, we present a new look at the problem of complex dynamics that can arise between a prey and a predator. This paper investigates scenarios of transition to chaos in the predator–prey model with age structure for prey. We use a slight modification of the Nicholson–Bailey model to describe the interaction between predator and prey. We assume the population size is regulated by decreasing juvenile survival rate with growth of age class sizes. The model considered may be written as a system of three equations:

$$\begin{cases} X_{n+1} = rY_n \exp(-bZ_n), \\ Y_{n+1} = \exp(-\alpha X_n - \beta Y_n)X_n + vY_n, \\ Z_{n+1} = crY_n(1-\exp(-bZ_n)), \end{cases} \tag{1}$$

where n is a reproductive season number; X and Y are the population size of juveniles and adults of prey (hosts), respectively; Z is the number of predators (parisitoids); r is the birth rate of prey (hosts); v is the survival rate of prey adults; b is the attack rate of the predator; c is a measure of the “conversion” of hosts (prey) into predators the following year. The survival rate of immature individuals of prey is selected as the Ricker model: $exp(-\alpha X_n - \beta Y_n)$, where α and β are the parameters describing the intensity of intrapopulation competition.

We made the analytical and numerical research of the mathematical model. For making the numerical experiments, we have elaborated the software systems to construct of bifurcation diagrams, attraction basins, and charts of dynamic modes, eigenvalues and Lyapunov exponents.

Conditions for sustainable coexistence of interacting species are described. It is shown that the coexistence of species becomes possible if there are a transcritical or saddle–node (tangential) bifurcations. Due to the saddle–node bifurcation there is bistability in the system of interacting species: predator either coexists with prey or dies depending on the initial conditions.

It is shown, with changing parameters' values and transition through the stability domain boundary the stability loss of the fixed point may occur according to both scenarios: the period doubling and the Neimark–Sacker bifurcation. Consequently depending on the values of the model parameters, the transition to chaos can be realized through the period doubling or through the destruction of the invariant curve.

The characteristic of the first scenario is that the eigenvalues pair of a fixed point is almost always imaginary (a fixed point is a saddle–focus). Therefore, the loss of stability of the 2–cycle is rarely accompanied by a period doubling cascade. The 2–cycle loses stability due to the Neimark–Sacker bifurcation. As a result, two invariant curves are formed around each periodic point. Further, there are two possible options for complicating of the attractor. First, a single doubling of the invariant curve occurs and a two–component strange attractor of the torus–chaos type emerges. Secondly, when parameters pass through resonant cycles, the invariant curve is strongly deformed and complexly wraps around the initial saddle cycle and a super–spiral attractor is formed. In both cases, a strange homoclinic attractor of period 2 emerges. Then the model trajectories are strongly mixed. As a result, in addition to periodic points, the new completely non–periodic attractor contains fixed points as well.

The second scenario is associated with the formation of a non–orientable Shilnikov funnel based on a single invariant curve formed according to the classical Neimark–Sacker bifurcation scenario. As a consequence of the period doubling bifurcation, the curve becomes a saddle, and in its neighborhood there is a pair of stable invariant curves of period 2. Unlike the classical or orientable case, these curves do not wrap around the initial curve, but are located in its neighborhood. With a further change in the bifurcation parameter, two super–spiral attractors are formed on the basis of the doubled invariant curve. Their fusion forms a non–homoclinic attractor of the torus–chaos type, which does not contain a fixed point.

This work was performed in the framework of the State targets of the Institute of Complex Analysis of Regional Problem FEB RAS and partially supported by the Russian Foundation for Basic Research (no. 18–51–45004 IND_ a).

Unfolding a Bykov attractor: from an attracting torus to strange attractors

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We present a comprehensive mechanism for the emergence of strange attractors in a two-parametric family of differential equations acting on a three-dimensional sphere. When both parameters are zero, its flow exhibits an attracting heteroclinic network (Bykov network) made by two 1-dimensional and one 2-dimensional separatrices between two hyperbolic saddles-foci with different Morse indices. After slightly increasing both parameters, while keeping the one-dimensional connections unaltered, we focus our attention in the case where the two-dimensional invariant manifolds of the equilibria **do not intersect**.

Under some conditions on the parameters and on the eigenvalues of the linearization of the vector field at the saddle-foci, we prove the existence of many complicated dynamical objects, ranging from an attracting quasi-periodic torus, Newhouse sinks to Hénon-like strange attractors, as a consequence of the Torus Bifurcation Theory (developed by Afraimovich and Shilnikov).

Under generic and checkable hypothesis, we conclude that any analytic unfolding of a Hopf-zero singularity (within the appropriate class) contains strange attractors.

On Modeling of One Unstable Bifurcation in the Dynamics of Vortex Structures

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This report is devoted to the results of phase topology research on a generalized mathematical model, which covers such two problems as the dynamics of two point vortices enclosed in a harmonic trap in a Bose-Einstein condensate [1] and the dynamics of two point vortices bounded by a circular region in an ideal fluid [2]. This model leads to a completely Liouville integrable Hamiltonian system with two degrees of freedom, and for this reason, topological methods used in such systems can be applied.

In this talk, we analytically derive equations that define the parametric family of bifurcation diagrams of the generalized model, including bifurcation diagrams of the specified limiting cases. The dynamics of the bifurcation diagram in a general case is shown using its implicit parametrization. A stable bifurcation diagram, related to the problem of dynamics of two vortices bounded by a circular region in an ideal fluid, is observed for particular parameters' values. Interactive visualization of the bifurcation diagram was made by A. A. Shadrin based on the equations of a bifurcation diagram and reduction to a system with one degree of freedom in the general case [3] and [4].

New bifurcation diagrams are obtained and three-into-one and four-into-one tori bifurcations are observed for some values of the physical parameters of the model. The three-into-one tori bifurcation was previously encountered in the works of M. P. Kharlamov in studying the phase topology of the integrable Chaplygin-Goryachev-Sretensky case in the dynamics of a rigid body [5] and as one of the features in the form of a 2-atom of a singular layer of Liouville foliation in the works of A. T. Fomenko, A. V. Bolsinov, S. V. Matveev [6]. In the work of A. A. Oshemkov and M. A. Tuzhilin [7], devoted to the splitting of saddle singularities, such a bifurcation turned out to be unstable and its perturbed foliations, one of which is realized in the integrable model under consideration, are given.

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Various aspects of mixed dynamics in the perturbed Chirikov map.

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In this work, we show how to perturb the Chirikov standard map breaking conservativity but keeping reversibility of this map and discuss various mixed dynamics phenomena that appear after the perturbation.

Chirikov standard map is one of the best-known examples of reversible area-preserving (moreover, symplectic) maps. It is well known that this map demonstrates typical for symplectic map transition from integrable dynamics to Hamiltonian chaos when changing a parameter. We consider this map in the following form:

$$T : \bar{x} = x + y,\ \bar{y} = y + K \sin(2\pi\bar{x}) \bmod 1. \tag{1}$$

where x and y are 1-periodic phase variables and K is a parameter. Note, that this map is reversible with respect to the involution

$$I : \bar{x} = x + y,\ \bar{y} = -y. \tag{2}$$

Moreover the map under consideration has the following symmetry:

$$S : \bar{x} = -x,\ \bar{y} = -y. \tag{3}$$

At $K = 0$, the map is integrable, its phase space is foliated into invariant tori. When K increases some tori become resonant and the pairs of symmetrical saddle and elliptic orbits appear. The neighborhood of each elliptic periodic orbit, in this case, is surrounded by a continuum of KAM-curves. These KAM-curves are separated by resonant zones with garlands originated near the alternating saddle and elliptic periodic orbits which appear in the neighborhood of resonant elliptic point with multipliers $e^{\pm ip/q}$. For a typical two-dimensional symplectic map, such garlands are formed near a pair of saddle and elliptic orbits. However, for Chirikov map the structure of some resonances differs from the typical one. Here garlands around elliptic points belonging to $Fix(S)$ and corresponding to odd q contain two pairs of elliptic orbits and two pairs of saddle orbits [1]. Using normal form theory we explain such specific structure of these resonant orbits.

The second part of this work is connected with the study of a perturbed Chirikov map of the following form:

$$F : \bar{x} = x + y,\ \bar{y} = y + K \sin(2\pi\bar{x}) + \varepsilon Q(y, \bar{y}) \bmod 1. \tag{4}$$

The function $Q(y, \bar{y})$ is chosen so the map F is also reversible with respect to the involution I or $I \circ S$ but is not conservative (area-preserving) for $\varepsilon \neq 0$. Non-symmetrical orbits here, which does not belong to the lines $Fix(I)$ or $Fix(I \circ S)$, are not longer conservative for $\varepsilon \neq 0$. We show that depending on the functions $Q(y, \bar{y})$ this map can demonstrate three general types of resonances for odd q near the symmetrical elliptic fixed point O:

- Non-isolating resonances with sinks and sources. In this case, a garland around resonant elliptic point consists of two pairs of symmetrical saddle orbits and a pair of a sink and a source.

- Isolating resonances. As in the previous case, a garland around O consists of two pairs of symmetrical saddle orbits and a pair of sink and source. However stable and unstable manifolds of the symmetrical saddles form an impassable for ε-orbits region (see [2]). In this case elliptic point O is stable with respect to the permanently acting perturbation and thus it is an example of a *reversible core* – such a stable set which belongs to both an attractor and a repeller.

- Resonances containing Lamb-Stenkin heteroclinic cycles. Here, the garland around O consists of a pair of symmetrical elliptic orbits and a pair of area-contracting and area-expanding saddle orbits. One pair of invariant manifolds of the saddle orbits intersects transversally while another pair forms quadratic tangency. As it is proved in [3] near such heteroclinic cycles there exist infinite sets of sinks, sources and elliptic points whose closure forms a non-empty intersection, i.e. chaos inside regions with such cycles is mixed.

Finally, we show in this work that the perturbed Chirikov map demonstrates the merger of a strange attractor and a strange repeller phenomenon which leads to the emergence of strongly dissipative mixed dynamics [4, 5]. After such a merger the attractor and the repeller of the system have a non-empty intersection but are different from each other and this difference does not seem to vanish with a reasonable increase in the computation time. The corresponding bifurcations leading to the emergence of strongly dissipative mixed dynamics are studied in detail.

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Features of the study of chaotic attractors in radiophysical generators

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Complex oscillations are widespread in radiophysics [1-2]. Oscillatory regimes can be classified into: periodic, quasiperiodic, and chaotic. Quasiperiodic oscillations are a class of oscillations widespread in science and technology [3]. Recently, much attention has been paid to this type of oscillation, since, on the one hand, they are not as complex as chaotic, but also non-periodic, which makes them interesting for analysis. The issues of diagnosis of such oscillations are very relevant and in the experiment are quite difficult to implement.

In the frame of this work we will provide an overview of the techniques for distinguishing complex oscillatory modes. For the numerical analysis of such systems, the most effective is the analysis of the full spectrum of Lyapunov exponents, which makes it possible to distinguish between chaotic and quasiperiodic oscillations, also to identify hyperchaotic oscillations and to classify quasiperiodic oscillations with a different number of incommensurable frequencies. It is enough difficult task to calculate the spectrum of Lyapunov exponents by time series in an experiment; examples of analysis of the Fourier spectra of various signals will be shown, as well as a methodology for constructing an invariant curve using a multiple Poincaré section [4].

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Transcendental First Integrals of Dynamical Systems

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In this paper, we examine the existence of transcendental first integrals for some classes of systems with symmetries. We obtain sufficient conditions of existence of first integrals of second-order nonautonomous homogeneous systems that are transcendental functions (in the sense of the theory of elementary functions and in the sense of complex analysis) expressed as finite combinations of elementary functions.

The results of the present paper develop previous studies, including some applied problems of the rigid-body dynamics (see [Sham1, Sham2, Sham3], where complete lists of transcendental first integrals expressed as finite combinations of elementary functions were obtained). Later, this fact allowed one to perform an analysis of all phase trajectories and to indicate rough properties that are preserved for systems of a more general form. The complete integrability of such systems is associated with hidden symmetries.

As is well known, the concept of integrability, generally speaking, is quite vague. It is necessary to consider the sense in which it is meant (i.e., a certain criterion that allows one to conclude that trajectories of a dynamical system have an especially "attractive and simple structure"), and in which class of functions first integrals are taken, and so on (see also [Sham4, Sham5]).

In this paper, we accept an approach in which the class of first integrals consists of elementary transcendental functions. Here the transcendence is meant not only in the sense of the elementary functions (e.g., trigonometric) but in the sense of complex analysis, i.e., as functions of a complex variable possessing essential singular points. In this case these functions must be formally continued in the complex domain.

Of course, in the general case, the construction of any integration theory of such nonconservative systems (even of low dimension) is quite difficult. But in some cases where the systems studied possess additional symmetries, one can find first integrals as finite combinations of elementary functions.

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Applications of intrinsic shape to dynamical system

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The intrinsic shape of various types of sets that appear in Dynamical systems gives an important information about the behavior of a dynamical system.

A brief introduction to intrinsic shape and a comparison with homotopy type will be given. Several results will be presented about intrinsic shape of chain recurrent set and non-saddle set generalizing previous results about attractors.

Linearization and integrability of nonlinear non–autonomous oscillators

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In this talk we consider the following family of nonlinear oscillators

$$y_{zz} + f(z,y)y_z^2 + g(z,y)y_z + h(z,y) = 0, \tag{1}$$

where f, g and h are sufficiently smooth functions and $g(z, y) \not\equiv 0$. Particular cases of (1) often appear in various applications in mechanics, physics and biology (see, e.g. [1, 2]). We study linearizability conditions for (1) into

$$w_{\zeta\zeta} + \beta w_{\zeta} + \alpha w = 0, \tag{2}$$

via the following nonlocal transformations

$$w = F(z, y), \quad d\zeta = G(z, y)dz. \tag{3}$$

Here $\alpha, \beta \neq 0$ are arbitrary parameters and F and G are sufficiently smooth functions satisfying $F_y G \neq 0$. Linearization of (1) into (2) via (3) was studied previously only for some particular cases of (3), namely $F_y = G_y = 0$ and $F_z = 0$ (see [3, 4, 5] and references therein). Here we consider the general case of transformations (3) and provide linearizability conditions for (1) via (3) in the explicit form. We show that in the linearizable case of (1) not only can we obtain the general solution of the corresponding equation with the help of the general solution of (2), but we also can explicitly construct a first integral for (1) via a known first integral for (2). We also demonstrate that there are several interesting examples of both autonomous and nonautonomous oscillators that can be linearized via (3) with $F_z \neq 0$. In particular we consider a generalization of the Duffing–van der Pol oscillator, a cubic Liénar oscillator with linear damping and some other examples and construct their general solutions in the parametric form along with their first integrals.

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Minimal generating sets and the structure of Wreath product of groups with non-faithful action

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We denote by $d(G)$ the minimal number of generators of the group G [1, 4]. A diffeomorphism $h : M \to M$ is said to be f-preserving if $f \circ h = f$. This is equivalent to the assumption that h is invariant for each level-set, i.e. $f^{-1}(c)$, $c \in P$ of f, where P denotes either the real line R or the circle S^1.

Let G be a group. The *commutator width* of G [3], denoted by $cw(G)$, is defined to be the least integer n, such that every element of G' is a product of at most n commutators if such an integer exists, and otherwise is $cw(G) = \infty$. The following Lemma imposes the Corollary 4.9 of [2].

Lemma 1. *An element of form* $(r_1, \dots, r_{p-1}, r_p) \in W' = (B \wr C_p)'$ *if and only if the product of all* r_i *(in any order) belongs to* B'*, where* $p \in \mathrm{N}$*,* $p \geq 2$*, where* $r_i = h_i g_{a(i)} h^{-1}_{ab(i)} g^{-1}_{aba^{-1}(i)}$*,* $h, g \in B$ *and* $a, b \in C_p$*.*

Lemma 2. *For any group* B *and integer* $p \geq 2$*, if* $w \in (B \wr C_p)'$*, then* w *can be represented by making use of the following wreath recursion*

$$w = (r_1, r_2, \dots, r_{p-1}, r_1^{-1} \dots r_{p-1}^{-1} \prod_{j=1}^{k} [f_j, g_j]),$$

where $r_1, \dots, r_{p-1}, f_j, g_j \in B$ *and* $k \leq cw(B)$*.*

Theorem 1. *If the orders of cyclic groups* $\mathbb{C}_{n_i}$*,* $\mathbb{C}_{n_j}$ *are mutually coprime* $i \neq j$*, then the group* $G = C_{i_1} \wr C_{i_2} \wr \dots \wr C_{i_m}$ *admits two generators, namely* β_0*,* β_1*.*

Let $\underset{j=0}{\overset{n}{\wr}} C_{i_j}$ be generated by β_0 and β_1 and $\underset{l=0}{\overset{m}{\wr}} C_{k_l} = \langle \alpha_0, \alpha_1 \rangle$. Denote an order of g by $|g|$.

Theorem 2. *If* $(|\alpha_0|, |\beta_0|) = 1$ *and* $(|\alpha_1|, |\beta_1|) = 1$*, or if* $(|\alpha_0|, |\beta_1|) = 1$ *and* $(|\alpha_1|, |\beta_0|) = 1$*, then there exists generating sets of two elements for the wreath-cyclic group* $G = (\underset{j=0}{\overset{n}{\wr}} C_{i_j}) \times (\underset{l=0}{\overset{m}{\wr}} C_{k_l})$*, where* i_j *are orders of* C_{i_j}*.*

We have found an upper bound for the generator number of G'. Let $\mathcal{A}$ be a group and $\mathcal{B}$ a permutation group, i.e. a group $\mathcal{A}$ acting upon a set X, where the active group $\mathcal{A}$ can act not faithfully.

Theorem 3. *If* $W = (\mathcal{A}, X) \wr (\mathcal{B}, Y)$*, where* $|X| = n$*,* $|Y| = m$ *and active group* $\mathcal{A}$ *acts on* X *transitively, then*

$$d\left(G'\right) \leq (n-1)d(\mathcal{B}) + d(\mathcal{B}') + d(\mathcal{A}').$$

We consider when the active group can be either finite or infinite and consider a center of such group. This consideration is a generalization of Theorem 4.2 from the book [2]. Let $X = \{x_1, x_2, \dots, x_n\}$ be an $\mathcal{A}$-space. If a non-faithful action by conjugation determines a shift of copies of $\mathcal{B}$ from the direct product $\mathcal{B}^n$, then we do not have the standard wreath product $(\mathcal{A}, X) \wr \mathcal{B}$ which is a semidirect product of $\mathcal{A}$ and $\prod\limits_{x_i \in X} \mathbb{B}$, i.e. $\mathcal{A} \ltimes_{\varphi} (\mathcal{B})^n$.

Corollary 1. *A center of the group* $(\mathcal{A}, X) \wr \mathcal{B}$ *is the direct product of the normal closure of the center of the diagonal of* $Z(\mathcal{B}^n)$*, i.e.* $(E \times Z(\triangle(\mathcal{B}^n)))$*, trivial an element, and the intersection of* $(\mathcal{K}) \times E$ *with* $Z(\mathcal{A})$*. In other words, we have*

$$Z((\mathcal{A}, X) \wr \mathcal{B}) = \langle (1;\ \underbrace{h, h, \dots, h}_{n}),\ e,\ Z(\mathcal{K}, X) \wr \mathcal{E} \rangle \simeq (Z(\mathcal{A}) \cap \mathcal{K}) \times Z(\triangle(\mathcal{B}^n)),$$

where $h \in Z(\mathcal{B})$*,* $|X| = n$*.*

For the restricted wreath product with n non-trivial coordinates, we have $Z((\mathcal{A}, X) \wr \mathcal{B}) =$ $\langle (1;\ \dots, h, h, \dots, h, \dots),\ e,\ Z(\mathcal{K}, X) \wr \mathcal{E} \rangle \simeq (Z(\mathcal{A}) \cap \mathcal{K}) \times Z(\triangle(\mathcal{B}^n))$*.*

In case of unrestricted wreath product, we have $Z((\mathcal{A}, X) \wr \mathcal{B}) =$ $\langle (1;\ \dots, h_{-1}, h_0, h_1, \dots, h_i, h_{i+1}, \dots,),\ e,\ Z(\mathcal{K}, X) \wr \mathcal{E} \rangle \simeq (Z(\mathcal{A}) \cap \mathcal{K}) \times Z(\tilde{\triangle}(\mathcal{B}))$*.*

Remark 1. *The quotient group of a restricted wreath product $G = Z \wr_X Z$ by a commutator subgroup is isomorphic to $\mathbb{Z} \times \mathbb{Z}$. Making use of previous conditions, we have if $G = A \wr_X B$, then $G/G' = A/A' \times B/B'$. If $G = Z_n \wr Z_m$, where $(m,\ n) = 1$, then $d(G/G') = 1$. If $G = Z \wr Z$ is an unrestricted regular wreath product, then $G/G' \simeq Z \times E \simeq Z$.*

The minimal set of generators for the fundamental group $\pi_1(O_f, f)$ of orbits of one function f with respect to the action of the group of diffeomorphisms of non-moving ∂M has been found here.

Theorem 4. *The group $H \simeq \mathbb{Z} \ltimes (\mathbb{Z})^n = \langle \rho,\ \tau \rangle$ with defined above homomorphism in $AutZ^n$ has two generators and non trivial relations, namely*

$$\rho^n \tau \rho^{-n} = \tau^{-1},\ \rho^i \tau \rho^{-i} \rho^j \tau \rho^{-j} = \rho^j \tau \rho^{-j} \rho^i \tau \rho^{-i},\ 0 < i, j < n.$$

This group admits another presentation which makes use of generators and relations, namely

$$\left\langle \rho, \tau_1, \ldots, \tau_n \left| \rho \tau_{i(\ mod\ n)} \rho^{-1} = \tau_{i+1(\ mod\ n)}\ ,\ \ \tau_i \tau_j = \tau_j \tau_i,\ i,\ j \leq n \right. \right\rangle.$$

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Mathematical Models of Rogue Waves

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The rogue wave problem originates in the ocean-related application, though today it is understood in broader sense. Abnormally large waves which occur on the sea surface have been discovered not long ago, see reviews and the discussion of promising researches in [1, 2]. The nonlinear modulational instability have been suggested as a regular mechanism which can alter the probability distribution function for wave heights and result in a larger likelihood of extreme events. This mechanism is related to the formation of nonlinear coherent wave patterns, which possess their own dynamic features. Within the frameworks of idealized evolution equations which can be analyzed with the help of the Inverse Scattering Transform (the Korteweg – de Vries equation, KdV, and the nonlinear Schrödinger equation and their generalizations), the simplest long-living coherent patterns correspond to solitons and envelope solitons respectively.

In this paper we discuss the effects of the dynamics of ensembles of solitons in long-wave models (KdV equation and modified KdV, mKdV, and Gardner equations of the focusing type), which may lead to the generation of extremely large waves. These problems belong to so-called 'soliton turbulence', though instead of the consideration of the density of the 'soliton gas' within the kinetic approach, we focus on the wave amplitudes. The role of multiple soliton and breather interactions in the formation of very high waves is disclosed. The discovered scenario depends crucially on the soliton polarities and breather phases, and is not described by kinetic models.

In particular, conditions of optimal (synchronized) collisions of any number of solitons and breathers are studied within the framework of the Gardner equation (GE) with positive cubic nonlinearity, which in the limits of small and large amplitudes tends to the classic and the modified Kortewe-de Vries equations respectively. To this end the N-soliton-M-breather solution is considered for any natural numbers N and M. The solution is constructed with the help of the Darboux transform, following the technique suggested in [3]. The wave amplitude in the focal point is calculated exactly. It exhibits a linear superposition of partial amplitudes of the solitons and breathers (see details in [4, 5]). The crucial role of the choice of proper soliton polarities and breather phases on the cumulative wave amplitude in the focal point is demonstrated. Solitons are most synchronized when they have alternating polarities. The straightforward link to the problem of synchronization of envelope solitons and breathers in the focusing nonlinear Schrödinger equation is discussed (then breathers correspond to envelope solitons propagating above a condensate).

The soliton dynamics in the focal point is essentially nonlinear and may suffer from weak disturbances, including inaccuracy of a numerical code. This effect strengthens when the number of colliding solitons grows [6]. As a result, solitons of opposite polarities with close velocities may transform to breathers, which seem to be more stable nonlinear wave structures.

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Derivatives of generalized measures on infinite dimensional spaces and quantum anomalies

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One says about the quantum anomaly if after the quantization of the Lagrangian or Hamiltonian system, whose action is invariant with respect to a transformation, one gets a quantum system dynamics of which is not invariant with respect to the same transformation. Recently (2003 and 2006) Oxford University Press and Cambridge University Press published two books in which alternative explanations of the origin of the quantum anomalies were suggested; moreover in the second book it was said that the explanation, given in the first book, is wrong. In the talk a new approach to the analysis of the quantum anomalies will be suggested. It shows that the result of the first book is correct.

Canonical second quantization via generalized measures

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Under the standard Hilbert symplectic space we understand the orthogonal sum $E = Q \oplus P \cong Q \times P$ of the two mutually isomorphic real Hilbert spaces Q and P (each often regarded as a dual to the another, $P \cong Q'$ and vice versa) endowed with the unitary operator $I : E' \to E$ ($E' \cong P \times Q$) defined by $I\binom{p}{q} = \binom{q}{-p}$. We call the space $\mathcal{E}(E)$ of all infinitely (Hadamard) smooth real functions on the symplectic space (E, I) the Poisson algebra if it is endowed with the "Poisson braces" bilinear operation $\{\cdot,\cdot\} : \mathcal{E}(E) \times \mathcal{E}(E) \to \mathcal{E}(E)$ defined by $\{f, g\}(z) = f'(z)(Ig'(z))$ where $f'(z) \in E'$ is the derivative of f at $z \in E$. We call a function $f \in \mathcal{E}(E)$ quadratic, if $f''' \equiv 0$, and we call a subalgebra of the Poisson algebra non-trivial Poisson subalgebra, if it contains: a) all constants and continuous linear functionals, b) dense, w.r.t. the topology of the pointwise convergence, subspace of non-linear quadratic functions and c) infinite dimensional set of non-quadratic functions.

Under the canonical $\hbar$-quantization of a non-trivial Poisson subalgebra $\mathcal{A}$ (elements of the algebra are called the classical observables, and real number $\hbar > 0$ plays here the role of the Planck constant h divided by 2π) we understand any linear operator $\hat{} : f \mapsto \hat{f}$ from the Poisson algebra into the some complex Lie algebra of self-adjoint operators (acting in some auxiliary complex Hilbert space H and having common dense invariant subspace $D_0 \subset H$ in their domains) such that $\hat{1}\psi = \psi$ and on the quadratic functions subalgebra $\mathcal{A}_2 = \{f \in \mathcal{A} : f''' \equiv 0\}$ we have the homomorphic "canonical commutation relations": $i\hbar\widehat{\{f, g\}} = [\hat{f}, \hat{g}] \equiv \hat{f}\hat{g} - \hat{g}\hat{f}$).

We refer to the canonical $\hbar$-quantization as to the Schroedinger one if the auxiliary complex Hilbert space H is a completion of an invariant, w.r.t. all isometric changes of variables, space $S(Q)$ of smooth complex functions, square-integrable w.r.t. some generalized measure on Q.

If Q is infinite dimensional, such quantizations are called second quantization.

We construct a Schroedinger canonical second $\hbar$-quantization of a non-trivial Poisson subalgebra $\mathcal{A}$ for each real $\hbar > 0$ using infinite dimensional pseudo-differential operators.

Bifurcations of multiple attractors in a predator-prey system

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This is an update of the presentation from previous conference inculding many questions. The system of n competing predators feeding on the same prey is of the type

$$X_i' = p_i\,\varphi_i(S)\,X_i - d_i\,X_i, \quad i = 1, ..., n, \tag{1a}$$

$$S' = H(S) - \sum_{i=1}^{n} q_i\,\varphi_i(S)\,X_i, \tag{1b}$$

where the variable S represents the prey and the variables X_i represent the predators. They are, of course, non-negative. The function φ_i is assumed non-decreasing.

We consider the case where

$$H(S) = r\,S\,\left(1 - \frac{S}{K}\right),\ \varphi_i(S) = \frac{S}{S + A_i}, \tag{2}$$

and where the parameters r, K and A_i are positive.

The dynamics in the coordinate planes representing one of the predators and the prey is well known and there is no more than one cycle. The system has no equilibrium, where predators coexist (in non-degenerate cases). But the predators can coexist in a cyclic and complicated way. There exists multiple attractors of cyclic and different chaotic chaos including "spiral-like" chaos. This happens even in cases, where the populations do not become unrealistic low. We present new discovered phenomena and discuss the possible bifurcations of these and contours from where they could develop.

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Classification of chaotic attractors in non-autonomous Rössler system

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Problems of control and stabilization of various oscillation modes of nonlinear systems attract a lot of attention due to both its theoretical value and great practical importance [1-2]. Situations when external force on the system with unstable mode not only leads to its stabilization, but also initiates a system of periodic and quasi-periodic modes with the classical system of Arnold's tongues on the parameter plane of period vs amplitude of the external force are very interesting and attractive for researchers.

One of the classical object for investigation in nonlinear dynamics is Rössler system. At certain parameters the Rössler system up to the threshold of the saddle-node bifurcation of equilibrium states demonstrates the mode when phase trajectories go to infinity. It has been shown in [3-4] that applying of periodic impulse signal to the system can stabilize oscillations in it and initiate synchronous response.

In the frame of the present work analysis of the complex dynamics in pulsed forced Rössler system is carried out. Using the full spectrum of Lyapunov exponents three types of chaotic dynamics were classified. Various routs leading to appearance of different type of chaotic attractors are discussed.

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Analysis of Some Non-Smooth Bifurcations with Applications to Ship Maneuvering

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The super/subcriticality of a Hopf bifurcation in a generic smooth 2D system can be readily determined by the sign of the first Lyapunov coefficient. However, for a system of continuous but non-smooth equations this cannot be applied in general. We show new results for autonomous systems of arbitrary finite dimension with focus on non-smooth nonlinearities of the form $|u_i|u_j$. This is motivated mainly by models for ship maneuvering and its control. We present the unfolding of Hopf-type bifurcations for such systems and discuss generalizations to bifurcations at switching points for continuous piecewise smooth systems.

Algebraic Constructions Generated By Causal Structure Of Space-times

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General relativity and quantum field theory both require systematic approach for working with families of causal subsets of space-times (suffice it to mention constructing the causal hierarchy of space-times, singularity theorems [1] and causal nets of operator algebras [2]). Here we shall confine ourselves to organizing subsets of Minkowski space-times M_1^D into algebraic systems (lattices) and describing their interrelations.

Let's consider the families of upper and lower cones on Minkowski space-time

$$\overset{+}{Con}\left(M_1^D\right) \equiv \left\{ \emptyset,\ \underset{x}{\overset{+}{con}} \middle| x \in M_1^D, x \cdot x \geq 0, x^0 \geq 0 \right\},$$
$$\overset{-}{Con}\left(M_1^D\right) \equiv \left\{ \emptyset,\ \underset{x}{\overset{-}{con}} \middle| x \in M_1^D, x \cdot x \geq 0, x^0 \leq 0 \right\},$$

where $x \cdot x \equiv \eta_{\alpha\beta} x^\alpha x^\beta, \quad \alpha, \beta = \overline{0, D-1}, \quad (\eta_{\alpha\beta}) \equiv diag\,(+, -, -, ..., -)$.

Now we define binary operations on $\overset{+}{Con}$ ("addition" $\overset{+}{\vee}$ and "multiplication" $\overset{+}{\wedge}$) as follows

$$\overset{+}{\vee}:\ \overset{+}{Con}\left(M_1^D\right) \times \overset{+}{Con}\left(M_1^D\right) \to \overset{+}{Con}\left(M_1^D\right)$$
$$\left(\underset{x}{\overset{+}{con}},\ \underset{y}{\overset{+}{con}}\right) \mapsto \underset{x}{\overset{+}{con}} \overset{+}{\vee} \underset{y}{\overset{+}{con}} \equiv \begin{array}{l} \textit{smallest upper cone,} \\ \textit{containing } \underset{x}{\overset{+}{con}} \text{ and } \underset{y}{\overset{+}{con}} \end{array}$$
$$\overset{+}{\wedge}:\ \overset{+}{Con}\left(M_1^D\right) \times \overset{+}{Con}\left(M_1^D\right) \to \overset{+}{Con}\left(M_1^D\right)$$
$$\left(\underset{x}{\overset{+}{con}},\ \underset{y}{\overset{+}{con}}\right) \mapsto \underset{x}{\overset{+}{con}} \overset{+}{\wedge} \underset{y}{\overset{+}{con}} \equiv \underset{x}{\overset{+}{con}} \bigcap \underset{y}{\overset{+}{con}}$$

It's easy to identify the properties of idempotency, commutativity, associativity of both operations, and also the validity of absorption identities

$$\underset{x}{\overset{+}{con}} \overset{+}{\wedge} \left(\underset{x}{\overset{+}{con}} \overset{+}{\vee} \underset{y}{\overset{+}{con}}\right) = \underset{x}{\overset{+}{con}} = \underset{x}{\overset{+}{con}} \overset{+}{\vee} \left(\underset{x}{\overset{+}{con}} \overset{+}{\wedge} \underset{y}{\overset{+}{con}}\right) \quad \forall\, x, y \in M_1^D$$

and conclude that $\left(\overset{+}{Con}\left(M_1^D\right), \overset{+}{\wedge}, \overset{+}{\vee}\right)$ is a lattice for which the property of distributivity is also true. In the same way we establish the distributivity of the lattice $\left(\overset{-}{Con}\left(M_1^D\right), \overset{-}{\wedge}, \overset{-}{\vee}\right)$, where "multiplication" $\overset{-}{\wedge}$ and "addition" $\overset{-}{\vee}$ of the lower cones are defined as in the previous case. A bijection between these lattices transforms one cone into another without changing the vertex

$$T:\ \left(\overset{+}{Con}\left(M_1^D\right), \overset{+}{\wedge}, \overset{+}{\vee}\right) \to \left(\overset{-}{Con}\left(M_1^D\right), \overset{-}{\vee}, \overset{-}{\wedge}\right)$$
$$\underset{x}{\overset{+}{con}} \mapsto T\left(\underset{x}{\overset{+}{con}}\right) \equiv \underset{x}{\overset{-}{con}}$$

Let's consider the family of diamonds on M_1^D

$$Dmd\left(M_1^D\right) \equiv \left\{ \begin{array}{l} \underset{x}{\overset{y}{dmd}} = \underset{x}{\overset{+}{con}} \bigcap \underset{y}{\overset{-}{con}} = \underset{x}{\overset{+}{con}} \bigcap T\left(\underset{y}{\overset{+}{con}}\right) = \\ = T^{-1}\left(\underset{x}{\overset{-}{con}}\right) \bigcap \underset{y}{\overset{-}{con}} \end{array} \middle| x, y \in M_1^D \right\},$$

and define “addition” $\vee$ and “multiplication” $\wedge$ as follows:

$$\begin{aligned} \vee:\ & Dmd\left(M_1^D\right) \times Dmd\left(M_1^D\right) \rightarrow Dmd\left(M_1^D\right) \\ & \left(\underset{'x}{\overset{'\mathrm{y}}{dmd}}, \underset{''x}{\overset{''\mathrm{y}}{dmd}}\right) \mapsto \underset{'x}{\overset{'\mathrm{y}}{dmd}} \ \vee \ \underset{''x}{\overset{''\mathrm{y}}{dmd}} \equiv \left(\underset{'x}{\overset{+}{con}} \overset{+}{\vee} \underset{''x}{\overset{+}{con}}\right) \bigcap \left(\underset{'y}{\overset{-}{con}} \overset{-}{\vee} \underset{''y}{\overset{-}{con}}\right) \end{aligned}$$

$$\begin{aligned} \wedge:\ & Dmd\left(M_1^D\right) \times Dmd\left(M_1^D\right) \rightarrow Dmd\left(M_1^D\right) \\ & \left(\underset{'x}{\overset{'\mathrm{y}}{dmd}}, \underset{''x}{\overset{''\mathrm{y}}{dmd}}\right) \mapsto \underset{'x}{\overset{'\mathrm{y}}{dmd}} \ \wedge \ \underset{''x}{\overset{''\mathrm{y}}{dmd}} \equiv \left(\underset{'x}{\overset{+}{con}} \overset{+}{\wedge} \underset{''x}{\overset{+}{con}}\right) \bigcap \left(\underset{'y}{\overset{-}{con}} \overset{-}{\wedge} \underset{''y}{\overset{-}{con}}\right) \end{aligned}$$

The properties of “addition” and “multiplication” on cones allow extracting the properties of operations on double cones. These causal subsets of the Minkowski spacetime, partially ordered by inclusion $\supseteq$, are directed sets, what makes them potentially interesting as a start when constructing the nets of C^*-algebras.

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Complicated dynamics in a reversible Hamiltonian system near a symmetric heteroclinic contour

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Hamiltonian systems arise as mathematical models in many branches of physics, chemistry, engineering. Such systems as their study shows have usually a rather complicated structure that leads to great difficulties in their examination. Therefore one of the fruitful method of their investigation is the study of the orbit behavior near some specific structures which can be distinguished by simple conditions. The study of a system near a homoclinic orbits or contours made up of several heteroclinic orbits and equilibria or periodic orbits is undoubtedly one of such problem.

We study the dynamics of an analytic reversible Hamiltonian system X_H with two degrees of freedom assuming the system has a heteroclinic contour involving a symmetric saddle-center equilibrium p (its eigenvalues are nonzero numbers $\pm i\omega$, $\pm\lambda$, $\omega, \lambda \in \mathbf{R}$), an orientable symmetric saddle periodic orbit γ lying in the same level of Hamiltonian $H = H(p)$ and two nonsymmetric heteroclinic orbits Γ_1, Γ_2 joining p with γ and interchanged by the involution L, $\Gamma_2 = L(\Gamma_1)$. The reversible involution L is supposed to have a smooth two-dimensional set $Fix(L)$ of its fixed points. Such a system are met in generic one-parameter families of reversible Hamiltonian systems.

Saddle periodic orbit γ belongs to a 1-parameter family γ_c of saddle periodic orbits in all close levels $H = c$ forming a symplectic cylinder. Reversible Hamiltonian systems possessing the above mentioned contour can be of two different types in dependence on how the involution acts locally near a saddle-center.

Our results demonstrate the existence in such a system:

- countable set of transverse 1-round homoclinic orbits to γ and related to them non-uniformly hyperbolic subsets;
- appearance for $c > 0$ of two transverse heteroclinic contours involving γ_c, a small Lyapunov periodic orbit l_c near p and four heteroclinic orbits $\Gamma_1^{\pm}$ and $\Gamma_2^{\pm} = L(\Gamma_1^{\pm})$ and related with them uniform hyperbolic subsets;
- a finite set of transverse 1-round homoclinic orbits to γ_c for $\mid c \mid$ close to $H(p)$ and uniformly hyperbolic sets related with them;
- a countable set of values $c_n < 0$ accumulating at $c = 0$ such that on the level $H = c_n$ the system has a tangent homoclinic orbit to γ_{c_n} and bifurcations nearby orbits related to this tangency;
- countable sets of saddle and elliptic periodic orbits.

Some other bifurcation phenomena will be discussed when generic one parameter reversible unfoldings of such a system are considered.

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Fast convergent Chernoff approximations to the solution of heat equation with variable coefficient of thermal conductivity

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The research presented is devoted to C_0-semigroups and their approximations. C_0-semigroup is an operator-valued function which enjoys a number of properties, see the textbook [1].

C_0-semigroups have many useful applications. One of them is that they provide solutions for a certain class of linear parabolic partial differential equations. Accordingly, approximations to the C_0-semigroup will be approximations to the solution of these equations. The Chernoff operator-valued function is an object that forms a sequence converging to the C_0-semigroup. The question arises in article [2]: with what speed does this sequence converge and on what conditions does this speed depend. The result presented in the talk is one of the first contributions to the field of studies of the convergence rate of Chernoff approximations.

In the research presented, a heat equation with variable thermal conductivity coefficient is considered; this equation is a particular case of a parabolic differential equation. The author of the talk applies the conditions of the Remizov's conjecture [3], according to which the approximations converge faster. It seems that we have reasons to hope that methods presented can be applied to a wider class of equations.

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Hopf bifurcation and stability of whirl and whip in rotor systems

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Rotors of high-speed turbo machines are commonly supported by hydrodynamic journal bearings. Many authors wrote about an effect of self-oscillations for this class of bearings, which in this context are usually referred as "oil whirl" and "oil whip". This effect has been extensively studied experimentally as well as analytically [1-10]. At the center position of the rotor loses its stability and a stable limit-cycle appears (oil whirl). The stability loss of the equilibrium position of a rigid rotor at the speed has been widely investigated both analytically and numerically by many authors [1, 6, 7-10]. In this paper we consider the problem of stability/instability effect oil whirl in the motion of the rotor in the fluid. The instability problem of rotor/seal system has been extensively analyzed by the linearization of model around the equilibrium position of the rotor. Such authors as [1, 5, 6, 7, 8, 9] wrote, that this method is not useful, since the nature of the whirling motion after onset of instability cannot be analyzed using only the linearized model. Non-linear analysis of the problem is very difficult [1, 6, 7, 9]. An analysis of the self-excited oscillations of a rotor is presented. We consider a symmetric Jeffcott rotor mounted at both rigid ends . The seal fluid force is assumed to be acting on the disk of the shaft [1, 2, 3, 10]. It is shown that Hopf bifurcation theory may be used to investigate small-amplitude periodic solutions of the nonlinear equations of motion for rotor speeds.

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Local and Global Inverse Problems for Difference Equations

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We present the new proof of the local inverse problem for systems of linear difference equations in the neighborhood of the infinity. This proof is based on almost complex structures. Using solution of the local problem we apply holomorphic vector bundles with meromorphic additive shift to studying of generalized Riemann–Hilbert–Birkhoff problem for difference systems.

As the application of this approach we obtain a generalization of Birkhoff's existence theorem. We prove that for any admissible set of characteristic constants and monodromy there exists a system

$$Y(z+1) = A(z)Y(z), \tag{1}$$

which has the given monodromy and characteristic constants and rational matrix $A(z)$.

An unguided tour started from chirality

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We will survey an unguided mathematical tour of research by topologists at Peking University and their collaborators over many years. The tour starts with work on chirality and, drawn by questions related to attractors, goes via a zigzag path across topology and dynamics.

Regularity of attractors of transversally similar Riemannian foliations

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Recall that a transformation $f \in Diff(N)$ of a q-dimensional Riemannian manifold (N, g) is called a similarity if $f^*g = \lambda g$ where λ is a positive constant. The set of all similarities of a Riemannian manifold (N, g) forms the Lie group $Sim(N, g)$. A foliation (M, F) of a codimension q on n-dimensional manifold M, where $0 < q < n$, is called a *transversally similar Riemannian* foliation if the transformations of foliated transversal coordinates are local similarities of some q-dimensional Riemannian manifold (N, g) whose connectivity is not assumed.

A subset of a manifold M is said to be saturated if it is a union of leaves of the foliation (M, F). A closed saturated nonempty subset $\mathcal{M}$ of M for which there exists an open neighborhood $\mathcal{U}$ such that the closure of every leaf from $\mathcal{U} \setminus \mathcal{M}$ contains $\mathcal{M}$, is called an *attractor* of this foliation. The neighborhood $\mathcal{U}$ is defined by the above condition, is called the basin of the attractor $\mathcal{M}$ and is denoted by $Attr(\mathcal{M})$. If, moreover, $Attr(\mathcal{M}) = M$, then the attractor $\mathcal{M}$ is called global.

An attractor $\mathcal{M}$ of a foliation (M, F) is called *regular* if it is a smooth submanifold of M. Recall that a minimal set of a foliation (M, F) is referred to as a nonempty closed saturated subset $\mathcal{K}$ of M such that every leaf belonging to $\mathcal{K}$ is dense in $\mathcal{K}$.

Minimal sets and attractors of foliations (M, F) largely determine the topology of (M, F). Therefore, the investigation of the existence and the structure of minimal sets and attractors is one of the main problems of both topological dynamics and qualitative theory of foliations. Attractors that are minimal sets of conformal and Weyl foliations were investigated, in particular, in [1], [2] and [3].

The purpose of this work is to describe the structure of global attractors of transversally similar Riemann foliations of an arbitrary codimension on n-dimensional manifolds. The following statement is one of the main results of this work.

Theorem *Every complete transversally similar Riemannian foliation* (M, F)*, which is not Riemannian, has a regular global attractor.*

The application of this theorem made it possible to describe the global structure of such foliations. In particular, it is proved that transversally similar non-Riemannian foliations exist only on non-compact manifolds. Also, the leaf closures are not submanifolds of M, in general. A leaf of a foliation (M, F) is called proper if it is an embedded submanifold of M. A foliation (M, F) is referred to as proper if all its leaves are proper. In the case where (M, F) is a proper non-Riemannian transversally similar Riemannian foliation, its global attractor is a closed leaf.

Examples are constructed.

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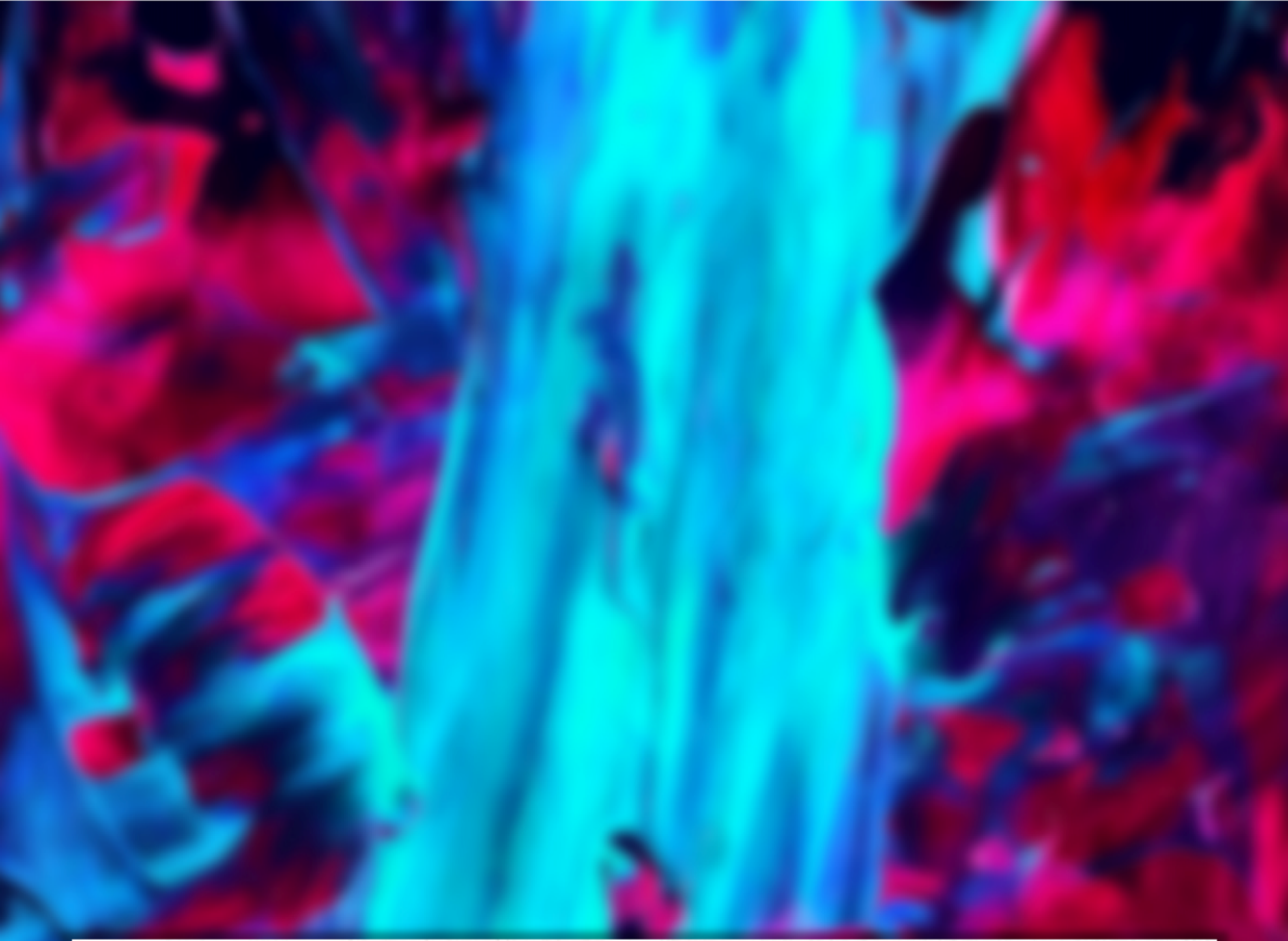

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