12-13 DEC

Conference

Book of abstracts

"Topological methods in dynamics and related topics. Shilnikov workshop"

NIZHNY NOVGOROD

International Conference
Topological Methods in Dynamics and Related Topics

Book of Abstracts

National Research University
Higher School of Economics

Nizhny Novgorod,
12-13 December, 2020

Organizers

National Research University Higher School of Economics, Nizhny Novgorod

Organising Committee

- Olga Pochinka
- Dmitry Turaev
- Vyacheslav Grines

Acknowledgment. The conference is partially supported by Laboratory of Dynamical Systems and Applications NRU HSE, of the Ministry of science and higher education of the RF grant ag. №075-15-2019-1931

Contents

On an asymptotic property of the M-integral for oriented 3-components links in $\mathbf{R}^3$

Akhmet'ev P. M. [1,2]

1. National Research University "Higher School of Economics", Moscow, Russia,
2. IZMIRAN, Moscow, Russia.

The linking coefficient of oriented 2-component links in $\mathbf{R}^3$ satisfies an asymptotic property. This is a property, which is required to define the ergodic Hopf invariant of trajectories of divergence-free vector fields in a bounded domain, see [1].

In magneto-hydro-dynamics the asymptotic ergodic Hopf invariant determines lower bounds of magnetic energy. In more complicated problems, for example, for relaxation of magnetic fields near equilibrium states, more strong invariants, which are not expressions of linking coefficients and which admit asymptotic properties, are required. Higher asymptotic invariants of links are also useful for non-dissipative dynamical systems, because of numerical characterization its bifurcations.

In [2] an integral, which is called the M-integral, for 3-components oriented links (an order of components is unrequited) $\mathbf{L} = L_1 \cup L_2 \cup L_3$ in $\mathbf{R}^3$ is defined. This invariant satisfies the following asymptotic property:

$$M(\lambda_1 L_1, \lambda_2 L_2, \lambda_3 L_3) = \lambda_1^4 \lambda_2^4 \lambda_3^4 M(L_1 \cup L_2 \cup L_3), \tag{1}$$

where $\lambda_1 L_1, \lambda_2 L_2, \lambda_3 L_3$ is the 3-component link, which is defined as a parallel winding of components of $\mathbf{L}$ with prescribed multiplicities; an isotopy class of the winding is not important, the only integer factors $\lambda_1, \lambda_2, \lambda_3$ are needed.

In [3],[4] (with references [5],[6]) is proved, that the M-integral determines a combinatorial invariant of 3-component oriented links in $\mathbf{R}^3$ (which is not a function of pairwise linking numbers of components) with the asymptotic property (1). This combinatorial invariant is expressed using the first two coefficients c_0, c_1 of the Conway polynomial of all proper links of $\mathbf{L}$. But the complete formula of this combinatorial invariant remains unknown.

Define M-invariant for 3-component oriented links $\mathbf{L} = L_1 \cup L_2 \cup L_3$ by the formula:

$$\begin{gathered} M(\mathbf{L}) = (1;2)(2;3)(3;1)\gamma(\mathbf{L}) + \\ -((1;2)^2(1;3)^2\beta(L_2 \cup L_3) - (2;3)^2(2;1)^2\beta(L_3 \cup L_1) - (2;3)^2(2;1)^2\beta(L_3 \cup L_1)) + \\ P((1;2),(2;3),(3;1)), \end{gathered} \tag{2}$$

where

$$\begin{gathered} \gamma(\mathbf{L}) = c_1(\mathbf{L}) + \\ -((1;2)(2;3) + (2;3)(3;1) + (3;1)(1;2))(c_1(L_1) + c_1(L_2) + c_1(L_3)) \\ -((3;1) + (2;3))(c_1(L_1 \cup L_2) - (1;2)(c_1(L_1) + c_1(L_2))) \\ -((1;2) + (3;1))(c_1(L_2 \cup L_3) - (2;3)(c_1(L_2) + c_1(L_3))) \\ -((2;3) + (1;2))(c_1(L_3 \cup L_1) - (3;1)(c_1(L_3) + c_1(L_1))) \end{gathered}$$

is the Melikhov invariant,

$$\beta(L_i \cup L_j) = c_1(L_i \cup L_j)) - c_0(L_i \cup L_j)(c_1(L_i) + c_1(L_j))$$

is the generalized Sato-Levine invariant $i, j = 1, 2, 3,\ i \neq j$,

$$(i;j) = c_0(L_i \cup L_j)$$

is the linking coefficient of the corresponding pair of components.

The polynomial P is defined by the formula:

$$\begin{gathered} P((1;2),(2;3),(3;1)) = +\frac{1}{6}(1;2)^2(2;3)^2(3;1)^2[(1;2) + (2;3) + (3;1)] \\ -\frac{1}{12}(1;2)(2;3)(3;1)[(1;2)(2;3) + (2;3)(3;1) + (3;1)(1;2)]. \end{gathered}$$

Theorem

The combinatorial M–invariant given by (2), satisfies the equation (1) and admits an integral formula, which is defined in [2] and is modified in [3].

We present a fragment of a proof of Theorem, following arguments of [4]. Steps of the proof are simplified and we check (with no references to [2]) that the M-integral in [3] is gauge invariant with respect to volume-preserved diffeomorphisms of the domain. Then we prove (with no reference to [6]) that jumps of the M-integral and jumps of the combinatorial invariant in (2) by homotopies (intersections of different components are not possible, but self-intersections of components are possible) coincide. The total proof of Theorem will be presented somewhere else.

Theorem is required for applications. In [7] an example of asymptotic calculations is presented. It is interesting to get a numeric proof of (1) using only the formula (2).

References

[1] Arnold V. I. and Khesin B. A. Topological Methods in Hydrodynamic // Applied Mathematical Science 125, (2013).

[2] Akhmetiev P. M. On a new integral formula for an invariant of 3-component oriented links // J. Geom. Phys. vol53 (2), 180-196, (2005) N2, 180-196.

[3] Akhmet'ev P. M. On a higher integral invariant for closed magnetic lines// J. Geom. Phys. vol. 74, (2013) 381-389.

[4] Akhmet'ev P. M. On combinatorial properties of a higher asymptotic ergodic invariant of magnetic lines// Journal of Physics: Conference Series, 544, 012015, (2014).

[5] Melikhov S. A. Colored finite type invariants and a multi-variable analogue of the Conway polynomial// http://arXiv:math/0312007v2, (2003).

[6] P. M. Akhmet'ev P. M. and Kunakovskaya O. V. Integral Formula for a Generalized Sato-Levine Invariant in Magnetic Hydrodynamics// Math. Notes, 85:4 (2009) 503-514.

[7] Akhmet'ev P. M. Knot Invariants in Geodesic Flows// Differential equations and dynamical systems, Collected papers, Proc. Steklov Inst. Math., 308 (2020), 42-55.

On the number of the classes of topological conjugacy of Pixton diffeomorphisms

Akhmet'ev P.M.[1], **Pochinka O.V.**[2]

1. Pushkov Institute of Terrestrial Magnetism, Ionosphere and Radio Wave Propagation Russian Academy of Sciences (IZMIRAN)

2. National Research University Higher School of Economics

Invariants of oriented knots in 3-manifold M^3 equipped with chosen cohomology class $\xi \in H^1(M^3; G)$ (here G is an abelian group) were studied from various points of view. The case $M^3 \cong S^2 \times S^1$ with the generator (positive) cohomology class $\xi^* \in H^1(S^2 \times S^1; \mathbb{Z})$ whose knots are called *Hopf knots* is of special interest for the theory of dynamical systems. In 3-dimensional dynamics the manifold $S^2 \times S^1$ appears in the natural way as the space of wandering orbits in the basin of a hyperbolic sink while the class ξ^* bears the comprehensive information on the dynamics of the system in this basin. In particular, a 1-dimensional saddle separatrix in the basin of this sink has a corresponding Hopf knot in $S^2 \times S^1$. B. Mazur in [2] constructed the Hopf knot which is non-equivalent and non-isotopic to trivial knot.

For the class of dynamical systems known as *Pixton diffeomorphisms*, this knot (up to a diffeomorphism of $S^2 \times S^1$) completely defines the class of topological conjugacy of the Pixton diffeomorphism [1] and, moreover, any Hopf knot can be realized as some Pixton diffeomorphism. Thus we have the complete topological classification of Pixton diffeomorphisms. Nevertheless the problem of the cardinality of the set of topological conjugacy classes of these diffeomorphisms is still open and it can be reduced to finding invariants of Hopf knots.

In the present paper we state the existence of an invariant of the first order for Hops knots. This allows to model countable families of pairwise non-equivalent Hopf knots and, therefore, infinite set of topologically non-conjugate Pixton diffeomorphisms.

Acknowledgments. O.V. Pochinka was supported by International Laboratory of Dynamical Systems and Applications of National Research University Higher School of Economics, Russian Federation Government grant 075-15-2019-1931.

References

[1] C. Bonatti, V. Grines. Knots as topological invariants for gradient-like diffeomorphisms 694 of the sphere S^3. J. Dyn. Control Syst. 2000. V. 6 (4), 579–602.

[2] B. Mazur. A note on some contractible 4-manifolds. Ann. Math. 1961. V. 79 (1), 221–228.

Bistability in a Hodgkin-Huxley-type of model with a communication defect

Bagautdinova E. R.[1,2], **Stankevich N. V.**[1]

1. National Research University Higher School of Economics

2. Chernyshevsky Saratov State University

Burst-spiking dynamics is widespread in various biophysical processes [1]. This dynamic behavior is typical for neurons, pancreatic β-cells, cardiomyocytes, etc. The functioning of such models are based on the Hodgkin-Huxley formalism, which describes the dynamics of the electric potential of the cell membrane during the transport of potassium, calcium, chlorine, and sodium ions through ion channels in the membrane. This type of model is characterized by a number of nonlinear effects including multistability, which is the coexistence of different modes of functioning of the cell model. Of particular interest among all types of multistability is the bistability between the silent state and burst state, which under certain parameters, can be implemented in the simplified model of the leech neuron proposed in [2].

In the present work, we consider a modification of the Sherman model [3] based on the Hodgkin-Huxley formalism, in which a bistability occurs between the steady state and bursting attractor. The modification [4] consists in the taking into account an additional potassium ion channel, the opening function of which is never equal to 1.0, in contrast to the typical potassium channel. Such feature of the model can be interpreted as a communication defect. In the frame of this work the probabilistic characteristics of such modified model are studied in dependence of the defect ion channel parameters.

The work was supported by a grant from the Russian Science Foundation (No. 20-71-10048).

References

[1] Izhikevich E.M. Dynamical systems in neuroscience. MIT Rress, (2007).

[2] Malashchenko T., Shilnikov A., Cymbalyuk G. PLoS One. **6** (2011) e21782.

[3] Sherman A., Rinzel J., Keizer J. **54**(3) (1988) 411-425.

[4] Stankevich N.V., Mosekilde E. Chaos. **27**(12) (2017) 123101.

Homoclinic chaos in the Rosenzweig-MacArthur model

Bakhanova Yu. V.

Laboatory of topological methods in dynamics
National Research University Higher School of Economics

The main goal of the work is to explain topical issues of the spiral attractors related to the formation of the loop to the saddle-focus in the three-dimensional Rosenzweig-MacArthur system. It was determined that homoclinic chaos in the system arises according to the Shilnikov's scenario [1]. Shilnikov's attractor, containing saddle-focus $(1, 2)$, and saddle Shilnikov's attractor, containing saddle $(1, 2)$ equilibrium point, were found in the system and studied in detail.

Shilnikov's attractor

Rosenzweig-MacArthur model was studied in detail in the work [2], it describes the dynamics of interaction of three populations in the prey – predator – superpredator food chain:

$$\begin{cases} \dot{x} = x\left(r\left(1 - \dfrac{x}{K}\right) - \dfrac{5y}{1+3x}\right) \\ \dot{y} = y\left(\dfrac{5x}{1+3x} - \dfrac{0.1z}{1+2y} - 0.4\right) \\ \dot{z} = z\left(\dfrac{0.1y}{1+2y} - 0.01\right) \end{cases} \tag{1}$$

Chaotic attractors were first discovered in this system in the works [3, 4, 5]. In papers [2, 6], it was shown that strange attractors here can have a spiral structure due to the appearance of homoclinic loops to a saddle-focus equilibrium state. In 2018 it was found [7] that attractors in the system (1) arise according to the Shilnikov's scenario. When changing a parameter r a chain of bifurcations can be observed in the system: a stable equilibrium state – a limit cycle – a double period limit cycle – a feigenbaum-type attractor and, with a further increase in the value of the parameter, homoclinic trajectories to the saddle-focus arise. For this to happen, Shilnikov's condition should be met [6], which is a criteria of the spiral chaos existence. One of such attractors is shown in the figure 1a.

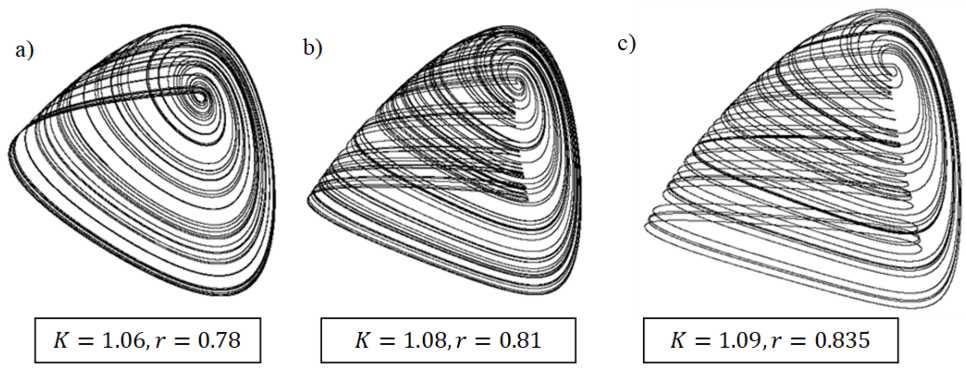


Figure 1: Three types of Shilnikov's attractor in the Rosenzweig-MacArthur system (1) found with different parameters (K, r)

Saddle Shilnikov's attractor

In the [2, 6] papers it was shown, that bifurcation curve h on the K and r control parameters plane correspond to the existence of a homoclinic loop to the S equilibrium state. Along this curve, although the attractor changes (its shape becomes more complicated due to the appearance of multi-pass homoclinic loops), it always contains the point S, that means that it is homoclinic, see Fig. 1a, 1b, 1c. In the point B with coordinates $(K, r) \approx (1.22, 4.026)$ on the bifurcation curve h on the parameters plane S equilibrium state has the pair of multiple eigenvalues, so the equilibrium state is saddle-focus $(1, 2)$ when parameters coordinates are below the point B on the parameters plane and is saddle $(1, 2)$ when coordinates are above the point B on the (K, r) parameters plane. The results of numerical calculations show that when passing through point B homoclinic attractor is preserved, but its type changes to a saddle Shilnikov one. An example of the saddle Shilnikov's attractor is shown in the Fig. 2 with parameters $(K, r) = (1.247, 4.31)$.

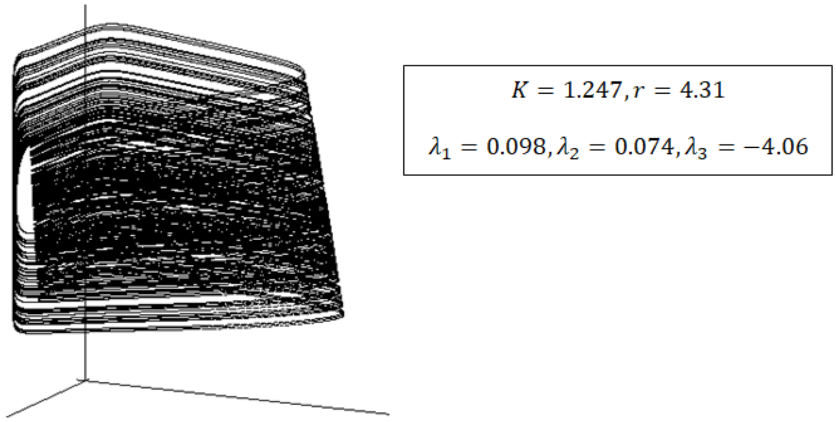


Figure 2: Saddle Shilnikov's attractor with parameters $(K, r) = (1.247, 4.31)$

The paper [7] considered the question of the pseudo-hyperbolicity of such an attractor. The result of checking the continuity of a stable manifold for continuity showed that such an attractor is not pseudohyperbolic. The question of the existence of the pseudohyperbolic attractor in the considered model, as well as the question of the existence of the pseudohyperbolic saddle attractor of Shilnikov, is currently open.

References

[1] Bakhanova Y.V., Kazakov, A.O., Korotkov A.G., Levanova T.A., Osipov G.V. Spiral attractors as the root of a new type of «bursting activity» in the Rosenzweig–MacArthur model // The European Physical Journal Special Topics. 2018. Vol. 227, no. 7–9. P. 959–970

[2] Kuznetsov Y.A., De Feo O., Rinaldi S. Belyakov homoclinic bifurcations in a tritrophic food chain model // SIAM Journal on Applied Mathematics. 2001. Vol. 62, no. 2. P. 462–487

[3] Hastings A., Powell T. Chaos in a three-species food chain // Ecology. 1991. Vol. 72, no. 3. P. 896–903

[4] Rai V., Sreenivasan R. Period-doubling bifurcations leading to chaos in a model food chain // Ecological modelling. 1993. Vol. 69, no. 1–2. P. 63–77

[5] Kuznetsov Y.A., Rinaldi S. Remarks on food chain dynamics // Mathematical biosciences. 1996, vol. 134, no. 1. P. 1–33

[6] Deng B., Hines G. Food chain chaos due to Shilnikov's orbit // Chaos: An Interdisciplinary Journal of Nonlinear Science. 2002. Vol. 12, no. 3. P. 533–538

[7] Bakhanova Yu.V., Kazakov A.O., Karatetskaia E.Yu., Kozlov A.D., Safonov K.A. About homoclinic attractors of three-dimensional flows // Izvestiya VUZ. Applied Nonlinear Dynamics, 2020, vol. 28, no. 3, pp. 231–258

Invariant manifolds of homoclinic orbits: super-homoclinics and multi-pulse homoclinic loops

Bakrani, Sajjad [1]
(joint work with Dmitry Turaev and Jeroen S. W. Lamb)

1. Faculty of engineering and natural sciences
Kadir Has University, Turkey

Consider a Hamiltonian flow on $\mathbb{R}^4$ with a hyperbolic equilibrium O and a transverse homoclinic orbit Γ. In this presentation, we discuss the dynamics near Γ in its energy level when it leaves and enters O along strong unstable and strong stable directions, respectively. In particular, we provide necessary and sufficient conditions for the existence of the local stable and unstable invariant manifolds of Γ. We then consider the case in which both of these manifolds exist. We globalize them and assume they intersect transversely. We show that near any orbit of this intersection, called super-homoclinic, there exist infinitely many multi-pulse homoclinic loops.

Foliated Devaney chaos

Barral Lijó, R.

Research Organization of science and technology Ritsumeikan University

This talk is based on work that is still in progress. Devaney introduced in [3] the following definition of chaos.

Definition 1. *(Devaney Chaos). A continuous map $f\colon X \to X$ on a metric space $(X; d)$ is chaotic if,*

(TT) for every pair of non-empty open sets U and V , there is n 2 N such that $f^n(U) \cap V \neq \varnothing$ (f is topologically transitive),

(DPO) the set of periodic points is dense in X (f has density of periodic orbits), and

(SIC) there is $c > 0$ such that, for every $x \in X$ and $r > 0$, there are $y \in B(x; r)$ and $n \in N$ with $d(fn(x); fn(y)) \geqslant c$ (f is sensitive to initial conditions).

It was proved in [1] that, in fact, (TT) and (DPO) imply (SIC); this result was later generalized to topological semigroup actions. Our aim is to find analogous definitions for pseudogroups on locally compact Polish spaces and to investigate under which conditions a similar result holds. Since the dynamics of foliated spaces are modelled by pseudogroups, this will provide information on chaotic foliated spaces. We should mention the very recent paper by Bazaikin, Galaev, and Zhukova studying chaos in the setting of Cartan foliations [2]. A foliated space is a topological space X endowed with a partition into connected topological manifolds called leaves satisfying that the leaves are locally stacked together as a product space. The most notable examples are foliations, where the ambient space is actually a manifold. Foliated spaces are generalized dynamical systems, where the leaves take the role of the orbits. A dynamical model for the foliated space X is given by the holonomy pseudogroup, acting on a union of local tranversal models; this pseudogroup 1 representation is not unique, but it can be shown to be unique up to etal´e equivalence. In this way, there is a one-to-one correspondence between leaves in the foliated space and orbits of the holonomy pseudogroup. In this talk, we will first introduce analogues of the conditions (TT), (DPO), and (SIC) for pseudogroups. While (TT) is straightforward, the other two conditions need a more subtle approach involving generating sets of the pseudogroup. We will also prove that these conditions are invariant by etal´e equivalences, and therefore it is well-defined whether a foliated space is chaotic in the sense of Devaney. Secondly, we will introduce the notion of compact generation and the following result.

Theorem 1. *If G is compactly generated pseudogroup satisfying (TT)+(DPO), then it also satisfies (SIC).*

The holonomy pseudogroup of a compact foliated space is always compactly generated, so we obtain the following application.

Corollary 1. *If X is a topologically transitive compact foliated space with a dense set of compact leaves, then X is sensitive to initial conditions.*

Finally, we will show using simple examples that the implication

$$(TT) + (DPO) \to (SIC)$$

does not hold for either non-compactly generated pseudogroups or for noncompact foliated spaces.

References

[1] G. Banks, J. Brooks, G. Cairns, G. Davis, and P. Stacey. On Devaney's

[2] Y.V. Bazaikin, A.S. Galaev, and N.I. Zhukova. Chaos in Cartan foliations. Chaos, 30, 103116, 2020.

[3] R.L. Devaney. An Introduction to Chaotic Dynamical Systems. Addison- Wesley, 3rd edition, 1989.

On the Godbillon-Vey-Losik class for the Reeb foliations

Bazaikin Ya.V.[1,2], Galaev A.S.[1], Gumenyuk P.A.[3]

1. University of Hradec Králové, Czech Republic

2. Sobolev Institute of Mathematics, Novosibirsk, Russia

3. Politecnico di Milano, Italy

The classical Godbillon-Vey class is an important invariant of a codimension-one foliation, and it is tightly related to the dynamics of the leaves of the foliation [3]. For all Reeb foliations on the 3-dimensional sphere, the Godbillon-Vey class is zero. Following Losik [4], we consider in [1,2] a modified Godbillon-Vey class that we call the Godbillon-Vey-Losik class (GVL class). This class takes values in the cohomology $H^3(S(M/F))$ of the second order frame bundle over the leaf space of the foliation. The GVL class is defined by the form

$$dx_0 \wedge dx_1 \wedge dx_2,$$

where x_0, x_1, x_2 are certain natural coordinates on $S(M/F)$. There is a map $H^3(S(M/F)) \to H^3(M)$ sending the GVL class to the classical Godbillon-Vey class.

The main result from [2] states that the GVL class is non-trivial for some Reeb foliations and it is trivial for some other Reeb foliations. In particular, the GVL class can distinguish non-diffeomorphic foliations and it provides more information than the classical Godbillon-Vey class.

A Reeb foliation on the 3-dimensional sphere is uniquely defined by a germ at 0 of diffeomorphisms of $\mathbb{R}$ infinitely tangent to the identity. Fix $\alpha > 0$ and consider the following vector field on $\mathbb{R}$:

$$V_\alpha(x) = \begin{cases} e^{-\frac{1}{|x|^\alpha}}, & \text{for } x \neq 0, \\ 0, & \text{for } x = 0. \end{cases}$$

The vector field V_α defines the flow φ_t^α. Denote by $\mathcal{R}_\alpha$ the Reeb foliation defined by the germ of the diffeomorphism φ_1^α.

Theorem 1. If $\alpha \in \mathbb{N}$ is odd, then the GVL class of $\mathcal{R}_\alpha$ is non-trivial.

Theorem 2. If $\alpha \in \mathbb{N}$ is an even, then the GVL class of $\mathcal{R}_\alpha$ is trivial.

Corollary. If $\alpha \in \mathbb{N}$ is odd and $\beta \in \mathbb{N}$ is even, then the foliations $\mathcal{R}_\alpha$ and $\mathcal{R}_\beta$ are not diffeomorphic.

These results show that the GVL class is very sensitive to the dynamics of the non-compact leaves in the following sense. Dynamics of the non-compact leaves in a neighborhood of the compact leaf $\mathcal{L}$ is described by the holonomy group of $\mathcal{L}$. For a fixed $x \in \mathcal{L}$ and a transversal T to F through x, the holonomy group of $\mathcal{L}$ consists of the germs at x of local diffeomorphisms of T defined by loops in $\mathcal{L}$ starting at x. This group can be included to a 1-parameter group of the germs of local diffeomorphisms of the transversal with one fixed point x. Even if this group consists of the germs of diffeomorphisms infinitely tangent to the identity, there is a notion of the order of the convergence (a 1-parameter group has greater order of convergence than another 1-parameter group, if the fraction of their generating vector fields is smooth and has zero value at x). It turns out that in some situations the GVL class distinguishes this order.

References

[1] Ya.V. Bazaikin and A.S. Galaev, Losik classes for codimension-one foliations, arXiv:1810.01143. To appear in J. Inst. Math. Jussieu.

[2] Ya.V. Bazaikin, A.S. Galaev, and P. Gumenyuk, Non-diffeomorphic Reeb foliations and modified Godbillon-Vey class, arXiv:1912.01267.

[3] S. Hurder, Dynamics and the Godbillon-Vey class: a History and Survey, In Foliations: Geometry and Dynamics (Warsaw, 2000), World Scientific Publishing Co. Inc., River Edge, N.J., 2002, 29–60.

[4] M. V. Losik, Categorical differential geometry. Cahiers de topol. et geom. diff. cat. 35 (1994), no. 4, 274–290.

Classical soliton theory for studying the dynamics and evolution of toroidal vortices shock waves in passive dispersed and active relaxation media

Belyakin S. T.[1], **Shuteev S. A.**[2]

1. Department of General Physics
Physics Faculty, Lomonosov Moscow State University, Moscow, Russia

2. Laboratory of dynamic systems
Physics Faculty, Lomonosov Moscow State University, Moscow, Russia

In this paper, we consider the possibility of using the dynamic classical soliton model to study the behavior of the evolution and dynamics of toroidal vortex flows. The main model used is that of two coupled van der Pol generators. This robot, based on the nonlinear dynamics of the Bernoulli map, presents the evolution of the torus. The use of the dynamic classical soliton model makes it possible to trace the dynamics of vortex flows. [1]

Mathematical dynamic model of a soliton for a toroidal vortex

The mathematical classical dynamic model of the soliton is represented by the equation [2]:

$$\begin{cases} \ddot{x} - (A\cos\omega t - x^2)\dot{x} + \omega_0^2 x = \varepsilon y \cos\omega_0 t, \\ \ddot{y} - (-A\cos\omega t - y^2)\dot{y} + \omega_0^2 y = \varepsilon x^2, \\ \dot{z} = xy \sinh x \sinh y \sin\omega_0 t \cos\omega_0 t. \end{cases}$$

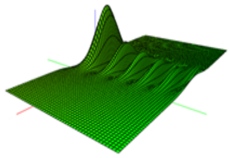
Here x, y, z - dynamic variables, A, ε and k - coefficient of connection, ω and ω_0 -inherent frequency oscillations.

The use of this hyperbolic model translates into a practical plane the problem of comparative research of hyperbolic and non-hyperbolic chaos in theory and experiment. This system can be used to study toroidal vortex processes using the classical soliton theory.

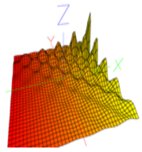
References

[1] S.T. Belyakin, S.A. Shuteev. Dynamics and Evolution of toroidal vortices by analog models. Annals Reviews and Research, 5, 2020, No4, Pp.0049-0056.

[2] S.P.Kuznetsov. A system of three non-Autonomous oscillators with hyperbolic chaos. A model with dynamics on the attractor described by the mapping on the torus "Arnold's cat". Izvestiya vuzov. Applied nonlinear dynamics, 20, 2012, No6, Pp.56-66.



(a) Figure1.



(b) Figure2.

Mukai-Fourier Transforms as Solutions to Field Equations: Spectrum as Higgs-Oscillations of H-States in the Space-Time

Prof. Dr. Francisco Bulnes

IINAMEI, Research Department in Mathematics and Engineering
TESCHA Chalco, Mexico.

The Mukai-Fourier transform is an equivalence between derived categories (with arbitrary decorations +, -, b). Thus is feasible construct a Fourier-Mukai equivalence given for $D_{Coh}(T^{\vee}A) \cong D_{Coh}(A^{\vee} \times \mathbf{H})$, where exists a distinguished deformation of the category $D_{Coh}(T^{\vee}A)$, which is non-commutative deformation of $T^{\vee}A$ defined by a natural symplectic form, that is its quatization [1] . Then $T_0^{\vee}A$, results a 1-parameter deformation A^b, of the space $A^{\vee} \times \mathbf{H}$, to an affine bundle over $A^{\vee}$, classified by $H^1(A^{\vee}; \mathcal{O} \otimes \mathbf{H})$. Then the Fourier-Mukai equivalence relative to the projection $T_0^{\vee}A$, deforms an equivalence between the deformed categories $D_{Coh}\mathcal{D}_A-$mod, and $D_{Coh}(A^b)$. Then we use the deformed version of the Mukai-Fourier transform that results on D_A- modules and we characterize to A, as a Picard variety of a curve C[1]. Then a Hecke functor is defined as the integral transform $\Phi^1 : D_{Coh}(\mathrm{Pic}(C), \mathcal{D}) \to D_{Coh}(C \times \mathrm{Pic}(C), \mathcal{D})$, to D- modules on ${}^L\mathrm{Bun}$. But using the classical limit conjecture is had the equivalence through the interpretation of Higgs sheaves, given in the category $D_{Coh}({}^L\mathrm{Higgs}_0, \mathcal{O})$, which can be extended to the corresponding Langlands correspondence $\mathfrak{c}$ of the "quantum" sheaves given by $\mathfrak{c} = quant_{Bun} \circ \Phi \circ quant_C^{-1}$, where Φ, is the Fourier-Mukai transform that we need. Then we have as integral transform the integral transforms composition [2] $\mathfrak{c} \circ \Phi^{\mu} =^L \Phi^{\mu}$, which is solution of the field equations, $\mathrm{Isom}d\mathbf{h} = \mathbf{0}$, where $\mathbf{h}$, are the mentioned cotangent vectors (Higgs fields). Then by superposing of these states, considering the field corresponding ramifications (connections), we have

$$\mathcal{H} = H^0(\omega_C) \oplus H^0(\omega_C^{\otimes 2}) \oplus \cdots \oplus H^0(\omega_C^{\otimes n}) \tag{1}$$

Likewise, a graphical representation through a 2-dimensional model of (1) considering a re-interpretation as the energy density expressed through the $\mathbf{H}-$ states, which can be written using the superposing principle for each connection $\omega_C^{\otimes j}$, with C, the curve that describe the corresponding dilaton (gauging particle) in field external presence. Likewise, in Hamilton densities space [3] we have the figure 1, considering a Hitchin basis. In the case of a spinor representation the corresponding $\mathbf{H}-$ states can be as spinor waves (figure 2) which can be consigned in oscillations in the space-time to a microscopic deformation measured [4],[5] in $\mathcal{H}$.

References

[1] E. Frenkel, C. Teleman Geometric Langlands correspondence near opers, J. of the Ramanujan Math. Soc. pp. 123–147.

[1] In a physical context (could be taken $\mathbb{M} = \mathrm{Pic}(C)$, where $\mathbb{M}$, is the space-time), this represent a trace of particles in the symplectic geometry that can be characterized in a Hamiltonian manifold.

[2] F. Bulnes Extended d - Cohomology and Integral Transforms in Derived Geometry to QFT–equations Solutions using Langlands Correspondences, *Theoretical Mathematics and Applications*, Vol. 7 (2), pp 51–62.

[3] F. Bulnes, *Integral Geometry Methods in the Geometrical Langlands Program*, SCIRP, USA, 2016.

[4] Bulnes, F. , Stropovsvky, Y. and Rabinovich, I. (2017) Curvature Energy and Their Spectrum in the Spinor-Twistor Framework: Torsion as Indicium of Gravitational Waves. *Journal of Modern Physics*, 8, 1723-1736. doi: 10.4236/jmp.2017.810101.

[5] Francisco Bulnes (2017). Detection and Measurement of Quantum Gravity by a Curvature Energy Sensor: H-States of Curvature Energy, Recent Studies in Perturbation Theory, Dr. Dimo Uzunov (Ed.), InTech, DOI: 10.5772/68026.

On the structure of homoclinic bifurcation curves in three-dimensional systems with Shilnikov attractors.

Burdygina T. K.[1], **Bobrovsky A. A.**[1], **Malykh S. M.**[1]

National Research University Higher School of Economics

In this paper we consider dynamical systems, which demonstrate Shilnikov attractors. It is worth reminding that, by definition, Shilnikov attractor contains a saddle-focus equilibrium state with two-dimensional unstable manifold.

In particular, a bifurcation set of homoclinic loops of such saddle-focus equilibrium state is studied in detail. We consider homoclinic orbits along with multi-round homoclinic orbits arising near it. From the paper [5] it is know, that the primary homoclinic curve has the U-shape. From works [6] and [7] it is known that curves of double, triple, and multi-round loops are located between a pair of branches of the primary homoclinic curve. Our main result is a visualization of dynamical regimes and the structure of homoclinic curves in Arneodo-Coullet-Tresser [3] and Rössler systems [1], [2], which, as well-known, demonstrate Shilnikov attractors.

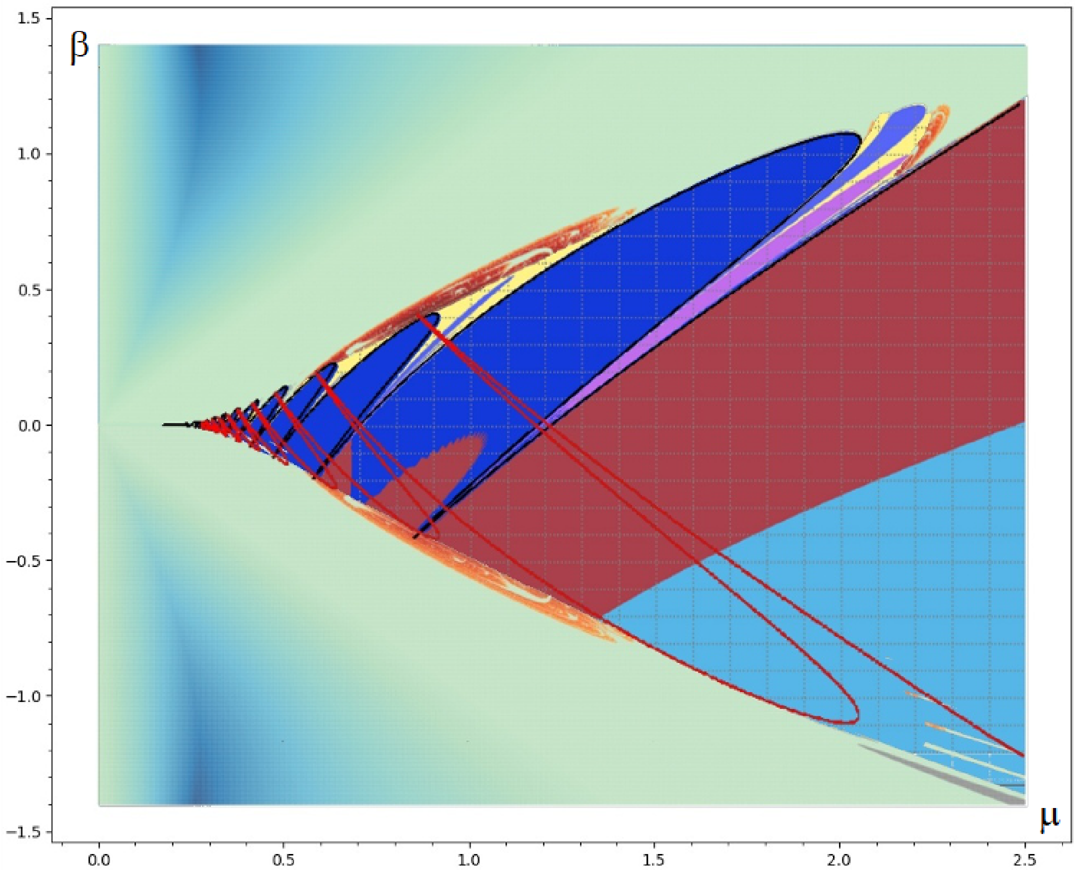


Figure 1. Bifurcation diagram for the Arneodo - Coullet - Tresser system, which consist of chart of symbolic dynamics, two curves corresponding to primary homoclinic orbits for equilibrium states (1,0,0) and (0,0,0), and charts of maximal Lyapunov exponent in forward and backward time.

In Fig. 1 two curves of primary homoclinic orbits for the Arneodo-Coullet-Tresser system are presented. Red curve corresponds to the loop of the equilibrium state (1,0,0), on which the attractor appearance is based, black one corresponds to the homoclinic loop of the second equilibrium (0,0,0). Moreover, Fig. 1 shows the chart of maximal Lyapunov exponent above the chart of symbolic dynamics. The black colored curve and chart of maximal Lyapunov exponent for $\beta > 0$ are calculated in forward time, while, the red colored curve and the chart of maximal Lyapunov exponent for $\beta < 0$ calculated in backward time. Points in the chart of maximal Lyapunov exponent are marked by blue color, when the exponent is negative (the darker the color, the lower its value), when the exponent is positive, the points are marked by shades of red color. Grey points indicate that orbits go to infinity for this parameter values. Kneading charts help to efficiently visualize the behavior of an unstable separatrix of a equilibrium state saddle-focus. In Fig.1 one can see the results obtained with help of kneading diagrams are in good agreement with bifurcation curves obtained using Matcont toolkit.

These studies were supported by the Laboratory of Dynamical Systems and Applications NRU HSE (Grant No. 075-15-2019-1931) from the Ministry of Science and Higher Education of Russian Federation.

References

[1] O. E. Rössler, "An equation for continuous chaos," Physics Letters A 57, 397–398 (1976).

[2] O. E. Rössler, "An equation for hyperchaos," Physics Letters A 71, 155–157 (1979).

[3] A. Arneodo, P. Coullet, and C. Tresser, "Oscillators with chaotic behavior: an illustration of a theorem by Shilnikov," Journal of Statistical Physics 27, 171–182 (1982).

[4] L. P. Shilnikov, "Bifurcation theory and turbulence," Methods of the Qualitative Theory of Differential Equations, 150–163 (1986)

[5] Y. A. Kuznetsov, O. De Feo, and S. Rinaldi, "Belyakov homoclinic bifurcations in a tritrophic food chain model," SIAM Journal on Applied Mathematics 62, 462–487 (2001).

[6] P. Gaspard, "Generation of a countable set of homoclinic flows through bifurcation", Physics Letters A 97, 1–4 (1983).

[7] S. V. Gonchenko, D. V. Turaev, P. Gaspard, and G. Nicolis, "Complexity in the bifurcation structure of homoclinic loops to a saddle-focus," Nonlinearity 10, 409 (1997).

Topological renormalization in the Hénon family

Crovisier S.[1]

1. Laboratoire de mathématiques d'Orsay
CNRS - Université Paris-Saclay (France)

I will describe how maps in the real Hénon family bifurcate from zero to positive entropy. When the Jacobian is close to 0, this has been done by De Carvalho, Lyubich and Martens, who have built a renormalization operator by a perturbative approach from the one-dimensional quadratic family. With E. Pujals and C. Tresser we extend some of these results up to Jacobian 1/4: any such Hénon map with zero entropy can be renormalized. As a consequence, we obtain a two-dimensional version of Sharkovsky's theorem about the set of periods of interval maps. Our techniques use mainly topological arguments. [1]

References

[1] S. Crovisier, E. Pujals, C. Tresser. Mildly dissipative diffeomorphisms of the disk with zero entropy. Preprint (2020) *ArXiv:2005.14278.*

Algebraic invariants for ordinary differential equations: definitions and applications

Demina M.V.

Department of applied mathematics,
HSE University, Moscow, Russia

maria_ dem@mail.ru

The problem of establishing integrability or solvability for a given ordinary differential equation is a classical problem of analysis. Let us consider an autonomous algebraic n-th order ordinary differential equation with $n \geq 2$. The aim of this talk is to address the following questions.

1. Does there exist an autonomous algebraic first–order ordinary differential equation compatible with the original equation?

2. If yes, how to find all such equations?

Bivariate polynomials producing autonomous algebraic first–order ordinary differential equations compatible with the equation under consideration are called algebraic invariants. The main difficulty in deriving irreducible algebraic invariants lies in the fact that their degrees are not known in advance. The problem of establishing an upper bound on the degrees of irreducible algebraic invariants in the case $n = 2$ is now commonly refereed to as the Poincaré problem [1]. At the moment only partial solutions to this problem are available.

We shall introduce a method, which can be used to find all the irreducible algebraic invariants [2]. The main idea of the method is to consider the factorizations of invariants over the algebraically closed field of fractional power series. As an example, we shall present the complete classification of irreducible algebraic invariants for several physically relevant ordinary differential equations including the famous Duffing – van der Pol equation [3] and the traveling–wave reduction of the modified Kuramoto–Sivashinsky equation [2].

Let us note that in the two–dimensional case algebraic invariants are key objects in establishing Darboux and Liouvillian integrability of the original ordinary differential equation [4]. Indeed, if one knows the complete set of irreducible algebraic and exponential invariants, then it is straightforward to decide whether an autonomous second–order ordinary differential equation is integrable or non–integrable with a Darboux or Liouvillian first integral.

In addition, algebraic invariants can be used to perform the classification of meromorphic solutions for wide families of ordinary differential equations. We plan to pay some attention to this problem.

The research reported in this talk is supported by Russian Science Foundation grant 19–71–10003.

References

[1] Ilyashenko Yu., Yakovenko S. *Lectures on Analytic Differential Equations.* — Graduate Studies in Mathematics – American Mathematical Society — Volume 86, 2008.

[2] Demina M. V. Classifying algebraic invariants and algebraically invariant solutions, *Chaos, Solitons and Fractals*, **140**, 110219 (2020).

[3] Demina M. V. Invariant algebraic curves for Liénard dynamical systems revisited, *Appl. Math. Lett.*, **84**, 42–48 (2018).

[4] Singer M. F. Liouvillian first integrals of differential systems, *Trans. Amer. Math. Soc.*, **333**, 673–688 (1992).

Smooth Small Perturbations of Skew Products and the Partial Integrability Property

Efremova L.S.

National Research Lobachevsky State University of Nizhni Novgorod

Moscow Institute of Physics and Technology, Moscow region, Dolgoprudny, Russia

lefunn@gmail.com

We discuss the integrability property of maps in the plane and formulate the criterion of integrability for these maps [1] – [2].
Then we introduce the concept of the partial integrability for maps in the plane and consider maps of the form

$$F(x,\, y) = (f(x) + \mu(x,\, y),\, g(x,\, y)) \quad \text{for any } (x,\, y) \in I$$

($I = I_1 \times I_2$; I_1, I_2 are closed intervals) satisfying some additional conditions.
We prove sufficient conditions of the partial integrability for maps under consideration and give the example of the partially integrable map [3] – [4].

References

[1] *Belmesova S.S., Efremova L.S.* On the Concept of Integrability for Discrete Dynamical Systems. Investigation of Wandering Points of Some Trace Map. Nonlinear Maps and their Applic.// Springer Proc. in Math. and Statist. 2015. 112. 127-158.

[2] *Efremova L. S* Dynamics of skew product of interval maps. Russian Math. Surveys. 2017. 72, 1. 101-178

[3] *Efremova L.S.* Small C^1-smooth perturbations of skew products and the partial integrability property// Applied Math. Nonlinear Sci. 2020 (to appear).

[4] *Efremova L.S.* Small Perturbations of Smooth Skew Products and Sharkovsky's Theorem// J. Difference Eq. and Applic. Special issue on the occasion of the 82-nd birthday of Olexander M. Sharkovsky, 2020. 26, 8. 1192-1211.

Optimizing discounted income for a structured population subject to harvesting

Egorova A. V.

Department of Functional Analysis and its Applications
Vladimir State University named after Alexander and Nikolay Stoletovs

The problem of rational use of renewable natural resources has remained relevant for many years. This is confirmed by a large number of works devoted to studies of the development of structured populations, divided into age groups or typical groups, and the problems of optimal harvesting [1]. The problems of optimal resource collection in probabilistic models [2] and the problem of maximizing the net discounted income from fishing for the [3] population are of great interest.

In this abstract we investigate the models of the dynamics of an exploited structured population. The main task is to calculate the discounted income from the extraction of a renewable resource over a finite period of time and to construct a control for which this function is maximal.

Define $x_i(k)$, $i = 1, \dots, n$ the number of resources of each of the $n \geq 2$ of the species or classes at the moment $k = 0, 1, 2, \dots$. We will consider the model of the exploited population in the form

$$x(j+1) = F\big((1-u(j))x(j)\big), \quad j = 0, 1, 2, \dots, k-1,$$

where $x(j) = \big(x_1(j), \dots, x_n(j)\big) \in R^n_+$, $R^n_+ \doteq \big\{x \in R^n : x_1 \geq 0, \dots, x_n \geq 0\big\}$, $u(j) = \big(u_1(j), \dots, u_n(j)\big) \in [0,1]^n$ — control that can be varied to achieve the best collection result, $(1 - u_i(j))x_i(j)$ — number of remaining resource of the i-th species at the moment k after harvesting, $F(x) = \big(f_1(x), \dots, f_n(x)\big)$, $f_i(x)$ — real non-negative functions defined for all $x \in R^n_+$, $f_i(0) = 0$, $f_i \in C^2(R^n_+)$, and Jacobi matrix $\left(\frac{\partial f_i}{\partial x_j}\right)_{i,j=1,\dots,n}$ is nondegenerate for all $x \in R^n_+$.

We will assume that the costs of a conventional unit of each of the classes are equal to the constant $C_1, \dots, C_n$, where $C_i \geq 0$, $i = 1, \dots, n$. Suppose that the cost of all production at the moment $k = j$ is $h_\alpha(j) = \sum\limits_{i=1}^{n} C_i x_i(j) u_i(j) e^{-\alpha j}$, where $\alpha > 0$ — is the discount factor. For any $\overline{u} \in U$ and $x(0) \in R^n_+$, we introduce into consideration the function

$$H_\alpha\big(\overline{u}, x(0)\big) \doteq \sum_{j=0}^{\infty} h_\alpha(j) = \sum_{j=0}^{\infty}\sum_{i=1}^{n} C_i x_i(j) u_i(j) e^{-\alpha j},$$

which we will call *discounted income* from resource extraction.

Theorem 1. *Let the function $D(x) \doteq \sum\limits_{i=1}^{n} C_i\big(f_i(x) - x_i e^\alpha\big)$ reaches the maximum value in the only one point $x^* \in R^n_+$ u $x^*_i \leq f_i(x^*) \neq 0$ for any $i = 1, \dots, n$. Then for any $x(0) \in R^n_+$ such that $x_i(0) \geq x^*_i$, $i = 1, \dots, n$, function $H_\alpha\big(\overline{u}(k), x(0)\big)$ reaches the maximum value*

$$H_\alpha\big(\overline{u}^*(k), x(0)\big) = D(x^*)\frac{e^{-\alpha(k-1)} - 1}{1 - e^\alpha} + \sum_{i=1}^{n} C_i x_i(0)$$

on multiple $[0,1]^{kn}$ at the following exploitation mode:

1) *if $k = 1$, then $u^*(0) = (1, \dots, 1)$;*

2) *if $k = 2$, then $\overline{u}^*(2) = \big(u^*(0), u^*(1)\big)$, $u^*(0) = \left(1 - \frac{x^*_1}{x_1(0)}, \dots, 1 - \frac{x^*_n}{x_n(0)}\right)$, $u^*(1) = (1, \dots, 1)$;*

3) *if* $k \geq 3$, *then* $\overline{u}^*(k) = \big(u^*(0), \ldots, u^*(k-1)\big)$, *where* $u^*(0) = \Big(1 - \frac{x_1^*}{x_1(0)}, \ldots, 1 - \frac{x_n^*}{x_n(0)}\Big)$; $u^*(j) = \Big(1 - \frac{x_1^*}{f_1(x^*)}, \ldots, 1 - \frac{x_n^*}{f_n(x^*)}\Big)$ *npu* $j = 1, \ldots, k-2$; $u^*(k-1) = (1, \ldots, 1)$.

The work was carried out under the guidance of Professor of the Department of Functional Analysis and its Applications of Vladimir State University L.I. Rodina.

References

[1] G. P. Neverova, A. I. Abakumov, E. Ya. Frisman. Dynamic modes of exploited limited population: results of modeling and numerical study, Mat. Biolog. Bioinform., 2016, V. 11, №1, PP. 1-13 (in Russian).

[2] L. I. Rodina. Optimization of average time profit for a probability model of the population subject to a craft, Vestn. Udmurtsk. Univ. Mat. Mekh. Komp. Nauki, 2018, V. 28, №1, PP. 48–58 (in Russian).

[3] A. O. Belyakov, V. M. Veliov. On optimal harvesting in age-structured populations // Dynamic Perspectives on Managerial Decision Making, 2016, PP. 149–166.

[4] A. V. Egorova, L. I. Rodina. On optimal harvesting of renewable resource from the structured population // Vestn. Udmurtsk. Univ. Mat. Mekh. Komp. Nauki, 2019, V. 29, №4, PP. 501–517 (in Russian).

New Cantor sets with high-dimensional projections

Frolkina O.D.[1]

1. Faculty of Mechanics and Mathematics
M.V. Lomonosov Moscow State University

L. Antoine described a Cantor set in plane all of whose projections coincide with those of a regular hexagon [2, **9**, p.272; fig.2, p.273]. For each $n \geqslant 2$, K. Borsuk constructed a Cantor set in $\mathbb{R}^n$ such that its projection onto every hyperplane contains an $(n-1)$-dimensional ball, equivalently, has dimension $(n-1)$ [4]. J. Cobb gave an example of a Cantor set in $\mathbb{R}^3$ such that its projection onto every 2-plane is one-dimensional, and posed a general question: "Could there be Cantor sets all of whose projections are connected, or even cells? ...given $n > m > k > 0$, does there exist a Cantor set in $\mathbb{R}^n$ such that each of its projections into m-planes is exactly k-dimensional?" [5]. In case of $m = k$, a positive answer is given by [4]. For cases $k = m - 1$ and $m = n - 1$, such sets were constructed in [6] and [3], correspondingly; both papers extend Cobb's ideas. Cobb's method is rather sophisticated; the resulting Cantor set (and also the sets from [4], [3], [6]) is tame in the following sense.

Definition. A zero-dimensional compact set $K \subset \mathbb{R}^n$ is called *tame* if there exists a homeomorphism h of $\mathbb{R}^n$ onto itself such that $h(K)$ is a subset of a straight line in $\mathbb{R}^n$; and it is called *wild* otherwise.

In $\mathbb{R}^2$ each zero-dimensional compactum is tame [1, **75**, p. 87–89]. L. Antoine constructed a family of Cantor sets in $\mathbb{R}^3$ which are now widely known as Antoine's necklaces [1, **78**, p. 91–92] and proved that they are wild [1, Part 2, Chap. III]; this was extended to $\mathbb{R}^n$, $n \geqslant 4$, independently by A.A.Ivanov and W.A.Blankinship.

Applying the theory of tame and wild Cantor sets, we prove (see [7]):

Theorem 1. *Let $K \subset \mathbb{R}^n$ be any Cantor set, $n \geqslant 2$. For each $\varepsilon > 0$ there exist*

1) an ε-isotopy $\{h_t\} : \mathbb{R}^n \cong \mathbb{R}^n$ such that for each $m \leqslant n-1$ the projection of $h_1(K)$ into any m-plane is m-dimensional;

2) an ε-isotopy $\{g_t\} : \mathbb{R}^n \cong \mathbb{R}^n$ such that the projection of $g_1(K)$ into any $(n-1)$-plane is $(n-2)$-dimensional.

For the case $(3, 2, 1)$, we present another very simple series of Cantor sets in $\mathbb{R}^3$ all of whose projections are connected and one-dimensional [8]. These are self-similar Antoine's necklaces which satisfy some additional conditions; self-similarity omits the necessity of additional isotopical transformations. (All necessary notions will be explained in the talk.)

Theorem 2. *Let $\mathcal{A}(T; S_1, \ldots, S_k) \subset \mathbb{R}^3$ be a self-similar Antoine's necklace, and let s_i be the similarity coefficient of S_i for $i = 1, \ldots, k$. If $s_1^2 + \ldots + s_k^2 < 1$, then projections of $\mathcal{A}(T; S_1, \ldots, S_k)$ into all planes and straight lines are connected one-dimensional sets. In particular, this holds for a regular self-similar Antoine's necklace $\mathcal{A}(T; S_1, \ldots, S_{2m})$ with $2ms^2 < 1$; here s is the similitude coefficient of S_i's.*

We will prove that necklaces which satisfy Theorem 2 do exist, constructing an infinite series of them.

Supported by Russian Foundation of Basic Research; Grant No. 19–01–00169.

References

[1] L. Antoine. Sur l'homéomorphie de deux figures et de leurs voisinages. — Thèses de l'entre-deux-guerres, vol. 28, 1921. http://eudml.org/doc/192716

[2] L. Antoine. Sur les voisinages de deux figures homéomorphes // Fund. Math. 5 (1924) 265–287.

[3] S. Barov, J.J. Dijkstra, M. van der Meer. On Cantor sets with shadows of prescribed dimension // Topol. Appl. 159 (2012) 2736–2742.

[4] K. Borsuk. An example of a simple arc in space whose projection in every plane has interior points // Fund. Math. 34 (1947) 272–277.

[5] J. Cobb. Raising dimension under *all* projections // Fund. Math. 144 (1994) 2, 119–128.

[6] O. Frolkina. A Cantor set in with "large" projections // Topol. Appl., 157 (2010) 4, 745–751.

[7] O. Frolkina. Cantor sets with high-dimensional projections // Topol. Appl. 275 (2020) 107020.

[8] O. Frolkina. A new simple family of Cantor sets in $\mathbb{R}^3$ all of whose projections are one-dimensional // Topol. Appl. (in press) 107452.

Approximation of semigroups of operators by compositional powers of high-order Chernoff functions

Galkin O. E., Remizov I. D.

International Laboratory of Dynamical Systems and Applications
National research University Higher school of Economics

We propose a theorem (which was previously a conjecture of the second author), which gives a practical way of estimating the rate of convergence of Chernoff approximations to a semigroup of operators. It is known that these semigroups give the solution to the Cauchy problem for linear evolution equation. Then, the proposed theorem is a method of proving that the so-called rapidly converging Chernoff approximations indeed have the stated high convergence rate.

This report is devoted to C_0-semigroups of operators in Banach space (for example, the books [2, 5] can be used as a source of definitions and facts in this area). We will be interested in approximating operators from these semigroups using constructions based on Chernoff's theorem [4]. A review of a significant part of this (already quite extensive) topic can be found in the article [3]. We will estimate of the rate of convergence of such approximations. This is a very young field of research that is rapidly gaining popularity [6, 1, 8].

Let X be a real or complex Banach space, $\mathcal{L}(X)$ is the set all of linear bounded operators in X. Let also A be a linear operator with the domain of definition $D(A) \subset X$ which is dense in X. Suppose that A is a generator of C_0–semigroup $(e^{tA})_{t\geq 0} \subset \mathcal{L}(X)$ of linear bounded operators acting in X.

By *Chernoff function* (*of order* 1) for the semigroup $(e^{tA})_{t\geq 0}$, we mean any operator-valued function $S\colon [0,T] \to \mathcal{L}(X)$ (where T is some positive number) satisfying the conditions of Chernoff's theorem (see [2, 5, 4]). In particular, the Chernoff function must satisfy the Chernoff tangency condition (of the order of 1): for all x from the essential domain of the operator A we have $G(t)x = x + tAx + o(t)$ as $t \to +0$. Chernoff's theorem states that using compositional powers of the Chernoff function, one can approximate the semigroup $(e^{tA})_{t\geq 0}$ as follows: $\lim_{n\to\infty} \|e^{tA}x - [S(t/n)]^n x\| = 0$ for any $t > 0$ and $x \in X$, where $[S(t/n)]^n$ is a compostion on n copies of linear bounded operator $S(t/n)$.

By *Chernoff function* (*of order* $m > 1$) for the semigroup $(e^{tA})_{t\geq 0}$, we mean the operator-valued function $S\colon [0,T] \to \mathcal{L}(X)$, which for all $x \in D(A^m)$ satisfies the Chernoff tangency condition of order m: $G(t)x = x + tAx + \ldots + t^m/m! \cdot A^m x + o(t^m)$ for $t \to +0$, as well as some other conditions that will be seen from the following theorem. The Chernoff tangency condition of order $m > 1$ was introduced in the hope [7] that when it is met then the value $\|e^{tA}x - [S(t/n)]^n x\|$ for $n \to \infty$ will tend to zero faster (at least for some x). Our main theorem shows that this hope was justified:

Theorem. *Let $(e^{tA})_{t\geq 0}$ be a C_0-semigroup with the generator $(A, D(A))$ in the Banach space X, and number $T > 0$ is given. Suppose that $(e^{tA})_{t\geq 0}$ for some $M_1 \geq 1$, $w_1 \geq 0$ satisfies the condition $\|e^{tA}\| \leq M_1 e^{w_1 t}$ for all $t \in [0,T]$. Let, in addition, suppose that the mapping $S\colon [0,T] \to \mathcal{L}(X)$ satisfies the following inequality for some natural m, some positive function $C_m(t)$ $(0 < t \leq T)$ and all $x \in D(A^{m+1}) \subset X$, $t \in [0,T]$:*

$$\left\| S(t)x - \sum_{k=0}^{m} \frac{t^k A^k x}{k!} \right\| \leq \frac{C_m(t) \cdot t^{m+1}}{(m+1)!} \|A^{m+1}x\|.$$

Let us also assume that at least one of the conditions a), b) holds:

a) there is $w_2 \geq 0$ such that $\|S(t)\| \leq e^{w_2 t}$ for all $t \in [0,T]$;

b) there are $w_2 \geq 0$ and $M_2 \geq 1$ such that $\|S(t)^k\| \leq M_2 e^{k w_2 t}$ for all $t \in [0,T]$ and natural k.

Then: 1) Condition a) implies condition b) with M_2=1.

2) Taking into account 1), for $w_1 \neq w_2$, all $t \in [0,T]$ and natural n, the following estimate will hold: $\|S(t/n)^n x - e^{tA}x\| \leq$

$$M_1 M_2 (C_m(t/n) + M_1 e^{w_1 t/n})\|A^{m+1}x\| \frac{t^{m+1}(e^{w_2 t} - e^{w_1 t})}{n^{m+1}(m+1)! e^{w_1 t/n}}.$$

3) Taking into account 1), for $w_1 = w_2 = w$, all $t \in [0,T]$ and natural n, the following estimation will hold:

$$\|S(t/n)^n x - e^{tA}x\| \leq (C_m(t/n) + M_1 e^{wt/n}) \frac{M_1 M_2 t^{m+1} e^{wt}}{n^m (m+1)!} \|A^{m+1}x\|.$$

Example Let $\|e^{tA}\| \leq e^t$, $\|S(t)\| \leq e^t$, $S(t)x = x + tAx + t^2/2 \cdot A^2 x + o(t^2)$ for $t \to 0$, and $\|S(t)x - x - tAx - t^2/2 \cdot A^2 x\| \leq Ct^{2+\varepsilon}\|A^3 x\|$ for some $\varepsilon \in (0,1]$ and $C > 0$ for all $t \in [0,1]$. Then $m = 2$, $C_m(t) = 6Ct^{\varepsilon-1}$, $M_1 = M_2 = w_1 = w_2 = 1$, and point 3) of the above theorem states that for any $t \in [0,1]$, $x \in D(A^3)$ and for every natural n, the next estimate is valid

$$\|S(t/n)^n x - e^{tA}x\| \leq \left(Ce \cdot \frac{t^{2+\varepsilon}}{n^{1+\varepsilon}} + \frac{e^2}{6} \cdot \frac{t^3}{n^2} \right) \cdot \|A^3 x\|.$$

Gratitudes. This work was supported by the International Laboratory of Dynamical Systems and Applications of the National Research University Higher School of Economics, grant of the Ministry of Science and Higher Education of the Russian Federation, agreement No. 075-15-2019-1931.

References

[1] C. Batty, A. Gomilko, Yu. Tomilov *A Besov algebra calculus for generators of operator semigroups and related norm-estimates.* // arXiv:1810.11799 [math.FA]. 2019.

[2] V.I. Bogachev, O.G. Smolyanov *Real and Functional Analysis.* Springer. 2020. 586 p.

[3] Ya.A. Butko *The Method of Chernoff Approximation.* // In: Banasiak J., Bobrowski A., Lachowicz M., Tomilov Y. (eds) Semigroups of Operators — Theory and Applications. SOTA 2018. Springer Proceedings in Mathematics and Statistics. 2020. V. 325. P. 19-46.

[4] P.R. Chernoff *Note on product formulas for operator semigroups.* // J. Functional Analysis. 1968. V. 2, no. 2. P. 238-242.

[5] K.-J. Engel, R. Nagel *One-Parameter Semigroups for Linear Evolution Equations.* New-York: Springer-Verlag, 2000. 589 p.

[6] H. Neidhardt, A. Stephan, V. Zagrebnov *Operator-Norm Convergence of the Trotter Product Formula on Hilbert and Banach Spaces: A Short Survey.* // arXiv:2002.04483 [math.FA]. 2020.

[7] I.D. Remizov *On estimation of error in approximations provided by Chernoff's product formula.* // International Conference "Shilnikov Workshop-2018", abstracts. Nizhny Novgorod: UNN. 2018. P. 38.

[8] A.V. Vedenin, V.S. Voevodkin, V.D. Galkin, E.Yu. Karatetskaya, I.D. Remizov. *Speed of Convergence of Chernoff Approximations to Solutions of Evolution Equations.* // Math. Notes. 2020. V. 108. P. 451-456.

Synchronization and symmetry breaking in a model of two interacting ultrasound contrast agents

Garashchuk I.R.[1], Kazakov A.O[2], Sinelshchikov D.I.[1]

1. Department of Applied Mathematics
HSE University

2. Laboratory of Topological Methods in Dynamics
HSE University

In this talk we consider a model describing oscillations of two interacting microbubble contrast agents [1, 2]. Microbubble contrast agents are micrometer size encapsulated gas bubbles. Such bubbles can be used for various biomedical applications, for example, for enhancing ultrasound visualization of blood flow. It is known that contrast agents can demonstrate complex dynamics and its type is important for applications [1, 3, 4, 5].

The dynamics of two intercating bubbles is described by a non-autonomous system of four differential equations (or an equivalent autonomous system of five equations). If the equilibrium radii of both bubbles are the same, then the governing dynamical system is invariant with respect to the transformation: $R_1 \leftrightarrow R_2, \dot{R}_1 \leftrightarrow \dot{R}_2$, where $R_1(t)$ and $R_2(t)$ denote the first and second bubbles' radii respectively and dot is the derivative with respect to time. This symmetry leads to the existence of the three-dimensional invariant manifold $R_1 = R_2, \dot{R}_1 = \dot{R}_2$. Solutions embedded in this manifold are characterized by completely in-phase (synchronous) oscillations of both bubbles. Some of these solutions can be asymptotically stable (attractive). Various synchronous (periodic, chaotic) and asynchronous (periodic, quasiperiodic, chaotic and hyperchaotic) states were studied recently in work [4].

The main aim of this talk is to study the process of destruction of synchronous oscillations in the considered model and the influence of such destruction on various dynamical regimes. There are two typical ways that are responsible for the loss of synchronization. First and more obvious one is the destruction of the synchronization manifold, which can be done by breaking the corresponding symmetry, i.e. by considering nonidentical bubbles. Second, destruction of the synchronous oscillations is possible without symmetry breaking via the bubbling transition process.

In order to study the first way, we introduce a perturbation of the equilibrium radius of one of the bubbles which leads to the symmetry breaking. Since synchronous attractors are essentially defined by presence of the symmetry, it is natural to assume that they are in general more sensitive to the symmetry breaking. We show that the main factors determining stability or instability of a synchronous attractor are the presence/absence and the type of an asynchronous attractor coexisting with the synchronous attractor. On the other hand, asynchronous hyperchaotic attractors are stable with respect to the symmetry breaking. As far as the second way of destruction of synchronous oscillations is concerned, we propose a phenomenological mechanism responsible for such destruction of synchronization and numerically demonstrate its implementation in the studied model. We show that the appearance and expansion of transversally unstable areas in the synchronization manifold leads to transformation of a synchronous chaotic attractor into a hyperchaotic one. We also demonstrate that this bifurcation sequence is stable with respect to symmetry breaking perturbations.

This work is supported by RSF grant 19-71-10048.

References

[1] L. Hoff, Acoustic characterization of contrast agents for medical ultrasound imaging, Springer, Berlin, 2001.

[2] A.A. Doinikov, A. Bouakaz, Modeling of the dynamics of microbubble contrast agents in ultrasonic medicine: Survey, J. Appl. Mech. Tech. Phys. 54 (2013) 867–876.

[3] J.M. Carroll, M.L. Calvisi, L.K. Lauderbaugh, Dynamical analysis of the nonlinear response of ultrasound contrast agent microbubbles., J. Acoust. Soc. Am. 133 (2013) 2641–9.

[4] I.R. Garashchuk, D.I. Sinelshchikov, A.O. Kazakov, N.A. Kudryashov, Hyperchaos and multistability in the model of two interacting microbubble contrast agents, Chaos An Interdiscip. J. Nonlinear Sci. 29 (2019) 063131.

[5] I.R. Garashchuk, A.O. Kazakov, D.I. Sinelshchikov, Synchronous oscillations and symmetry breaking in a model of two interacting ultrasound contrast agents, Nonlinear Dyn. 101 (2020) 1199–1213.

Density of thin film billiard reflection pseudogroup in Hamiltonian symplectomorphism pseudogroup

Glutsyuk A.A.[1,2]

1. UMR 5669 (UMPA, ENS de Lyon), UMI 2615 (ISC J.-V.Poncelet), CNRS, France.

2. Faculty of Mathematics, HSE University, Moscow, Russia

Reflections from hypersurfaces act by symplectomorphisms on the space of oriented lines with respect to the canonical symplectic form. We consider an arbitrary C^∞-smooth hypersurface $\gamma \subset \mathbf{R}^{n+1}$ that is either a global strictly convex closed hypersurface, or a germ of hypersurface. We deal with the pseudogroup generated by compositional differences of reflections from γ and reflections from its small deformations. In the case, when γ is a global convex hypersurface, we show that the latter pseudogroup is dense in the pseudogroup of Hamiltonian diffeomorphisms between subdomains of the phase cylinder: the space of oriented lines intersecting γ transversally. We prove an analogous local result in the case, when γ is a germ. The derivatives of the above compositional differences in the deformation parameter are Hamiltonian vector fields calculated by Ron Perline. To prove the main results, we find the Lie algebra generated by the corresponding Hamiltonian functions and prove its C^∞-density in the space of C^∞-smooth functions.

The author is partially supported by Laboratory of Dynamical Systems and Applications NRU HSE, of the Ministry of science and higher education of the RF grant ag. No 075-15-2019-1931.

He is also partially supported by RFBR and JSPS (research project 19-51-50005).

References

[1] Glutsyuk, A. *Density of thin film billiard reflection pseudogroup in Hamiltonian symplectomorphism pseudogroup.* Preprint https://arxiv.org/abs/2005.02657

Generalizations of the Furstenberg Theorem on random matrix products

Gorodetski A.[1], **Kleptsyn V.**[2]

1. Department of Mathematics, University of California Irvine and National Research University Higher School of Economics

2. Institut de Recherche Mathematique de Rennes CNRS

Random products of matrices appear naturally in smooth dynamical systems, probability theory, spectral theory and mathematical physics, and geometric measure theory. The main questions are usually focused on the rate of growth of these products. In this context an important step was made in 1960 by Furstenberg and Kesten. They proved that products of random matrices generated by a stationary process have well defined asymptotic exponential growth rate. This rate of growth is usually called *Lyapunov exponent*. It corresponds exactly to the logarithm of the spectral radius when all the random matrices degenerate to a single matrix. Later Furstenberg showed that in most cases the Lyapunov exponent must be positive.

In this talk we will discuss several generalizations of the classical Furstenberg Theorem, namely

- Parametric version, see [1];
- Non-stationary version, [2];
- Parametric non-stationary version, [3].

As an application, we provide a proof of both spectral and dynamical Anderson Localization in 1D Anderson-Bernoulli Model, which was expected to hold by experts in spectral theory, but up to now was out of reach of the available methods.

References

[1] A.Gorodetski, V.Kleptsyn, Parametric Furstenberg Theorem on Random Products of $SL(2,\mathbb{R})$ matrices, to appear in *Advances in Mathematics*.

[2] A.Gorodetski, V.Kleptsyn, Non-stationary version of Furstenberg Theorem on random matrix products, work in progress.

[3] A.Gorodetski, V.Kleptsyn, Non-stationary version of Anderson Localization, work in progress.

On interrelations between trivial and nontrivial basic sets of structurally stable diffeomorphisms of surfaces

Grines V. Z.[1], **Mints D. I.**[1]

1. International Laboratory of Dynamical Systems and Applications
National Research University Higher School of Economics, Russian Federation

Let M^n be a closed smooth manifold of dimension $n \geq 1$.

In [1], it is proved that if the non-wandering set $NW(f)$ of a structurally stable diffeomorphism $f : M^n \to M^n$ $(n \geq 3)$ contains an expanding orientable attractor Ω of codimension one, then 1) manifold M^n is homotopically equivalent to n-dimensional torus $\mathbb{T}^n$; if $n \neq 4$, then M^n is homeomorphic to $\mathbb{T}^n$; 2) the set $NW(f)\backslash\Omega$ consists of a finite number of isolated sources and saddles.

For the case $n = 2$, this statement is not true. Namely, the ambient surface M^2 of a structurally stable diffeomorphism f whose non-wandering set $NW(f)$ contains an orientable attractor Ω is not necessarily a torus, but can be any orientable surface other than a sphere, and dynamics on the complement to the attractor Ω can be much more complicated.

The following theorem is the main result of the given report.

Theorem 2. *Let $f : M^2 \to M^2$ be a structurally stable diffeomorphism all trivial basic sets of which are source periodic points $\alpha_1, ..., \alpha_k$, where $k \geq 1$. Then the non-wandering set $NW(f)$ of the diffeomorphism f consists of points $\alpha_1, ..., \alpha_k$ and exactly one one-dimensional attractor Λ.*

In paper [2], it was announced a generalization of S. Smale's surgery on pseudo-Anosov diffeomorphisms of an arbitrary surface. This surgery leads to the appearance of a structurally stable diffeomorphism of the same surface with the non-wandering set consisting of exactly one one-dimensional attractor and a finite number of source periodic points. The diffeomorphisms obtained as a result of this surgery, as well as diffeomorphisms of a two-dimensional sphere whose non-wandering set contains Plykin's attractor and DA-diffeomorphisms of a two-dimensional torus, are examples of diffeomorphisms satisfying the conditions of theorem 2.

A nontrivial basic set Ω of an A-diffeomorphism $f : M^2 \to M^2$ is said to be widely situated if there are no loops homotopic to zero formed by a segment of the stable manifold and a segment of the unstable manifold of a point $x \in \Omega$.

Let g be genus of surface M^2.

Corollary 2. *Let the conditions of theorem 2 be satisfied and, in addition, the attractor Λ has no bunches of degree one and $g \geq 1$ if surface M^2 is orientable, $g \geq 3$ if surface M^2 is nonorientable. Then the attractor Λ is widely situated on surface M^2.*

Corollary 3. *Let the conditions of theorem 2 be satisfied and, in addition, $k = 1$ (that is the point α_1 is fixed). Then surface M^2 has genus $g \geq 1$ if it is orientable, $g \geq 3$ if it is nonorientable, the attractor Λ is widely situated on surface M^2 and has no bunches of degree one.*

In [3] (theorem 1), it is proved that if non-wandering set of structurally stable diffeomorphism of a surface contains one-dimensional attractor (repeller), then it also contains a source (sink) periodic point. The following theorem complements this statement.

Theorem 3. *If non-wandering set of a structurally stable diffeomorphism $f : M^2 \to M^2$ contains a nontrivial zero-dimensional basic set Ω, then it contains source and sink periodic points.*

The results of this report were obtained with the financial support of the Russian Science Foundation (project 17-11-01041).

References

[1] V. Z. Grines, E. V. Zhuzhoma, ”Structurally stable diffeomorphisms with basis sets of codimension one”, Izv. RAN. Ser. Mat., 66:2 (2002), 3-66; Izv. Math., 66:2 (2002), 223-284.

[2] A. Yu. Zhirov, R. V. Plykin, ”On the relationship between one-dimensional hyperbolic attractors of surface diffeomorphisms and generalized pseudo-Anosov diffeomorphisms”, Mat. Zametki, 58:1 (1995), 149-152; Math. Notes, 58:1 (1995), 779-781.

[3] V. Z. Grines, ”On the topological classification of structurally stable diffeomorphisms of surfaces with one-dimensional attractors and repellers”, Mat. Sb., 188:4 (1997), 57-94; Sb. Math., 188:4 (1997), 537-569.

On Topological Classification of Morse-Smale Diffeomorphisms on the Sphere S^n $(n > 3)$

Gurevich E.

NRU HSE

egurevich@hse.ru

The report is devoted to discussing of results of papers [1]-[2] obtained in collaboration with V. Grines, O. Pochinka, V. Medvedev and D. Malyshev.

The dynamics generated by Morse-Smale diffeomorphisms, structurally stable diffeomorphisms whose non-wandering set consists of a finite number of periodic points, is quite simple: the trajectory of any point of a phase space tends asymptotically to a periodic orbit. Moreover, the closures of the invariant manifolds of saddle periodic points of codimension 1 cut the phase space into components with the same asymptotic behavior of the trajectories. Therefore, at the first blush it seems that the topological conjugacy classes of such systems are well defined by the mutual arrangement of invariant manifolds of codimension one, and one cal use combinatorial invariants similar to scheme of Leontovich and Mayer or Peixoto graph. As follows from the classical works of V.Z. Grines and A.N. Moneyless, this fact is true in the case when the dimension of the phase space equals two, and the Morse-Smale diffeomorphism has a finite number of heteroclinic orbits. In the case of dimension three and higher, the situation is different, which is related to the possibility of a wild embedding of invariant manifolds of saddle periodic points (that fundamentally distinguishes flows from diffeomorphisms).

In this talk it is established that for the class G of Morse-Smale diffeomorphisms without heteroclinic intersections, defined on a sphere S^n of dimension four and higher, the complete invariant of topological conjugacy again can be described in combinatorial terms.

For any diffeomorphism $f \in G$, we define a two-color graph Γ_f that describes a mutual arrangement of invariant manifolds of saddle periodic points of the diffeomorphism f. We enrich the graph Γ_f by an automorphism P_f induced by dynamics of f and define the isomorphism notion between two colored graphs.

Theorem 1. *Two diffeomorphisms $f, f' \in G$ are topologically conjugated if and only if their graphs Γ_f, Γ'_f are isomorphic.*

It is not difficult to show that graph Γ_f of any $f \in G$ is a tree. The following theorem solves the problem of realization of all classes of topological conjugacy for class G.

Theorem 2. *For any two-color three Γ and any color preserving automorphism $P : \Gamma \to \Gamma$ there exists a diffeomorphism $f \in G$ whose graph Γ_f is isomorphic to Γ by means of an isomorphisms $\xi : \Gamma_f \to \Gamma$ such that $P_f = \xi^{-1} P \xi$.*

The research is supported by Russian Science Foundation, project 17-11-01041, and by Laboratory of Dynamical Systems and Applications of National Research University Higher School of Economics, grant of the Ministry of Science and Higher Education of the Russian Federation №075-15-2019-1931.

[1] Grines V., Gurevich E., Pochinka O., Malyshev D. On topological classification of Morse-Smale diffeomorphisms on the sphere $S^n (n > 3)$ // Nonlinearity. 2020. Vol. 33. No. 12. P. 7088-7113.

[2] Grines V., Gurevich E., Medvedev V. On realization of topolokical conjugacy classes of Morse-Smale cascades on the sphere S^n// Proc. Steklov Inst. Math., 310 (2020), 108-123

Rigitidy of Lie foliations with locally symmetric leaves

Hiraku Nozawa

Ritsumeikan University hnozawa@fc.ritsumei.ac.jp

This is a joint work with Gaël Meigniez (Université Bretagne-Sud).

Given a Lie group G, a *G-Lie foliation* is a foliation transversely modeled on G. The study of such foliations is motivated by Molino theory [Mo88]: the classification of Riemannian foliations is essentially reduced to that of Lie foliations. Yet little is known about the classification of Lie foliations.

Typical examples of Lie foliations are so-called *homogeneous* Lie foliations, which are foliations on double coset spaces with homogeneous leaves. Some rigidity results imply that Lie foliations are homogeneous under various conditions, which are useful for the classification: Caron-Carrière [CC80] showed that every minimal 1-dimensional Lie foliation is diffeomorphic to a linear flow on a torus. Matsumoto-Tsuchiya [MT92] proved that every 2-dimensional affine Lie foliation on closed 4-manifolds are homogeneous. Zimmer [Zi88] proved that if a minimal G-Lie foliation admits a Riemannian metric such that each leaf is isometric to a product of symmetric space of noncompact type of real rank ≥ 2, then it is obtained as a pull back of a homogeneous Lie foliation.

Our main result is a generalization of Zimmer's result:

Theorem 1. *Let X be a product of irreducible Riemannian symmetric spaces of non-compact type different from the Poincaré disk. Let $(M, \mathcal{F})$ be a connected closed manifold with a minimal Lie foliation. Assume that M admits a Riemannian metric of class C^0 whose restriction to every leaf is smooth and locally isometric to X. Then, $(M, \mathcal{F})$ is smoothly conjugate to a homogeneous Lie foliation.*

This result is not true for the case where X is the Poincaré disk. Indeed, there exists a non-homogeneous Lie foliation whose leaves are hyperbolic surfaces homeomorphic to the 2-sphere S^2 minus a Cantor set [HMM05].

The proof uses the barycentre mapping due to Besson-Curtois-Gallot [BCG96] and a rigidity theorem of Kleiner-Leeb [KL97].

Combining Theorem 1 with known results for lattices of semisimplie Lie groups, we recover other results of Zimmer in [Zi88].

References

[BCG96] G. Besson, G. Courtois and S. Gallot, Minimal entropy and Mostow's rigidity theorems, *Ergod. Th. & Dynam. Sys.* **16** (1996), 623–649.

[CC80] P. Caron and Y. Carrière, Flots transversalement de Lie $\mathbb{R}^n$, flots transversalement de Lie minimaux, *C. R. Acad. Sci. Paris Sér. A-B* **291** (1980), 477–478.

[HMM05] G. Hector, S. Matsumoto and G. Meigniez, Ends of leaves of Lie foliations, *J. Math. Soc. Japan* **57** (2005), 753–779.

[KL97] B. Kleiner and B. Leeb, Rigidity of quasi-isometries for symmetric spaces and Euclidean buildings, *Publ. Math. Inst. Hautes Études Sci.* **86** (1997), 115–197.

[MT92] S. Matsumoto and N. Tsuchiya, The Lie affine foliations on 4-manifolds, *Invent. Math.* **109** (1992), 1–16.

[Mo88] P. Molino, *Riemannian Foliations*, Progr. Math. **73**, Birkhäuser, Boston, MA, 1988.

[Zi88] R. Zimmer, Arithmeticity of Holonomy Groups of Lie Foliations, *J. Amer. Math. Soc.* **1** (1988), 35–58.

Von Neumann's ergodic theorem and Fejér sums for signed measures on the circle

Kachurovskii A. G.

Sobolev Institute of Mathematics
Novosibirsk, Russia

The Fejér sums for measures on the circle and the norms of the deviations from the limit in von Neumann's ergodic theorem are calculated, in fact, using the same formulas (by integrating the Fejér kernels) — and so, this ergodic theorem is a statement about the asymptotics of the Fejér sums at zero for the spectral measure of the corresponding dynamical system [1].

It made it possible, having considered the integral Holder condition for signed measures, to prove a theorem that unifies both following well-known results: classical S.N. Bernstein's theorem on polynomial deviations of the Fejér sums for Holder functions — and theorem about polynomial rates of convergence in von Neumann's ergodic theorem. On the way, a new proof of the Bernstein's theorem was obtained [2].

References

[1] Kachurovskii A. G., Podvigin I. V. Fejér Sums for Periodic Measures and the von Neumann Ergodic Theorem // Dokl. Math. 2018. Vol. 98, No. 1, pp. 344–347.

[2] Kachurovskii A. G., Lapshtaev M. N., Khakimbaev A. J. Von Neumann's ergodic theorem and Fejér sums for signed measures on the circle // Sib. Electron. Math. Reports. 2020. Vol. 17, pp. 1313-1321.

Asymptotics of the dynamic bifurcation saddle-node

Kalyakin L. A.

Department of differential equations
Institute of mathematics RAS, Ufa

The main object under consideration is a system of two differential equations with a small parameter $0 < \varepsilon \ll 1$

$$\varepsilon \frac{d\mathbf{x}}{d\tau} = \mathbf{f}(\mathbf{x}; \mathbf{a}); \quad \mathbf{x} \in \mathbb{R}^{\mathbf{2}}, \ \tau \in \mathbb{R}^{\mathbf{1}}.$$

The right-hand side depends on a parameter $\mathbf{a} \in \mathbb{R}^n$. There is an equilibrium state $\mathbf{x} \equiv \mathbf{p}(\mathbf{a})$ taken from functional equation

$$\mathbf{f}(\mathbf{x}; \mathbf{a}) = 0.$$

The saddle-node bifurcation takes place [1] on the surface $S \subset \mathbb{R}^n$. Let be $\mathcal{L} = \{\mathbf{a} \in \mathbb{R}^n : \ \mathbf{a} = \mathbf{A}(\tau)\}$ a smooth line, which crosses the bifurcation surface at some moment: $\mathbf{A}(0) = \mathbf{a}_0 \in S$

Problem for the non autonomous system with parameter depending on time $\mathbf{a} = \mathbf{A}(\tau)$ is considered. We study the solutions that, in the leading order term of the asymptotics in the small parameter, coincide with zeros of the right-hand side: $\mathbf{x} \equiv \mathbf{p}(\mathbf{A}(\tau))$ for $\tau < 0$. The aim of this work is to construct an asymptotic solution as $\varepsilon \to 0$ on a large time interval $\tau \in (-\delta, \delta)$, including the bifurcation moment $\tau = 0$. Numerical experiments show the presence of a narrow transition layer near the moment $\tau = 0$. In this layer, the solution is quickly rearranged from a value close to point, $\mathbf{a}_0$, to another equilibrium state. A description of such a rearrangement in the asymptotic approximation at small ε is presented. A simplest model was studied in [2].

References

[1] Bautin N. N. and Leontovich E. A. Methods and Techniques of Qualitative Research of Dynamical Systems in the Plane (Nauka, Moscow, 1990) [in Russian].

[2] Kalyakin L. A. Asymptotics of the Solution of a Differential Equation in a Saddle-Node Bifurcation. Computational Mathematics and Mathematical Physics, 2019, Vol. 59, 9, pp. 1454-1469

Hyperchaos and discrete Lorenz-shape attractors in the model of two coupled parabolas

Karatetskaia E.[1], Kuryzhov E.[1], Mints D.[1]

National Research University Higher School of Economics, 25/12 Bolshaya Pecherskaya Ulitsa, 603155 Nizhny Novgorod, Russia.

The present paper is devoted to the study of transition mechanisms from a stable fixed point to chaotic and even hyperchaotic attractors in two-dimensional endomorphism, which describes the dynamics of two coupled parabolas:

$$\begin{cases} \overline{x} = 1 - ax^2 + \varepsilon(x - y) \\ \overline{y} = 1 - ay^2 + \varepsilon(y - x) \end{cases} \tag{1}$$

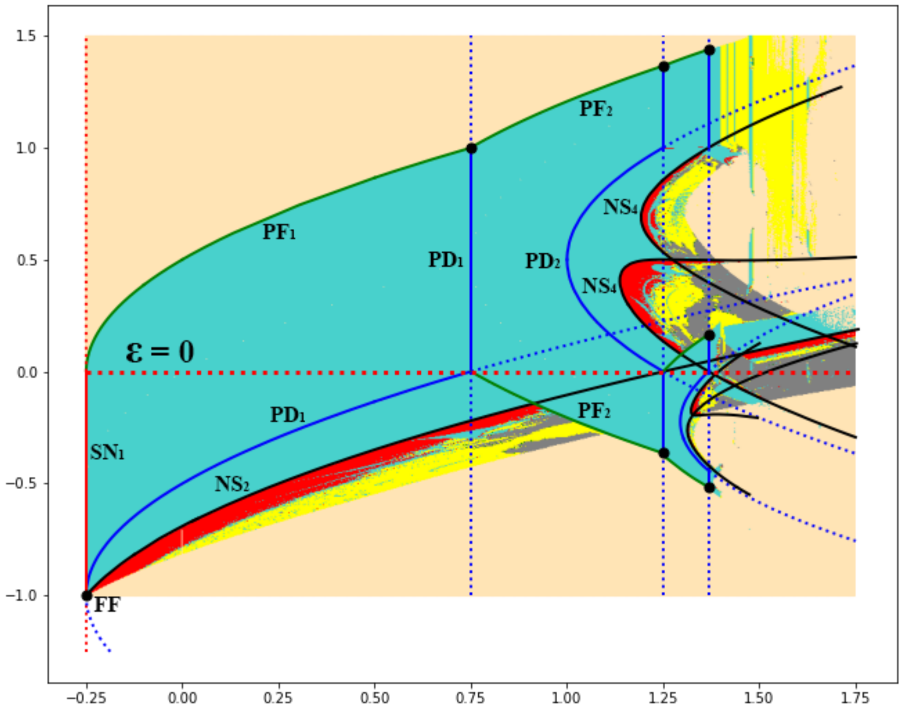


Figure 1. The chart of Lyapunov exponents and main bifurcation curves
PD - period-doubling bifurcation, PF - pitchfork bifurcation, SN - saddle-node bifurcation, NS - Neimark-Sacker bifurcation; FF - fold-flip bifurcation; lower index corresponds to the period of point which bifurcates; solid line for stable periodic orbits, dashed line - for unstable ones.

Local bifurcations in map (1) were studied in paper [1]. We note that when parameter corresponding to the coupling of two parabolas is equal to zero ($\varepsilon = 0$) each of two independent subsystems $\bar{x} = 1 - ax^2$ and $\bar{y} = 1 - ay^2$ shows a classical transition from stable fixed point to chaotic attractor through the Feigenbaum cascade of period-doubling bifurcations. Thereby, in the system (1) the appearance of hyperchaotic attractors is awaited at least for small values of ε.

In the first part of this work we investigated discrete Lorenz-shape attractors in model of two coupled parabolas (1). In papers [2], [3] for three-dimensional models it was shown, that discrete Lorenz-shape attractors can occur in the neighborhood of periodic orbits with a pair of multipliers $(+1, -1)$. Bifurcation diagrams near such point (the so-called fold-flip) were studied in detail in paper [4]. To develop this idea for the map (1) we computed a chart of Lyapunov exponents, superimposed with main bifurcation curves (see Fig. 1) and studied the main scenarios associated with the birth of Lorenz-shape attractors. Also, we discovered the existence of 2-, 4- and 8-period Lorenz-shape attractors. The mechanisms of appearance of 8-period Lorenz-shape attractor are demonstrated below (see Fig. 2).

The region of stability of fixed point from the bottom and the right side is bounded by the curves of period-doubling bifurcation, from above – by pitchfork bifurcation, from the left - by vertical line of saddle-node bifurcation at $a = -0.25$. For the period two orbit top and bottom borders are supercritical pitchfork bifurcation and period-doubling one from the right side. Pitchfork bifurcation curve crosses period-doubling bifurcation curve for stable period 2 orbit on a line when $a = 1.25$ in top and bottom parts of diagram. These points are also co-dimension two points (fold-flip), but their type differs from fold-flip point, mentioned previously.

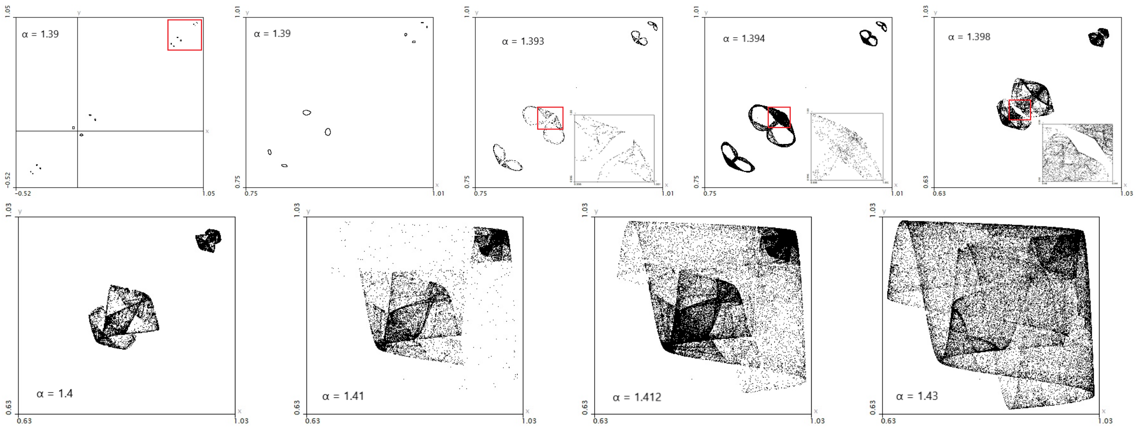


Figure 2. Illustration of mechanisms of appearance of two symmetric 8-component Lorenz-like attractors in map (1) along the route $\varepsilon = 0.045$

Second part of work is devoted to the study of scenarios of hyperchaotic attractors appearance in map (1). According to Fig. 1, when $\varepsilon \neq 0$ and along the way of increasing of parameter α, the cascade of period-doubling bifurcations is interrupted by Neimark-Sacker bifurcation. Consequently, respective stable periodic orbit O_p becomes completely unstable and stable invariant curve is born in its neighborhood. By further increasing of parameter α this curve breaks, attractors of type torus-chaos appear and then, as a result of absorption of completely unstable periodic orbit O_p, periodic Shilnikov attractor appears. After that, with a further growth of parameter α, this attractor absorbs the set of completely unstable orbits, which appear as a result of period-doubling bifurcations of saddle points, and hyperchaotic attractor appears.

This work was supported by the RSF grant No. 19-71-10048

References

[1] V. Biragov, I. M. Ovsyannikov, D. Turaev, "A study of one endomorphism of a plane", Methods of qualitative theory and theory of bifurcations, 72-86, (1988).

[2] S.V. Gonchenko, I. I. Ovsyannikov, Simó C, D. Turaev, "Three-dimensional Hénon-like maps and wild Lorenz-like attractors", International Journal of Bifurcation and Chaos 11, 3493-3508, (2005).

[3] A.S. Gonchenko, E.A. Samylina, "On the region of existence of a discrete Lorenz attractor in the nonholonomic model of celtic stone", Radiophysics and Quantum Electronics 5, 412-428, (2019).

[4] Y. A. Kuznetsov, H. G. E. Meijer, van Veen L., "The fold-flip bifurcation", International Journal of Bifurcation and Chaos 07, 2253-2282, (2004).

Non-alternating Hamiltonian Lie algebras of even characteristic: new series of simple Lie algebras

Kondrateva A. V.[1], Kuznetsov M. I.[2]

Institute of Information Technology, Mathematics and Mechanics
Nizhny Novgorod State University

The simplest non-alternating Lie algebra of characteristic 2 constructed in [1] consists of vector fields preserving the symmetric Hamiltonian form $\omega = dx_1^{(2)} + \ldots + dx_n^{(2)}$. This Lie algebra is analogous to the Hamiltonian Lie superalgebra $H(0|n)$ of characteristic 0. More general non-alternating Hamiltonian Lie algebras corresponding to Lie superalgebras $H(m|n)$ are studies in [2, 3]. They are constructed over divided powers algebra $O(\mathscr{F})$ and correspond to forms ω with polynomial coefficients. Here $\mathscr{F}: E = E_0 \supseteq E_1 \supseteq \ldots \supseteq E_r \supset E_{r+1} = \{0\}$ is a flag of a space $E = \langle x_1, \ldots, x_n \rangle$. Let $M = (\omega_{ij})$ be the matrix of the form ω, $M^{-1} = (\overline{\omega}_{ij})$. The matrix $M^{-1}(0)$ defines a dual form $\overline{\omega}(0)$ on the space E. It is shown in [4] that if $E_1 \not\subset E^0$, then the Hamiltonian Lie algebra $P(\mathscr{F}, \omega)$ is isomorphic to $P(\mathscr{F}, \omega(0))$. Here, E^0 is the subspace of isotropic vectors of E with respect to $\overline{\omega}(0)$. At present, normal shapes ω_i, $i = 1, 2, 3, 4$ of all non-alternating Hamiltonian forms in three variables are obtained:

$\omega_1 = dx_1 dx_2 + dx_3^{(2)}$,

$\omega_2 = dx_1 dx_2 + dx_2^{(2)} + dx_3^{(2)}$,

$\omega_3 = dx_1^{(2)} + dx_2^{(2)} + dx_3^{(2)}$,

$\omega_{4,1} = dx_1 dx_2 + dx_3^{(2)} + x_1^{(2^{m_1}-1)} x_3 dx_1 dx_3$,

$\omega_{4,2} = dx_1 dx_2 + dx_3^{(2)} + x_2^{(2^{m_2}-1)} x_3 dx_1 dx_3$,

where m_1, m_2, $m_1 \neq m_2$, are the heights of x_1, x_2. Lie algebras corresponding to different forms ω_i, ω_j, $i \neq j$, are pairwise non-isomorphic. Therefore, new series of simple Lie algebras corresponding to ω_2, ω_4 are found.

The work is supported by RFBR (grant 18-01-00900) and Ministry of Science and Higher education of Russian Federation (project 0729-2020-0055).

References

[1] Lin Lei, *Non-alternating Hamiltonian algebra $P(n, m)$ of characteristic two*, Commun. Algebra **21**(2), 399–411 (1993).

[2] Bouarrouj S., Grozman P., Lebedev A., Leites D., *Divided power (co)homology. Presentation of simple finite dimensional modular superalgebras with Cartan matrix*, Homology, Homotopy Appl. **12**(1), 237–248 (2010).

[3] Yier U., Leites D., Messaoudene M., Shchepochkina I. *Examples of simple vectorial Lie algebras in characteristic 2* J. Nonlin. Math. Phys. **17**(1), 311–374 (2010).

[4] Kuznetsov M. I., Kondrateva A. V., Chebochko N. G., *Non-alternating Hamiltonian Lie algebras in characteristic 2. I*, (http://arxiv.org/abs/1812.11213). Accessed 2018.

Topological conjugation of surface Morse-Smale flows

Kruglov V.E.

National Research University Higher School of Economics

Two flows $f^t, f'^t \colon M \to M$ on a manifold M are called *topologically equivalent* if there exists a homeomorphism $h \colon M \to M$ sending trajectories of f^t into trajectories of f'^t preserving orientations of the trajectories. Two flows are called *topologically conjugate* if $h \circ f^t = f'^t \circ h$, it means that h sends trajectories into trajectories preserving not only directions but in addition the time of moving. To find an invariant showing the class of topological equivalence or topological conjugacy of flows in some class means to get a *topological classification* for it.

The *Morse-Smale flows* were introduced on the plane for the first time in the classical paper of A.A. Andronov and L.S. Pontryagin in [1]. The non-wandering set of such flows consists of a finite number of hyperbolic fixed points and finite number of hyperbolic limit cycles, besides, saddle separatrices cross-sect only transversally (there are no *connection* – separatrices joined two saddle points). This important class of flows was generalized on arbitrary surfaces and classified up to topological equivalence for many times during the twentieth century. The most important combinatorial invariants are the *Leontovich-Maier's scheme* [2], [3] for flows on the plane, the *Peixoto's directed graph* [4] for Morse-Smale flows on any closed surface and the *Oshemkov-Sharko's three-colour graph* [5] for Morse-Smale flows on any closed surface.

Since the epoch-making work of J. Palis [6] it is known that the class of topological equivalence of a regular surface flow can contain several classes of topological conjugacy, describing by parameters called *moduli*. He proved that every connection gives a modulus equals to the ratio of eigenvalues of non-intersecting invariant manifolds of the saddles joined by the connection.

In [7] it was proved that for gradient-like flows (i.e. Morse-Smale flows without limit cycles) on surfaces classes of topological equivalence and topological conjugacy coincide. Obviously, any limit cycle generates a modulus equals to the period of one. Additionally, in [8] it was proved that the presence of a cell bounded by limit cycles gives infinite number of moduli connected with the uniqueness of invariant foliation in the basin of the limit cycle.

This research solves the problem of of finding the criterion of finiteness of the number of the moduli for Morse-Smale flows on surfaces and classification in sense of the topological conjugacy of such flows.

The main result of the present paper is that two Morse-Smale flows without a trajectory going from one limit cycle to another one are topologically conjugate iff they are topologically equivalent and the periods of corresponding limit cycles coincide.

Acknowledgments. The report was supported by International Laboratory of Dynamical Systems and Applications of National Research University Higher School of Economics, Russian Federation Government grant 075-15-2019-1931.

References

[1] Andronov A. A., Pontryagin L. S. Rough systems (in Russian). Doklady Akademii nauk SSSR. 1937;14(5):247–250.

[2] Leontovich E. A., Mayer A. G. About trajectories determining qualitative structure of sphere partition into trajectories (in Russian). Doklady Akademii nauk SSSR. 1937;14(5):251–257.

[3] Leontovich E. A., Mayer A. G. On a scheme determining the topological structure of the separation of trajectories (in Russian). Doklady Akademii nauk SSSR. 1955;103:557–560.

[4] Peixoto M. M. On the classification of flows on 2-manifolds. In Dynamical systems (Proc. Sympos., Univ. Bahia, Salvador, 1971). 1973;389–419.

[5] Oshemkov A. A., Sharko V. V. On the classification of Morse-Smale flows on two-dimensional manifolds. Mat. Sb. 1998;189(8):93–140.

[6] Palis J. A differentiable invariant of topological conjugacy and moduli of stability. Asterisque, 1978. V. 51, 335–346.

[7] Kruglov V. E. Topological conjugacy of gradient-like flows on surfaces. Dinamicheskie sistemy. 2018; 8(36)(1):15–21.

[8] Kruglov V., Pochinka O., Talanova G. On functional moduli of surface flows. Proceedings of the International Geometry Center. 2020;13(1):49–60.

Demonstration of the "billiard Maxwell's Demon" effect in the "corrugated waveguide" system with an oscillating boundary

Lyubchenko D. O.[1], Savin A. V.[1]

1. Faculty of Nonlinear Processes
Saratov State University

For quite a long time it was believed that unpredictable behavior can be observed only in systems with a large number of degrees of freedom. Statistical analysis is usually used to study such systems. However, the discovery of dynamic chaos in simple systems showed that unpredictability also occurs in systems with a small number of degrees of freedom.

The simple models with chaotic dynamics are the billiard-type models. These are systems in which the particle moves in a compact space and elastically collides with the boundaries. Such models with fixed boundaries are already well studied. In a situation in which the boundaries can oscillate according to a known law, nontrivial and rather curious effects appear. These effects are interesting both from the fundamental and from the applied point of view.

One of the nontrivial effects arising in such models is the Fermi acceleration effect [1] which is almost unlimited increase in the speed of a particle that moves between the walls. In [2], it was assumed that for the appearance of Fermi acceleration in the billiard-type systems with oscillating boundaries, it is sufficient that such a billiard with a fixed boundary has chaotic dynamics.

Another interesting effect was discovered in [3], in which the ensemble-averaged velocity of particles in a stadium-like billiard with oscillating boundaries was studied. In this system, we can observe that weak oscillations of the boundary lead to the appearance of a critical value of the initial velocity in the system, below which the average velocity begins to decrease. If the initial velocity is greater than the critical velocity, then the particles are accelerated on average. This phenomenon was called billiard Maxwell's Demon.

In this work, we undertake a research of such effects and phenomena on the example of another model with collisions, in which one of the walls is stationary, and the other wall has harmonic corrugation and oscillates according to a harmonic law. In fact, this is a modified model of the Tennyson-Lieberman-Lichtenberg model [4], to which bounder oscillations are added.

The research shows that the system demonstrates chaotic dynamics in the situation in which boundaries are fixed, and also conducted numerical studies showing the possibility of implementing the phenomenon of "billiard Maxwell's demon" in this system.

References

[1] Fermi E. On the origin of the cosmic radiation. Phys. Rev. 75 1169 (1949)

[2] Loskutov A., Ryabov A. B., Akinshin L. G. Phys. A: Math. Gen. 33 7973 (2000)

[3] Loskutov A., Ryabov A. J. Stat. Phys. 108 995 (2002)

[4] A. J. Lichtenberg, M. A. Lieberman Regular and Chaotic Dynamics. Applied Mathematical Sciences 38 692 (1992)

Combinatorial invariant for Morse-Smale diffeomorphisms on surfaces with orientable heteroclinic

Malyshev D., Morozov A., Pochinka O.V.

National Research University Higher School of Economics

In this paper, we consider a class of orientation-preserving Morse-Smale diffeomorphisms defined on an orientable surface. In papers by A.A. Bezdenezhnykh and V.Z. Grines showed that such diffeomorphisms have a finite number of heteroclinic orbits. In addition, the classification problem for such diffeomorphisms is reduced to the problem of distinguishing orientable graphs with substitutions describing the geometry of a heteroclinic intersection. However, such graphs generally do not admit polynomial discriminating algorithms. This work proposes a new approach to the classification of these cascades. For this, each diffeomorphism under consideration is associated with a graph that allows the construction of an effective distinguish algorithm. We also identified a class of admissible graphs, each isomorphism class of which can be realized by a diffeomorphism of a surface with an orientable heteroclinic. The results obtained are directly related to the realization problem of homotopy classes of homeomorphisms on closed orientable surfaces. In particular, they give an approach to constructing a representative in each homotopy class of homeomorphisms of algebraically finite type according to the Nielsen classification, which is an open problem today.

Recall that the diffeomorphism $f : M^n \to M^n$, given on a smooth closed connected n-manifold $(n \geq 1)$ M^n is called a *Morse-Smale diffeomorphism*, if

1) its non-wandering set Ω_f consists of a finite number of hyperbolic orbits;

2) the manifolds W^s_p, W^u_q intersect transversally for any non-wandering points p, q.

We denote by $MS(M^n)$ the set of such diffeomorphisms.

For orientation-preserving Morse-Smale diffeomorphisms f, given on an orientable surface M^2, let us recall the concept of an orientable heteroclinic as follows.

Let σ_i, σ_j — saddle points of difeomorphism f, such that $W^s_{\sigma_i} \cap W^u_{\sigma_j} \neq \emptyset$. For any heteroclinic point $x \in W^s_{\sigma_i} \cap W^u_{\sigma_j}$ we define an ordered pair of vectors $(\vec{v}^u_x, \vec{v}^s_x)$, where:

- $\vec{v}^u_x$ — is the tangent vector to the unstable manifold of the point σ_j at the point x and directed from x to $f^{m^u}(x)$, where m^u is a period of the unstable separatrix containing x;
- $\vec{v}^s_x$ — is the tangent vector to the stable manifold of the point σ_i at the point x and directed from x to f^{m^s}, where m^s is the period of the stable separatrix containing x.

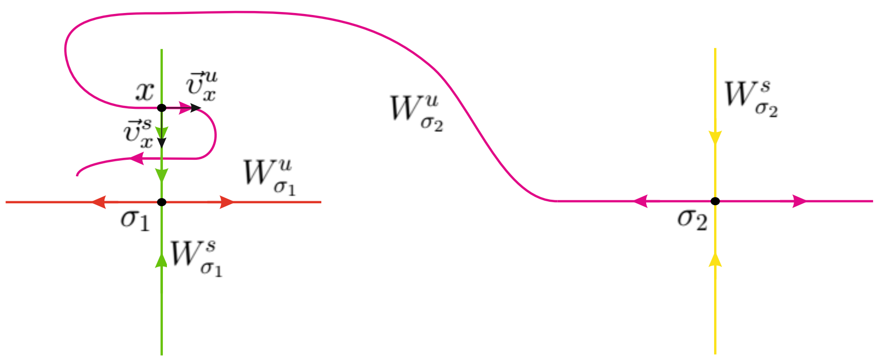


Figure 1: Non-orientable heteroclinic intersection

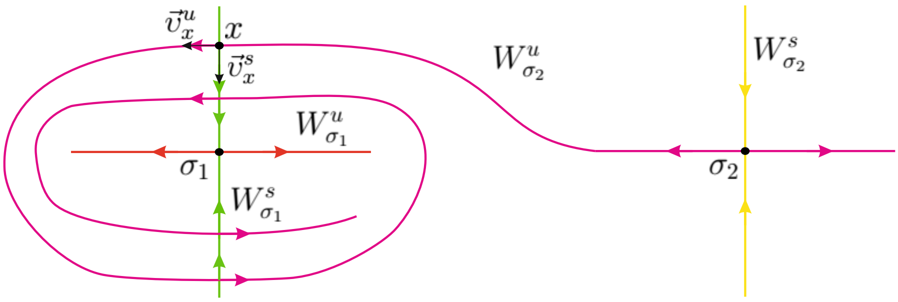


Figure 2: Orientable heteroclinic intersection

Heteroclinic intersection of diffeomorphism f is called *orientable*, if ordered pairs of vectors $(\vec{v}_x^u, \vec{v}_x^s)$ determines the same orientation of the ambient surface M^2 at every heteroclinic point x of the diffeomorphism f

Let's $G \subset MS(M^2)$ — is the class of orientation-preserving diffeomorphisms with orientable heteroclinic intersections.It was announced by V.Grines and A. Bezdenezhnyh[1] and was proved as the chapter of candidate's dissertation by A.Bezdenezhnyh [2] that any diffeomorphism $f \in G$ has a finite number of heteroclinic orbits. This result was also proved in[3] using another methods. In this paper, we described the graph T_f, P_f for diffeomorphisms $f \in G$, and also defined an admissible graph that admits the discrimination algorithm in polynomial time and by which a diffeomorphism of the class G can be realized.

The main result of this work is the following theorems:

Theorem 1. Diffeomorphisms $f, f' \in G$ are topologically conjugate if and only if their graphs (T_f, P_f), $(T_{f'}, P_{f'})$ are isomorphic.

Theorem 2. There is an efficient algorithm for establishing isomorphism of graphs (T_f, P_f), $(T_{f'}, P_{f'})$.

Theorem 3. For any admissible graph (T, P), there exists a diffeomorphism $f \in G$ such that the graph (T, P) and (T_f, P_f) are isomorphic.

Acknowledgments. Authors was supported by International Laboratory of Dynamical Systems and Applications of National Research University Higher School of Economics, Russian Federation Government grant 075-15-2019-1931.

Morozov A. and Pochinka O. was supported by a grant from the Fund of the Development of Theoretical Physics and Mathematics "BAZIS", contract 19-7-1-15-1.

References

[1] Bezdenezhnykh, A. N., and Grines, V. Z. (1985). Diffeomorphisms with orientable heteroclinic sets on two-dimensional manifolds. In: Qualitative Methods of the Theory of Differential Equations, Gorky, 139-152.

[2] Bezdenezhnich, A.N., (1985). Topological classification of Morse-Smale diffeomorphisms with an orientable heteroclinic set on two-dimensional manifolds, Gorky Order of the Red Banner Labor of the State University. N.I. Lobachevsky

[3] Morozov, A., and Pochinka, O. (2020). Morse-Smale surfaced diffeomorphisms with orientable heteroclinic. Journal of Dynamical and Control Systems, 1-11.

Two-dimensional attractors and fibered links on 3-manifolds

Medvedev V., Zhuzhoma E.

Department of mathematics
National Research University Higher School of Economics

Dynamical Axiom A systems (in short, A-systems) were introduced by Smale [3]. Here, we consider A-flows that are A-systems with continuous time. Due to Smale's Spectral Decomposition Theorem, the non-wandering set of any A-system is a disjoint union of closed, invariant, and topologically transitive sets called basic sets. It is natural to study two-dimensional basic sets (attractors) beginning with 3-manifolds M^3. The following statement holds.

Theorem 1. *Let f^t be an A-flow on an orientable closed 3-manifold M^3 such that the non-wandering set $NW(f^t)$ contains a 2-dimensional non-mixing attractor Λ_a. Then there is a compactification $M(\Lambda_a) = W^s(\Lambda_a) \cup_{i=1}^{k} l_i$ of the basin (stable manifold) $W^s(\Lambda_a)$ of Λ_a by the family of simple disjoint circles $l_1, \ldots, l_k$ such that*

- *$M(\Lambda_a)$ is a closed orientable 3-manifold;*
- *the flow $f^t|_{W^s(\Lambda_a)}$ is extended continuously to the nonsingular flow $\tilde{f}^t$ on $M(\Lambda_a)$ with the non-wandering set $NW(\tilde{f}^t) = \Lambda_a \cup_{i=1}^{k} l_i$ where $l_1, \ldots, l_k$ are repelling isolated periodic trajectories of $\tilde{f}^t$;*
- *the family $L = \{l_1, \ldots, l_k\} \subset M(\Lambda_a)$ is a fibered link in $M(\Lambda_a)$.*

The second result, in a sense, is reverse to the first one.

Theorem 2. *Let $\{l_1, \ldots, l_k\} \subset M^3$ be a fibered link in a closed orientable 3-manifold M^3. Then there is a nonsingular A-flow f^t on M^3 such that the non-wandering set $NW(f^t)$ contains a 2-dimensional non-mixing attractor and the repelling isolated periodic trajectories $l_1, \ldots, l_k$.*

Applying Alexander's statement on fibered links [1], one gets the following result.

Corollary 1. *Given any closed orientable 3-manifold M^3, there is a nonsingular A-flow f^t on M^3 such that the non-wandering set $NW(f^t)$ contains a two-dimensional non-mixing attractor.*

This corollary was proved by another method in [2].

Acknowledgments. This work is supported by the Russian Science Foundation under grant 17- 11-01041, except Theorem 2 supported by Laboratory of Dynamical Systems and Applications of National Research University Higher School of Economics, of the Ministry of science and higher education of the RF, grant ag. No 075-15-2019-1931.

References

[1] **Alexander J.** A lemma on systems of knotted curves. *Proc. Nat. Acad. Sci. USA*, **9**(1923), 93-95.

[2] **Medvedev V., Zhuzhoma E.** On two-dimensional expanding attractors of A-flows. *Math. Notes*, **107**:5(2020), 824-827.

[3] **Smale S.** Differentiable dynamical systems. *Bull. Amer. Math. Soc.*, **73**(1967), 747-817.

On the exactness of the stationary phase method for matrix coefficients of some real linear irreducible representations of compact Lie groups

Meshcheryakov M. V.

Department of Mathematical Analysis
Mordovia State University

mesh@math.mrsu.ru

It is well known that the asymptotic methods play an important role in the theory of dynamical systems and partial differential equations [1]. The purpose of the report – to show that the method of stationary phase for oscillatory integrals of the form $\int_G e^{itf}\,dg$, where t large enough, is exact for matrix coefficients of some class of real linear irreducible representations of compact connected groups G. The integral is taken over the group G with respect to the bi-invariant Haar measure. We will be use the notion of the exactness of the stationary phase method as defined at [2]. Let a Morse function $f : X \to \mathbb{R}$ with isolated critical points and different critical values be given on a compact n-dimensional manifold X with a fixed element of volume dm. Then the stationary phase method gives an approximation of the integral $\int_X e^{itf}\,dm$ in terms of the values of f and its derivatives at the critical points. The exactness of the stationary phase method means that for each critical point x there is a converging power series $A_x(t) = \sum\limits_{j\geqslant 0} a_{x,j}(it)^{-j}$ with the following property:

$$\int_X e^{itf}\,dm = \left(\frac{2\pi}{t}\right)^{n/2} \sum_{df(x)=0} e^{itf(x)} e^{(n-2\lambda(x))\pi i/4} A_x(t),$$

where the $\lambda(x)$ denotes the index of the function f at the point x. In the case of Morse-Bott functions, the stationary phase method gives an asymptotic expansion of the form:

$$\int_X e^{itf}\,dm = \left(\frac{2\pi}{t}\right)^{n/2} \sum_{1\leqslant j\leqslant n} e^{itf(C_j)} e^{(n-2\lambda(C_j))\pi i/4} A_x(t),$$

where $\lambda(C_j)$ is the index f in the along nondegenerate critical submanifold C_j, $j = 1, \ldots, p$.

Theorem (F. Kirvan). *If the method of stationary phase for the Morse function f is exact, then the index every critical point is even and f perfect Morse function. In particular, the dimension of X is even.*

Accordingly to [3], for Morse height functions on the orbits of the adjoint representation Ad of compact semisimple Lie groups G, the stationary phase method is exact. In other words, the stationary phase method is exact for the matrix coefficients of the adjoint representation on the group manifold G. In [4] Duistermaat and Heckman proved that the components of the moment map of Hamiltonian actions of compact groups G on closed symplectic manifolds also have the exactness property of the stationary phase method.

Our main result is the following

Theorem. *The stationary phase method for the matrix coefficients of real irreducible representations of compact groups G with a transitive action of G on even-dimensional spheres of the representation space is exact.*

The proof of the theorem is based on some results of geometric analysis related to Bessel functions (see [5]).

Another interpretation of this result connected with asymptotic behavior characteristic functions of the matrix coefficients, which considered as random variables with respect to Haar measure on G.

References

[1] Arnold V. I., Gusein-Zade S. M., Varchenko A. N. *Singularities of differentiable maps.* Vol. II. Monodromy and asymptotics of integrals, Monogr. Math., 83, Birkhauser Boston Inc., Boston, MA, 1988, 492 pp.

[2] Kirwan F. *Morse functions for which the stationary phase approximation is exact.* Topology, 1987, v. 26, №1, p. 37–40.

[3] Semenov-Tjan-Šanskiĭ M. A. *A certain property of the Kirillov integral.* Zap. Naucn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI), 1973, issue 37, p. 53–65.

[4] Duistermaat J. J., Heckman G. J. *On the variation in the cohomology of the symplectic form of the reduced phase spaces.* Invent. Math., 1982, v. 69, p. 259–268.

[5] Wolf J. *Spherical functions on Euclidean spaces.* Journal functional analysis, 2006, v. 239, p. 127–136.

On the exactness of the stationary phase method for matrix coefficients of some real linear irreducible representations of compact Lie groups

Meshcheryakov M. V.

Department of Mathematical Analysis
Mordovia State University
mesh@math.mrsu.ru

It is well known that the asymptotic methods play an important role in the theory of dynamical systems and partial differential equations [1]. The purpose of the report – to show that the method of stationary phase for oscillatory integrals of the form $\int_G e^{itf}\,dg$, where t large enough, is exact for matrix coefficients of some class of real linear irreducible representations of compact connected groups G. The integral is taken over the group G with respect to the bi-invariant Haar measure. We will be use the notion of the exactness of the stationary phase method as defined at [2]. Let a Morse function $f : X \to \mathbb{R}$ with isolated critical points and different critical values be given on a compact n-dimensional manifold X with a fixed element of volume dm. Then the stationary phase method gives an approximation of the integral $\int_X e^{itf}\,dm$ in terms of the values of f and its derivatives at the critical points. The exactness of the stationary phase method means that for each critical point x there is a converging power series $A_x(t) = \sum\limits_{j\geqslant 0} a_{x,j}(it)^{-j}$ with the following property:

$$\int_X e^{itf}\,dm = \Big(\frac{2\pi}{t}\Big)^{n/2} \sum_{df(x)=0} e^{itf(x)} e^{(n-2\lambda(x))\pi i/4} A_x(t),$$

where the $\lambda(x)$ denotes the index of the function f at the point x. In the case of Morse-Bott functions, the stationary phase method gives an asymptotic expansion of the form:

$$\int_X e^{itf}\,dm = \Big(\frac{2\pi}{t}\Big)^{n/2} \sum_{1\leqslant j\leqslant n} e^{itf(C_j)} e^{(n-2\lambda(C_j))\pi i/4} A_x(t),$$

where $\lambda(C_j)$ is the index f in the along nondegenerate critical submanifold C_j, $j = 1, \ldots, p$.

Theorem (F. Kirvan). *If the method of stationary phase for the Morse function f is exact, then the index every critical point is even and f perfect Morse function. In particular, the dimension of X is even.*

Accordingly to [3], for Morse height functions on the orbits of the adjoint representation Ad of compact semisimple Lie groups G, the stationary phase method is exact. In other words, the stationary phase method is exact for the matrix coefficients of the adjoint representation on the group manifold G. In [4] Duistermaat and Heckman proved that the components of the moment map of Hamiltonian actions of compact groups G on closed symplectic manifolds also have the exactness property of the stationary phase method.

Our main result is the following

Theorem. *The stationary phase method for the matrix coefficients of real irreducible representations of compact groups G with a transitive action of G on even-dimensional spheres of the representation space is exact.*

The proof of the theorem is based on some results of geometric analysis related to Bessel functions (see [5]).

Another interpretation of this result connected with asymptotic behavior characteristic functions of the matrix coefficients, which considered as random variables with respect to Haar measure on G.

References

[1] Arnold V. I., Gusein-Zade S. M., Varchenko A. N. *Singularities of differentiable maps.* Vol. II. Monodromy and asymptotics of integrals, Monogr. Math., 83, Birkhauser Boston Inc., Boston, MA, 1988, 492 pp.

[2] Kirwan F. *Morse functions for which the stationary phase approximation is exact.* Topology, 1987, v. 26, 1, p. 37–40.

[3] Semenov-Tjan-Šanskiĭ M. A. *A certain property of the Kirillov integral.* Zap. Naucn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI), 1973, issue 37, p. 53–65.

[4] Duistermaat J. J., Heckman G. J. *On the variation in the cohomology of the symplectic form of the reduced phase spaces.* Invent. Math., 1982, v. 69, p. 259–268.

[5] Wolf J. *Spherical functions on Euclidean spaces.* Journal functional analysis, 2006, v. 239, p. 127–136.

Singularities of special multi-flags at the crossroads of Algebraic Geometry and Differential Geometry

Mormul P.

Department of Mathematics, Informatics and Mechanics
University of Warsaw

There exist different approaches to the singularities of special multi-flags, also (misguidely) called ‘generalized Goursat flags’. They live in Monster Towers, also called Semple Towers. Colley et al have concluded, in [2], a series of earlier algebro-geometric constructions, by several authors, of fine stratifications of the stages of Semple Towers, eventually producing so-called ‘RV-classes of singularities’. In the meantime the present author constructed in the stages of Monster Towers, in [1], the so-called singularity classes, using purely differential and Lie-algebraic tools. It had been generally believed that the former classes (much more numerous) were a refinement of the latter ones. This belief now turns out to be false. The two approaches appear, to a sizeable degree, to mutually complement each other. And - important - complementary appear the very languages used in the two approaches. In fact, the charts used on the DG side are [by tradition] called ’Extended Kumpera-Ruiz’. Virtually the same charts used in the AG side are called ’C-charts’. However, the motivations underlying charts’ constructions in AG and DG are so distant that there has been, until recently, no identification procedure in sight. Such a procedure has been found over the last summer. With an effective translation going now both ways between the EKS’s and C-charts, the singularities seen on the AG side can now be interpreted on the DG side, and vice versa.

References

[1] P. Mormul; Singularities of special 2-flags. SIGMA 5 (2009), 102 (electronic).

[2] A. Castro, S. Colley, G. Kennedy, C. Shanbrom; A coarse stratification of the Monster Tower. Michigan Math. J. 66 (2017), 855 – 866.

On the phase change for perturbations of Hamiltonian systems with separatix crossing

Neishtadt A.[1], **Okunev A.**[1,2]

1. Department of Mathematical Sciences
Loughborough University, UK

2. Space Research Institute, Moscow

We consider general perturbations of Hamiltonian systems with one degree of freedom (the Hamiltonian depends on a slowly changing parameter) such that the orbits of the perturbed system cross separatices of the unperturbed system. We assume that for fixed value of the parameter the unperturbed system has a saddle with two separatix loops forming a figure eight. The solution of the perturbed system starts outside the union of the separatix loops and after crossing a separatrix proceeds inside one of the separatix loops. For study of such systems see, e.g., [1] and references therein. The point where the separtrix crossing occurs is described by a parameter called the pseudo-phase, it is known that this point depends on the initial conditions in a quasi-random way. Which separatix will be crossed is also determined by the pseudo-phase, so initial conditions with different outcomes are mixed in the phase space.

Formulas for the pseudo-phase were obtained (using the averaging method) in [2] for Hamiltonian systems with one degree of freedom and slow time dependence; in [3] for slow-fast Hamiltonian systems with one degree of freedom corresponding to fast motion; in [4] for perturbed strongly nonlinear oscillators. These formulas together with the formulas for the change of the adiabatic invariant allow to study trajectories with multiple separatrix crossings ([5], [6]).

We prove an asymptotic formula for the dependence of the pseudo-phase on the initial conditions. Such formula is a necessary ingredient for the study of quasi-random phenomena associated with multiple separatrix crossings. As a corollary of the formula for the pseudo-phase we prove that the "probabilities" of crossing each of the two separatrix loops are the same for both definitions of such probabilities suggested in [1]. In [1] the probabilities were computed for one of the definitions and for the other definition it was suggested without a proof that the probabilities are given by the same formulas.

Our proof is based on a detailed study of the averaged system of order 2. We also estimate how well the solutions of the averaged system of order 2 approximate the solutions of the perturbed system up to the separatrix. After necessary estimates related to the averaged system of order 2 are obtained, the proof of the formula for the pseudo-phase is close to the one in [3].

Acknowledgments. The work was supported by the Leverhulme Thrust (Grant No. RPG-2018-143).

References

[1] A. I. Neishtadt. "Averaging method for systems with separatrix crossing". Nonlinearity 30.7 (2017): 2871.

[2] J.R. Cary and R.T. Skodje. "Phase change between separatrix crossings". Physica D: Nonlinear Phenomena 36.3 (1989), pp. 287-316.

[3] A. I. Neishtadt and A. A. Vasiliev. "Phase change between separatrix crossings in slow-fast Hamiltonian systems". Nonlinearity 18.3 (2005): 1393.

[4] F. J. Bourland, R. Haberman, and W. L. Kath. "Averaging methods for the phase shift of arbitrarily perturbed strongly nonlinear oscillators with an application to capture". SIAM Journal on Applied Mathematics 51.4 (1991), pp. 1150-1167.

[5] A.I. Neishtadt, V.V. Sidorenko, and D.V. Treschev. "Stable periodic motions in the problem on passage through a separatrix". Chaos 7.1 (1997), pp. 2-11.

[6] A.A. Vasiliev, A.I. Neishtadt, C. Simo, and D.V. Treschev. "Stability islands in domains of separatrix crossings in slow-fast Hamiltonian systems". Proceedings of the Steklov Institute of Mathematics 259.1 (2007), p. 236.

On a stable arc connecting Palis diffeomorphisms on a surface

Nozdrinova E.V.

National Research University Higher School of Economics

In 1976, S. Newhouse, J. Palis, F. Takens [2] introduced the concept of a stable arc connecting two structurally stable systems on a manifold. Such an arc does not change its quality properties with a small perturbation. In the same year, S. Newhouse and M. Peixoto [4] proved the existence of a simple arc (containing only elementary bifurcations) between any two Morse-Smale flows. It follows from the result of G. Fleitas [1] that a simple arc constructed by Newhouse and Peixoto can always be replaced by a stable one [3].

Recall that a diffeomorphism f is *gradient-like* if its non-wandering set Ω_f consists of a finite number of hyperbolic points and the invariant manifolds of different saddle points do not intersect (the diffeomorphism f has no heteroclinic intersections). Consider the class $G(M^2)$ of gradient-like diffeomorphisms f on a closed orientable surface M^2, under the assumption that all non-wandering points are fixed and have positive orientation type.

Theorem 3. *Any diffeomorphisms $f, f' \in G(M^2)$ can be connected by a stable arc with a finite number of generically unfolding non-critical saddle-node bifurcations.*

The proof of this result is based on the construction of an arc without bifurcations connecting the diffeomorphism $f \in G(M^2)$ with the diffeomorphism $\phi_f \in G(M^2)$, which is a one-time shift of a generic gradient flow of some Morse function. By virtue of the works [4], [1], [3], any two such flows are connected by an arc with a finite number of saddle-node bifurcations.

Acknowledgements. Supported by Foundation for the Advancement of Theoretical Physics and Mathematics "BASIS" (project 19-7-1-15-1).

References

[1] Fleitas G. (1977). Replacing tangencies by saddle-nodes, Bol. Soc. Brasil. Mat. 8, No. 1, 47-51.

[2] Newhouse S., Palis J., Takens F. (1976). Stable arcs of diffeomorphisms, Bull. Amer. Math. Soc. 82, No.3, 499-502.

[3] Newhouse S.E., Palis J., Takens F. (1983). Bifurcations and stability of families of diffeomorphisms, Publications mathematiques de l' I.H.E.S. 57, 5-71.

[4] Newhouse S., Peixoto M. (1976).There is a simple arc joining any two Morse-Smale fows, Asterisque 31, 15-41.

Python visualization of Gröbli solution for three magnetic vortices

Palshin G. P.[1]
Research advisor: Ryabov P. E.[1]
[1] *Department of Data Analysis and Machine Learning*
Financial University under the Government of the Russian Federation

As known, integrable models of N point vortices hold a central position in the analytical dynamics of vortex structures. Suffice it to mention about the problem of N interacting point vortices in an ideal fluid, with point vortex $\alpha = 1, \ldots, N$ having strength Γ_α (which is constant according to Helmholtz's theorems) and position $\boldsymbol{r}_\alpha = (x_\alpha, y_\alpha)$. The problem consists in solving the system of $2N$ first-order nonlinear ordinary differential equations, so called the Helmholtz-Kirchhoff equations [1].

In present report, following the work [2], we will consider the generalized mathematical model for a case of point-like magnetic vortices in a magnetic film, which is described by the following differential equations [2]:

$$s_\alpha \dot{x}_\alpha = \frac{\partial H}{\partial y_\alpha}, \quad s_\alpha \dot{y}_\alpha = -\frac{\partial H}{\partial x_\alpha}, \quad \alpha = 1, \ldots N, \tag{1}$$

with the Hamilton function

$$H = -\sum_{\alpha,\beta}{}' \Gamma_\alpha \Gamma_\beta \ln \ell_{\alpha\beta}.$$

Here, $\ell_{\alpha\beta} = |\boldsymbol{r}_\alpha - \boldsymbol{r}_\beta|$ is the distance between vortices α and β, and the prime on the summation indicates omission of the singular term $\beta = \alpha$. Each vortex is characterized by a pair of indices: Γ_α is the vortex number ($+1$ or -1 for vortices and antivortices, respectively), and so-called the skyrmion number, $s_\alpha = \lambda_\alpha \Gamma_\alpha$, where the polarity λ_α specifies the direction of the magnetic vortex ($+1$ or -1 for magnetization pointing up or down, respectively).

In addition to Hamilton function H, the system (1) has three independent first integrals: the analog of linear momentum $\boldsymbol{P} = (P_x, P_y)$ and angular momentum L:

$$P_x = -\sum_\alpha s_\alpha y_\alpha, \quad P_y = \sum_\alpha s_\alpha x_\alpha, \quad L = \frac{1}{2}\sum_\alpha s_\alpha (x_\alpha^2 + y_\alpha^2).$$

Regardless of the values of the vortex number Γ_α, the integrals H, L and $P_x^2 + P_y^2$ are in *involution*, that is, the Poisson bracket between any two of them is zero, where *Poisson bracket* is defined by relations

$$\{f, g\} = \sum_\alpha \frac{1}{s_\alpha} \left(\frac{\partial f}{\partial x_\alpha} \frac{\partial g}{\partial y_\alpha} - \frac{\partial f}{\partial y_\alpha} \frac{\partial g}{\partial x_\alpha} \right)$$

for any two quantities f and g depending on the vortex positions. According to Liouville's theorem in analytical dynamics the Hamiltonian system (1) for $N = 3$ is completely integrable; that is, all vortex trajectories can be formally found by integration based on the conserved quantities of Hamiltonian, linear momentum and angular momentum.

In present report, we give a Python visualization of Gröbli solution for three magnetic vortices, which cannot be realized in ordinary fluids (Fig. 1).

References

[1] *V. V. Meleshko and Hassan Aref*, A bibliography of vortex dynamics 1858-1956 // Advances in Applied Mechanics, vol. 41, 2007, pp. 197–292.

[2] *S. Komineas and N. Papanicolaou*, Gröbli solution for three magnetic vortices, *Journal of Mathematical Physics*, vol. 51, no. 4, 2010, pp. 042705-1-18.

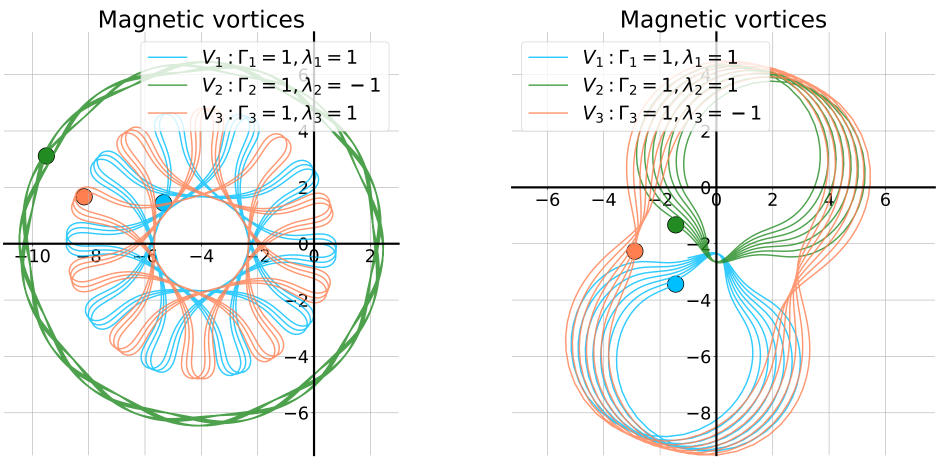


Figure 1: The trajectories for three magnetic vortices.

Disappearance bifurcation of a non-compact heteroclinic curve

Pochinka O.V., Shmukler V.I.

National Research University Higher School of Economics

In the present paper, we describe the scenario of the disappearance of a non-compact heteroclinic curve for a three-dimensional diffeomorphism. As a consequence, it is established that 3-diffeomorphisms with a single heteroclinic curve and fixed points of pairwise different Morse indices exist only on the 3-sphere. The described scenario is directly related to the reconnection processes in the solar corona, the mathematical essence of which, from the point of view of the magnetically charged topology, consists in the disappearance and birth of non-compact heteroclinic curves.

Consider the class G of orientation-preserving Morse-Smale diffeomorphisms f defined on a closed manifold M^3, the non-wandering set of which consists of exactly four points $\omega, \sigma_1, \sigma_2, \alpha$ with positive types of orientation and with Morse indices (dimensions of unstable manifolds) $0, 1, 2, 3$, respectively. Despite the simple structure of the non-wandering set, the class under consideration contains diffeomorphisms with wildly embedded saddle separatrices [1]. Moreover, two-dimensional saddle separatrices always intersect.

In work [3] it was proved that for any diffeomorphism $f \in G$ the set $H_f = W^s_{\sigma_1} \cap W^u_{\sigma_2}$ is not empty and contains at least one non-compact heteroclinic curve. It was also established that diffeomorphisms of the class under consideration admit the sphere S^3 and all lens spaces. On all the manifolds listed above, diffeomorphisms of class G are shifts per unit time of the gradient flow of the Morse function with exactly four critical points of pairwise different indices. Since, in the general case, the diffeomorphisms are not included even in the topological flow [2], then the question of the complete list of ambient manifolds for diffeomorphisms $f \in G$ is open.

In the present paper work, the following fact will be established.

Theorem 1. Let $f \in G$ and the set H_f linearly connected. Then M^3 is diffeomorphic to the 3-sphere.

Proof of the theorem 1 based on the construction of the following arc of diffeomorphisms.

Theorem 2. Let $f \in G$ and the set H_f linearly connected. Then f connected by a stable arc $\varphi_t : M^3 \to M^3, t \in [0, 1]$ with "source-sink" diffeomorphism, moreover φ_t has a single saddle-node bifurcation point.

Acknowledgments. The research was supported by the IL DSA, National Research University Higher School of Economics, grant from the government of the Russian Federation, contract 075-15-2019-1931.

References

[1] Bonatti C., Grines V., Pochinka O, Topological classification of Morse-Smale diffeomorphisms on 3-manifolds // Duke Mathematical Journal 2019. V. 168, N. 13. P. 2507–2558.

[2] Гринес В. З., Гуревич Е. Я., Медведев В. С., Починка О. В. О включении диффеоморфизмов Морса-Смейла на 3-многообразии в топологический поток // Математический сборник 2012. Т. 203, № 12. С. 81–104.

[3] Grines V. Z., Zhuzhoma, E. V. and Medvedev V. S. On Morse–Smale Diffeomorphisms with Four Periodic Points on Closed Orientable Manifolds. // Mathematical Notes 2003. V. 74, N.3. P. 352—366.

On self-similarity of the Dirichlet problem for the upper half-plane

Rassadin A. E.

Department of Fundamental Mathematics
Higher School of Economics

Let us consider the Dirichlet problem for the upper half-plane:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0\,, \qquad u(x,0) = U(x)\,.$$

Further let one suppose that Fourier transform $F[U]$ of boundary condition for this problem exists. Therefore the next theorem is valid.

Theorem. If supp$F[U] = [-\pi\,, \pi]$, then exact solution $u(x,y)$ of the Dirichlet problem for the upper half-plane can be represented as follows:

$$u(x,y) = \sum_{n=-\infty}^{+\infty} U(n)\, u_0(x-n,y)\,,$$

where

$$u_0(x,y) = \frac{1}{\pi}\,\frac{y}{x^2+y^2} + \frac{\exp(-\pi\, y)}{\pi}\,\frac{x\,\sin(\pi\, x) - y\,\cos(\pi\, x)}{x^2+y^2}\,.$$

On the upper half-plane this function obeys to the next relations:

$$u_0(x,y) = \sum_{n=-\infty}^{+\infty} \frac{N^2}{\pi\, n}\,\sin\frac{\pi\, n}{N}\, u_0(N\,x-n, N\,y)\,, \qquad N = 2, 3, 4, \ldots$$

and

$$\sum_{n=-\infty}^{+\infty} u_0(x-n,y) = 1\,.$$

Estimation of discounted profit for exploited populations

Rodina L. I.[1,2], **Woldeab M. S.**[2]

1. Department of Mathematics
National University of Science and Technology MISIS

2. Department of Functional Analysis and its Applications
A. G. and N. G. Stoletov Vladimir State University

We conside the model of a population subject to a craft from which some fraction of resource u_k is extracted at the time $\tau_k = kd$:

$$\dot{x} = g(x), \quad t \neq \tau_k,$$

$$x(\tau_k) = (1 - u_k) \cdot x(\tau_k - 0), \quad k = 1, 2, \ldots.$$

Here and below, $d > 0$ is the length of interval between the harvest times. Assume that the equation $\dot{x} = g(x)$ has the asymptotically steady solution $\varphi(t) \equiv K$ and (K_1, K_2) is the domain of attraction of this solution.

Denote $U \doteq \{\bar{u} : \bar{u} = (u_1, \ldots, u_k, \ldots)\}$, where $u_k \in [0, 1]$,

$$X_k = X_k\big(u_1, \ldots, u_{k-1}, x_0\big) = x(\tau_k - 0)$$

is the resource amount before harvest at the time $\tau_k = kd$, $k = 1, 2, \ldots$. We conside the following problem: how to control the workpiece shares $\bar{u} \in U$ in order that the profit from resource extraction

$$H_\alpha(\bar{u}, x_0) \doteq \sum_{k=1}^{\infty} X_k\big(u_1, \ldots, u_{k-1}, x_0\big) u_k e^{-\alpha k d}$$

with the discount coefficient $\alpha > 0$ to be maximal?

We denote by $\varphi(t, x_0)$ the solution to the differential equation $\dot{x} = g(x)$, satisfying the initial condition $\varphi(0, x_0) = x_0$.

Proposition. *Let the function $D(x) \doteq \varphi(d, x) - e^{\alpha d} x$ attain the largest value at a unique point $x^* \in (K_1, K)$. Then for any $x_0 \in (K_1, K_2)$ the maximal value of $H_\alpha(\bar{u}, x_0)$ is as follows:*

$$H_\alpha(\bar{u}^*, x_0) = \big(\varphi(k_0 d, x_0) - x^*\big) \cdot e^{-\alpha k_0 d} + \big(\varphi(d, x^*) - x^*\big) e^{-\alpha(k_0+1)d} / 1 - e^{-\alpha d}, \quad (1)$$

where $k_0 = k_0(x_0)$ is the least natural number such that $\varphi(k_0 d, x_0) > x^*$. The value $H_\alpha(\bar{u}^*, x_0)$ as attained in the following exploitation regime: $u_k^* = 0$ for all $k = 1, \ldots, k_0 - 1$, $u_{k_0}^* = 1 - x^*/\varphi(k_0 d, x_0)$ and $u_k^* = 1 - x^*/\varphi(d, x^*)$ for all $k > k_0$.

The proof of proposition you can see in [1]. In this work we also research the stochastic models of population dynamics.

Example. We consider the population model described by the equation

$$\dot{x} = a x \ln K/x, \quad t \neq \tau_k,$$

$$x(\tau_k) = (1 - u_k) \cdot x(\tau_k - 0), \quad k = 1, 2, \ldots,$$

where $a > 0$, $K > 0$, $\tau_k = kd$. We note that the equation $\dot{x} = a x \ln K/x$ has the asymptotically steady solution $\varphi(t) \equiv K$ with the domain of attraction $(0, +\infty)$. The function

$$D(x) = \varphi(d, x) - e^{\alpha d} x = K\big(x_0/K\big)^{e^{-ad}} - e^{\alpha d} x$$

attains the largest value at the point $x^* = K e^{(\alpha + a) d e^{ad} / 1 - e^{ad}}$. The largest value of $H_\alpha(\bar{u}^*, x_0)$ we can find from (1).

The study was funded by RFBR, project number 20-01-00293.

References

[1] L. I. Rodina, A. H. Hammadi. Optimization problems for models of harvesting a renewable resource // Journal of Mathematical Sciences. 2020. V. 250, pp. 113-122.

Hidden symmetries, coupled networks and equivariant degrees

Haibo Ruan

Institute of Mathematics
Technical University of Hamburg

haibo.ruan@tuhh.de

Hidden symmetries have been previously explored in the context of coupled cell networks and coupled cell systems. These include interior symmetry, quotient symmetry and quotient interior symmetry. We introduce here an equivariant degree theory that incorporates these different forms of hidden symmetry based on lattice structures of synchrony subspaces. The result is a unified theory capable of treating synchrony-breaking Hopf-bifurcation problems in coupled cell systems with various forms of hidden symmetries, which leads to full topological classifications of bifurcating branches arising from single bifurcation points under mutual influence of these hidden symmetries.

On the singular fibre with a doubly pinched torus and its realization in one mechanical system

Ryabov P. E.[a,1,2,3], **Sokolov S. V.**[b,2,3], **Kaverina V. K.**[a,1]

[a] *Department of Data Analysis and Machine Learning*
[1] *Financial University under the Government of the Russian Federation*
[2] *Institute of Machines Science, Russian Academy of Sciences*

[b] *Department of Theoretical Mechanics*
[3] *Moscow Institute of Physics and Technology (State University)*

In differential topology, the name doubly pinched torus corresponds to the surface of a torus which pinched in doubly points. In $\mathbb{R}^3(x, y, z)$, such a surface can be defined as an implicit equation

$$625(x^2+y^2-1)^2(x^2+y^2)^2+16[y^2-4(x^2+y^2)z^2]^2-$$
$$-200(x^2+y^2)(1+x^2+y^2)[y^2-4(x^2+y^2)z^2]=0.$$

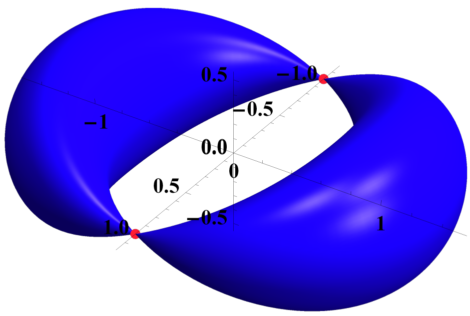


Figure 1: Doubly Pinched $\mathbb{T}^2$.

The present report is devoted to the realization of doubly pinched torus in one mechanical system that describes the dynamics of a rigid body in a uniform gravity field. One of the points of the body lying on the axis of symmetry (the suspension point) performs high-frequency vertical oscillations of small amplitude. In the works of A. P. Markeev [1],[2], a transformation is specified that leads the original equations of motion to a system that has the form of Euler-Poisson equations. The corresponding system of differential equations has the form

$$\dot{\boldsymbol{M}} = \boldsymbol{M} \times \frac{\partial H}{\partial \boldsymbol{M}} + \boldsymbol{\gamma} \times \frac{\partial H}{\partial \boldsymbol{\gamma}}, \qquad \dot{\boldsymbol{\gamma}} = \boldsymbol{\gamma} \times \frac{\partial H}{\partial \boldsymbol{M}} \tag{1}$$

with the Hamilton function

$$H = \frac{1}{2}\left(M_1^2 + M_2^2 + cM_3^2\right) + a\gamma_3 - \frac{1}{2}b\gamma_3^2. \tag{2}$$

Here, $\boldsymbol{M} = \{M_1, M_2, M_3\}$ and $\boldsymbol{\gamma} = \{\gamma_1, \gamma_2, \gamma_3\}$ denote the angular momentum and the unit vector of the vertical axis in the system of main axes connected to the rigid body. The parameters a, b, and c, according to [1],[2], have explicit physical meaning. The system (1) allows one additional first integral of motion $F = M_3$ (*Lagrange's integral*). The phase space $\mathcal{P}$ is given as a tangent bundle $T\mathbb{S}^2$ to a two-dimensional sphere $\mathbb{S}^2$: $\mathcal{P} = \{(\boldsymbol{M}, \boldsymbol{\gamma}) \ : \ (\boldsymbol{M}, \boldsymbol{\gamma}) = \ell, \, |\boldsymbol{\gamma}|^2 = 1\}$. We define *momentum map* $\mathcal{F} : \mathcal{P} \to \mathbb{R}^2$, assuming $(f, h) = \mathcal{F}(\boldsymbol{x}) = (F(\boldsymbol{x}), H(\boldsymbol{x}))$ and denote by $\mathcal{C}$ the set of all critical points of the momentum map, that is points where $\operatorname{rank} d\mathcal{F}(\boldsymbol{x}) < 2$. Set of critical values $\Sigma = \mathcal{F}(\mathcal{C} \cap \mathcal{P})$ is called *bifurcation diagram*.

For the Hamiltonian (2), there are two equilibrium positions (critical points of rank 0 of the momentum map $\mathcal{F}$). Stationary rotations of a rigid body around the axis of symmetry with a constant angular velocity correspond to the equilibrium positions. The corresponding values of the constants of the first integrals define two isolated points $P^{\pm}$ on the bifurcation diagram Σ. These points correspond to the *focus singularities* of rank 0 in $\mathcal{P}$. The case when the isolated points on the bifurcation diagram coincide occurs when the area integral ℓ is zero and $a = 0$; $b < 0$, i.e. when the center of mass is placed at the origin of the coordinate system associated with a rigid body, but the parameter b, which is responsible for the effect of the vibrating potential on the suspension point, must be negative. This leads to the existence of the singular fibre with a doubly pinched torus in the preimage of an isolated critical value $P^+ = P^- = P\left(0; -\frac{b}{2}\right)$ of the momentum map $\mathcal{F}$.

The research of P. E. Ryabov was partially supported by the Russian Science Foundation, project no. 19-71-30012. The work of S. V. Sokolov was carried out at MIPT under project 5-100 for state support for leading universities of the Russian Federation and also partially support by RFBR grant 18-29-10051-mk.

References

[1] Markeev A. P., On the theory of motion of a rigid body with a vibrating suspension // Doklady Physics, vol. 54, no. 8, 2009, pp. 392-396.

[2] Markeev A. P. On the motion of a heavy dynamically symmetric rigid body with vibrating suspension point // Mechanics of Solids, vol. 47, no. 4, 2012, pp. 373-379.

Analogs of the Lebesgue measure on a Hilbert space and its applications

Sakbaev V.Zh.

1. National Research Lobachevsky State University of Nizhni Novgorod

2. Moscow Institute of Physics and Technology, Moscow region, Dolgoprudny, Russia
sakbaev.vzh@phystech.edu

According to Weyl's theorem, the Lebesgue measure does not exist in an infinite-dimensional Hilbert space. Thus a measure is considered as an additive function of a set defined on a ring of subsets of a Hilbert space. The space of functions that are integrable with respect to the translationally invariant measure is introduced. It is shown that the result of averaging of shift operators on a random vector with Gaussian distributions is a semigroup of self-adjoint contractions. The criterion of strong continuity of a semigroup is established. Using the introduced diffusion semigroup, we define Sobolev spaces and the space of smooth functions on a Hilbert space. Conditions of the embedding and of the densely embedding of the space of smooth functions into the Sobolev space are obtained [1]. The absence of densely embedding of the space of smooth functions into the space of Sobolev function is called Lavrent'ev effect [2]. The applications of this effect to the variational description of solutions of boundary value problems are considered [2, 3].

References

[1] *Busovikov V.M., Sakbaev V.Zh.* Sobolev spaces of functions on a Hilbert space endowed with a translation-invariant measure and approximations of semigroups, Izv. Math., 84:4 (2020), 694–721.

[2] *Zhikov V.V.* The Lavrent'ev effect and averaging of nonlinear variational problems.// Differ. Equ., 27:1 (1991), 32–39

[3] *Busovikov V.M., Sakbaev V.Zh.* Dirichlet Problem for Poisson Equation on the Rectangle in Infinite Dimensional Hilbert Space// Applied Mathematics and Nonlinear Sciences. https://doi.org/10.2478/amns.2020.2.00016

Classes of integrable systems with dissipation on the tangent bundles of four-dimensional manifolds

Shamolin M. V.

Institute of Mechanics
Lomonosov Moscow State University

We prove the integrability of certain classes of dynamical systems on the tangent bundles of four-dimensional manifolds (systems with four degrees of freedom). The force field considered possessed so-called variable dissipation; they are generalizations of fields studied earlier. This paper continues earlier works of the author devoted to systems on the tangent bundles of two-and three-dimensional manifolds.

Configuration spaces of of many dynamical systems are four-dimensional smooth manifolds; naturally, their phase spaces are tangent bundles of these manifolds. For example, the motion of a five-dimensional generalized spherical pendulum in a nonconservative force field is described by a dynamical system on the tangent bundle of the four-dimensional sphere whose metric is induced by an additional symmetry group (see [1, 2, 3]). In this case, dynamical systems that describe the motion of such a pendulum possess variable dissipation, and a complete list of first integrals consists of transcendental functions that can be expressed as finite combinations of elementary functions.

In this activity, we prove the integrability of certain classes of dynamical systems on tangent bundles of smooth four-dimensional manifolds in the case of systems with variable dissipation (see [4, 5, 6]), which are generalizations of systems studied earlier. Similar results for manifolds of dimensions 2 and 3 were obtained by the author earlier.

References

[1] M. V. Shamolin, “Dynamical systems with variable dissipation: Approaches, methods, and applications,” Fundam. Prikl. Mat., **14**, No. 3, 3–237 (2008).

[2] M. V. Shamolin, “Variety of integrable cases in dynamics of low- and multi-dimensional rigid bodies in nonconservative force fields,” J. Math. Sci., **204**, No. 4, 379–530 (2015).

[3] M. V. Shamolin, “Integrable systems with variable dissipation on the tangent bundle of a multidimensional sphere and their applications,” Fundam. Prikl. Mat., **20**, No. 4, 3–231 (2015).

[4] O. I. Bogoyavlenskii, “Some integrable cases of Euler equation,” Dokl. Akad. Nauk SSSR, **287**, No. 5, 1105–1108 (1986).

[5] B. A. Dubrovin and S. P. Novikov, “On Poisson brackets of hydrodynamic type,” Dokl. Akad. Nauk SSSR, **279**, No. 2, 294–297 (1984).

[6] A. P. Veselov, “On integrability conditions for the Euler equations on so(4),” Dokl. Akad. Nauk SSSR, **270**, No. 6, 1298–1300 (1983).

Uniformization of holomorphic foliations with hyperbolic leaves

Shcherbakov A.A.
MCCME

We consider foliations of compact complex manifolds by analytic curves. We suppose that the line bundle tangent to the foliation is negative. We show that in a generic case the manifold of universal coverings with the base B is diffeomorphic to $B \times D$ (D is the unit disk) wit some almost complex structure quasiconformic on the fibers. Also there exists a finitely smooth homeomophism, holomorphic on the fibers and mapping fiberwise the manifold of universal coverings onto some domain in $B \times \mathbb{C}$ with continuous boundary. The problem can be reduced to a study of the Beltrami equation with parameters on the unit disk in the case, when derivatives of the corresponding Beltrami coefficient grow no faster than some negative power of the distance to the boundary of the disk.

Connectivity notions and properties - a new approach by coverings

Shekutkovski N.

University of St. Cyril and Methodius

The standard definition of connectedness was given in the beginning of 20th century by Riesz and Hausdorff. The main minus of the definition is that definition is given by negative sequence Another definition of connectedness was given by Cantor about 1880, equivalent with standard definition in the case of compact metric spaces.

(Cantor definition of connectedness) Space is connected if for any two points x and y and any $r > 0$ there is a finite number of points $x = x_1, x_2, , \ldots, x_n = y$ such that $d(x_i, x_{i+1}) < r$ for $1 \leq i \leq n-1$.

This definition can be reformulated by use of coverings. Suppose $\mathcal{F}$ is a family of subsets of X , and x and y are two points in X . A chain in $\mathcal{F}$ from x to y is a finite sequence $F_1, F_2, , ..., F_n$ of members of $\mathcal{F}$ such that $x \in F_1, y \in F_n$ and $F_i \cap F_{i+1} \neq \emptyset$, for $1 \leq i \leq n-1$. Definition. X is connected if for any two points x and y in X and any open covering of X there is a chain of members of the covering from x to y

This definition holds for all topological spaces. Using this definition we generalize connectivity properties in more general topological spaces.

Verbal width by set of squares of alternating group A_n and its subgroups

Skuratovskii Ruslan [1]

1. Institute for Applied and System Analysis, av. Pobedu 37, 03056 Kiev, (Ukraine)
Igor Sikorsky Kyiv Polytechnic Institute, Kiev

The **width of the verbal subgroups** $V(G)$ is equal to a least value of $m \in \mathrm{N} \cup \{\infty\}$ such that every element of the subgroup $V(G)$ is represented as the product of at most m values of words V.

The Verbal Width of a free group was investigated by Sucharit Sarkar [1]. Sarkar proved that an arbitrary commutator of free group of rank greater than 1 cannot be generated by only 2 squares. The conditions when a commutator can be presented as a product of 2 squares were further found by him too [1]. The commutator subgroup of Sylow 2-subgroups of an alternating group have been previously investigated by the author [2].

We consider a similar problem in the symmetric group S_n and alternating group A_n. Upon taking into account that the commutator subgroup of S_n is the alternating group A_n, when $n > 4$ [3], then the problem can be reformulated in terms of the alternating group.

Therefore, we research the verbal width by squares of A_n.

Theorem 1. *An arbitrary element of A_n can be presented in the form of a product of two squares of elements from A_n.*

Proposition 1. *If for an element $g \in A_n$, there exists $l \in \mathrm{N}$, such that $l = 2k,\ k \in \mathrm{N}$ and g can be presented in the form of a product of independent cycles with an odd number of l-cycles, then g cannot be presented in the form of square of h, $h \in A_n$, where h is presented as a product of independent cycles.*

Lemma 1. *The square of a cycle of even length L is a product of two cycles of length $\frac{L}{2}$.*

We denote by p_l the number of pairs of l-cycles in cyclic presentation of g. We define residue of g $(r(g) > 1)$ as the number of element of $\{1, ..., n\}$, where g acts trivial. We define m_i as the number of cycles of length i in cyclic structure.

Theorem 2. *If in cyclic structure of $g \in A_n$ every even cycle appears even times i.e. $m_{2k} \equiv 0 \pmod 2$ and one of the next conditions 1 or 2 holds:*

1)
$$\left[\begin{array}{c} a)\ r(g) > 1 \\ b)\ \max\limits_{k \in N} (m_{2k-1}) > 1, \end{array}\right.$$

2) $\sum\limits_{l}^{L} p_{2l} \equiv 0 \pmod 2$

then this g can be presented as $g = h^2$, $h \in A_n$. The vice versa is also true.

Theorem 3. *An arbitrary element of A_n can be presented in form of a product of 2 squares of elements from A_n.*

Lemma 2. *If an element $g \in A_n$, $\exists l \in \mathrm{N}$, $l = 2k,\ k \in \mathrm{N}$ can be presented in form of a product of independent cycles with odd number of independent l-cycles, then g can not be presented in form of square of h, $h \in A_n$, where h is presented as one cycle.*

Theorem 4. *The set of all squares S from A_n does not coincide with the whole alternating group A_n and does not form a proper subgroup of A_n.*

Proposition 2. *Element g having asymmetric type of cycle structure can be constructed in form of product of squares $g = g_1 \cdot \mathbf{g_2}$ with using 2 joint elements in cyclic representation of g_1, $\mathbf{g_2}$, where cycles has odd length.*

Theorem 5. The following statements are true.

1. An element $g = (g_1, g_2) \in G'_k$ iff $g_1, g_2 \in G_{k-1}$ and $g_1 g_2 \in B'_{k-1}$.

2. Commutator subgroup G'_k coincides with set of all commutators for $k \geq 1$

$$Syl'_2 A_{2^k} = \{[f_1, f_2] \mid f_1 \in Syl_2 A_{2^k}, f_2 \in Syl_2 A_{2^k}\}.$$

Corollary. Commutator width of the group $Syl_2 A_{2^k}$ equal to 1 for $k \geq 2$.

References

[1] Sucharit Sarkar. Commutators and squares in free group, *Algebra Geometry Topologe*, **4** (2004), 595-602.

[2] Ruslan V. Skuratovskii. Commutator subgroup of Sylow 2-subgroups of alternating group and the commutator width in the wreath product. *BASM* n.1(92), 2020, pp.3-16.

[3] John D. Dixon, Brian Mortimer Permutation groups. *Graduate texts in mathematics*; **163** (1996), P. 348.

Chaos with additional zero Lyapunov exponents in radiophysical generators

Stankevich N. V.

National Research University Higher School of Economics

One of the promising area of applied use of radiophysical generators is communication systems based on the synchronization of chaotic signals [1, 2]. Tasks of this kind present certain requirements for complex signal of generators. One of the most important properties is the broadband of the generated signal. The broadband property may be associated with a scenario resulting in a chaotic signal. For example, during the formation of a chaotic signal as a result of a cascade of bifurcations of doubling of a two-frequency torus, chaos may arise the spectrum of Lyapunov exponents of which contains an additional zero Lyapunov exponents [3]. This scenario is universal and has been found in various dynamic systems, including models of radiophysical generators. In this work, we study the scenario of appearance and spectral characteristics of such signals for various radiophysical generators.

We consider two the simplest examples, its are generators described by ordinary differential equations of forth order [4]. The next example of a radiophysical generator, in which such dynamics observed is two coupled generators of quasiperiodic oscillations [3], which are described by a system of ordinary differential equations of the sixth order. And the last example of a model in which it is possible to observe the development of chaos with an additional zero Lyapunov exponent in the spectrum is an ensemble of five van der Pol oscillators coupled in a ring [5].

The work presents the results of a comparative analysis of the study of different radiophysical generators in which chaotic oscillations characterized by a spectrum of Lyapunov exponents with additional zero exponents can be observed. A study of the power spectra of chaotic signals with an additional zero Lyapunov exponent, as well as comparison with chaos without additional zero and with hyperchaos.

The work was supported by grant of Russian Foundation for Basic Research (project No. 19-31-60030)

References

[1] Dmitriev A.S., Panas A.I. Dynamic chaos: new carriers of information for communication systems. Fizmatlit, (2002). (in Russia)

[2] Koronovskii A.A., Moskalenko O.I., Hramov A.E. // Physics-Uspekhi. **52**(12) (2009) 1213.

[3] Kuznetsov A.P., Kuznetsov S.P., Shchegoleva N.A., Stankevich N.V. // Physica D. **398** (2019) 1-12.

[4] Anishchenko V.S., Nikolaev S.M. // Technical physics letters. **31** (2005) 853-855.

[5] Stankevich N.V., Popova E.S., Kuznetsov A.P., Seleznev E.P. // Technical physics letters. **45** (2019) 1233-1236.

A Morse-Bott function for topological flows with a finite hyperbolic chain-recursive set

Zinina S. Kh.

National Research Mordovia State University

The Lyapunov Function of a dynamical system defined on a closed topological manifold M^n is called a continuous function $\varphi : M \to \mathbb{R}$, which is constant on each chain component of the system and decreases along its orbits outside the chain-recurrent set. Due to the results of C. Conley [1], the Lyapunov function exists for any dynamical system, and the fact of its existence is called the "Fundamental theorem of dynamical systems". Critical values of the φ Function. Conley named the numbers that belong to the image of a chain-recurrent set. However, for a smooth function, its critical value is usually called the image of the critical point (the point where the gradient of the function vanishes), which, generally speaking, does not have to belong to a chain-recurrent set. In this connection, along with the Lyapunov function, the smooth category uses the concept of *energy function*, that is, a smooth Lyapunov function whose set of critical points coincides with the chain-recurrent set of the system.

The first results on the construction of the energy function belong to S. Smale [2], who in 1961 proved the existence of the Morse energy function for gradient-like flows. K. Meyer [3] in 1968 generalized this result by constructing the Morse-Bott energy function for an arbitrary Morse-Smale flow. In the work [4] O. V. Pochinka and S. Kh. Zinina considered topological flows with a finite hyperbolic chain-recurrent set on closed surfaces and proved that they have an energy (continuous) Morse function.

We introduce a class G of continuous flows f^t on M^n that generalize the concept of Morse-Smale flows. Such flows have a hyperbolic (in the topological sense) chain-recurrent set R_{f^t} consisting of a finite number of orbits (*chain components*). Each non-wandering orbit is either a fixed point or a periodic orbit $\mathcal{O}$ for which the concept of stable $W^s_{\mathcal{O}}$ and unstable $W^u_{\mathcal{O}}$ manifolds is correctly defined. We can prove that the chain components of the considered flows do not form cycles, and can therefore be totally ordered $O_1 \prec \cdots \prec \mathcal{O}_k$ with preservation of the ratio of Smale: $W^s_{\mathcal{O}_i} \cap W^u_{\mathcal{O}_j} \neq \emptyset \Rightarrow i < j$.

True the following theorem that establishes basic dynamic properties of flows in class G.

Theorem Let $f^t \in G$. Then

1) $M = \bigcup\limits_{i=1}^{k} W^u_{\mathcal{O}_i} = \bigcup\limits_{i=1}^{k} W^s_{\mathcal{O}_i}$;

2) For any fixed point $\mathcal{O}_i$, there is a number $\lambda_i \in \{0, \dots, n\}$ (Morse index of the point $\mathcal{O}_i$) such that its unstable manifold $W^u_{\mathcal{O}_i}$ is a topological submanifold of a variety M that is homeomorphic to $\mathbb{R}^{\lambda_i}$ and a stable variety $W^s_{\mathcal{O}_i}$ is a topological submanifold of M, homeomorphic to $\mathbb{R}^{n-\lambda_i}$;

3) for a periodic orbit $\mathcal{O}_i$, there is a number $\lambda_i \in \{0, \dots, n-1\}$ (Morse index of the orbit $\mathcal{O}_i$) and a pair of numbers $\mu_i, \nu_i \in \{-1, +1\}$ (orbit type $\mathcal{O}_i$) such that its unstable manifold $W^u_{\mathcal{O}_i}$ is a topological submanifold of M, homeomorphic to $\mathbb{R}^{\lambda_i} \times \mathbb{S}^1$ for $\mu_i = +1$ and $\mathbb{R}^{\lambda_i} \tilde{\times} \mathbb{S}^1$ for $\mu_i = -1$; stable manifold $W^s_{\mathcal{O}_i}$ is a topological submanifold of a manifold M homeomorphic to $\mathbb{R}^{n-\lambda_i} \times \mathbb{S}^1$ for $\nu_i = +1$ and $\mathbb{R}^{n-\lambda_i} \tilde{\times} \mathbb{S}^1$ for $\nu_i = -1$;

4) $(cl(W^u_{\mathcal{O}_i}) \setminus W^u_{\mathcal{O}_i}) \subset \bigcup\limits_{j=1}^{i-1} W^u_{\mathcal{O}_j}$; $(cl(W^s_{\mathcal{O}_i}) \setminus W^s_{\mathcal{O}_i}) \subset \bigcup\limits_{j=i+1}^{k} W^s_{\mathcal{O}_j}$.

The main result is the following theorem.

Theorem Each flow $f^t \in G$ has a Morse-Bott energy function whose critical points are either nondegenerate or have a degeneracy degree of 1.

Acknowledgments. S.Kh. Zinina was supported by International Laboratory of Dynamical Systems and Applications of National Research University Higher School of Economics, Russian Federation Government grant 075-15-2019-1931, Foundation for the Advancement of Theoretical Physics and Mathematics "BASIS" (project 19-7-1-15-1) and the reported study was funded by RFBR, project number 20-31-90069.

References

[1] Conley C. Isolated invariant sets and the Morse index. CBMS Regional Conference Series in Math. Vol. 38. Providence, RI: AMS. 1978.

[2] Smale S. On gradient dynamical systems. Ann. of Math. 1961. vol. 74. pp. 199–206

[3] Meyer K. R. Energy functions for Morse Smale systems. Amer. J. Math. 1968. vol.90. pp.1031–1040

[4] Pochinka, O.V., Zinina, S.K. A Morse Energy Function for Topological Flows with Finite Hyperbolic Chain Recurrent Sets. Math Notes 107, 313–321 (2020). https://doi.org/10.1134/S0001434620010319