NRU HSE NIZHNY NOVGOROD 30 MAR. – 1 APR.

STUDENT EDUCATIONAL SCHOOL-CONFERENCE

MATHEMATICAL SPRING 2021

INVITATION TO
DYNAMICAL SYSTEMS

BOOK OF ABSTRACT

Student Educational School-Conference
Mathematical Spring 2021
Invitation to Dynamical Systems

Book of Abstracts

National Research University
Higher School of Economics

Nizhny Novgorod,
March 30 - April 2, 2021

Organizers

National Research University Higher School of Economics, Nizhny Novgorod

Organising Committee

- Olga Pochinka
- Vyacheslav Grines
- Elena Nozdrinova

Acknowledgment. The conference is partially supported by Laboratory of Dynamical Systems and Applications NRU HSE, of the Ministry of science and higher education of the RF grant ag. №075-15-2019-1931

Contents

Remark on Lower and Upper Bounds for Ratio of Complete Elliptic Integrals of the First Kind

Alekseeva E.S.

Nizhny Novgorod Mathematical Society
kometarella@mail.ru

The value

$$\mathbf{s}(k) = \frac{\mathbf{K}(\sqrt{1-k^2})}{\mathbf{K}(k)}\,, \tag{1}$$

where $\mathbf{K}(k) = \int_0^{\frac{\pi}{2}}(1-k^2\sin^2\varphi)^{-1/2}d\varphi$ is a complete elliptic integral of the first kind, often arises in different applications. For instance, the optimal number of iterations for the alternating-directions method in the commutative case [1] is expressed via function (1). Another example of appearance of ratio (1) is the problem of conformal mapping of rectangular domain on the upper half-plane [2].

In this report the following statement has been proven:

Theorem. *Under $k \in (0,1)$ function (1) obeys to the next inequalities:*

$$\mathbf{s}^-(k) < \mathbf{s}(k) < \mathbf{s}^+(k)\,,$$

where

$$\mathbf{s}^-(k) = \frac{\mathbf{K}^-(\sqrt{1-k^2})}{\mathbf{K}^+(k)}\,, \qquad \mathbf{s}^+(k) = \frac{\mathbf{K}^+(\sqrt{1-k^2})}{\mathbf{K}^-(k)}\,, \tag{2}$$

and

$$\mathbf{K}^-(k) = \frac{\pi}{4\,k}\left(\arcsin k + \ln\sqrt{\frac{1+k}{1-k}}\right)$$

$$\mathbf{K}^+(k) = \frac{\pi}{2\sqrt{2}\,k}\left(\arcsin\frac{k}{\sqrt{2}} + \ln\sqrt{\frac{1+k}{1-k}\,\frac{2-k+\sqrt{2-k^2}}{2+k+\sqrt{2-k^2}}}\right).$$

Graphs of lower and upper bounds (2) for function (1) are presented on Figure 1.

Graphs of relative error $\delta\ \mathbf{s}^-(k)$ of lower bound for function (1) and of relative error $\delta\mathbf{s}^+(k)$ of upper bound for this one are shown on Figure 2.

From these Figures one can observe that there is an interval of modulus k of complete elliptic integral of the first kind on which estimations (2) are quite exact for usage in engineering.

References

[1] A. A. Samarskii, E. S. Nikolaev, Numerical Methods for Grid Equations. Volume II. Iterative Methods, Basel, Boston, Berlin (1989), Birkhäuser.

[2] N. I. Akhiezer, Elements of the Theory of Elliptic Functions, Moscow (1970), translated into English as AMS Translations of Mathematical Monographs Volume 79 (1990) AMS, Rhode Island.

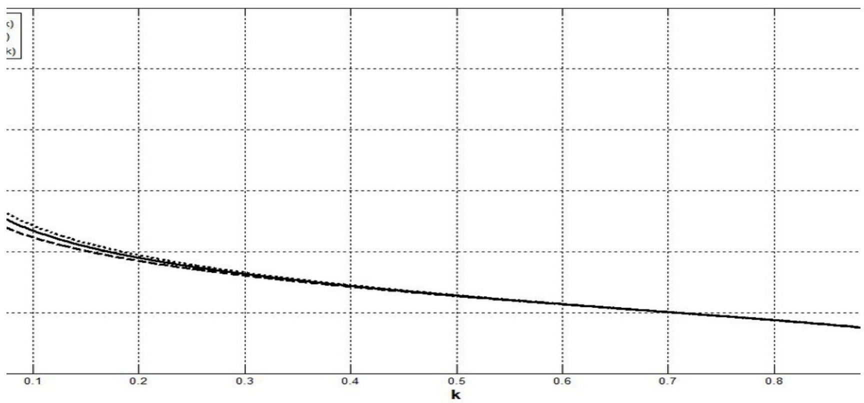

Figure 1: Graphs of $\mathbf{s}(k)$ and its lower and upper bounds

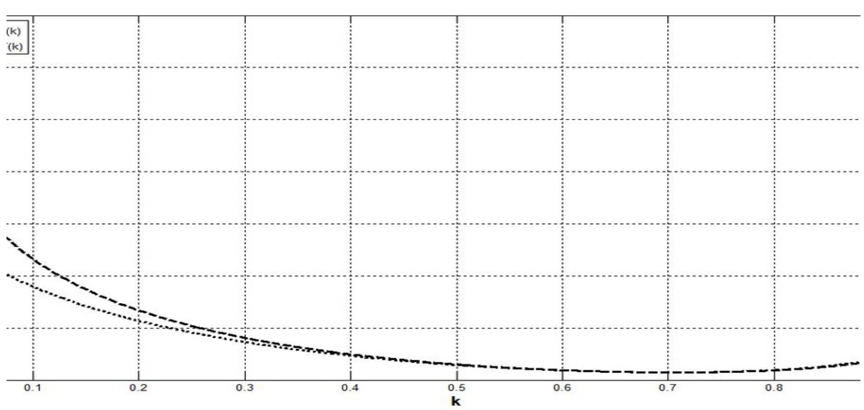

Figure 2: Graphs of relative errors of approximations of $\mathbf{s}(k)$

A topological classification of three-dimensional billiards bounded by confocal quadrics[1]

Belozerov G. V.[1]

1. *Department of differential geometry and applications*
Moscow State University

Let us consider a motion of material point inside a billiard table, i. e. inside a compact three-dimensional domain bounded by confocal quadrics. Also we assume that all dihedral angles on boundary of the billiard table are equal to $\frac{\pi}{2}$, no force acts on this material point and reflection is absolutely elastic. As it turns out that such billiards are integrable Hamiltonian systems (in the piecewise smooth sense). Let us consider rough Liouville equivalence relation of such billiards (isomorphism of bases of their Lagrangian fibrations with singularities). Author proved that there are exactly 25 types of rough Liouville nonequivalent billiards.

Also it turns out that we can determinate homeomorphism class of regular isoenergy surfaces, if we know shape of billard table. Author proved that each regular isoenergy surface of the billiard is homeomorphic to S^5 or $S^4 \times S^1$ or $S^3 \times S^2$.

References

[1] A. V. Bolsinov and A. T. Fomenko, *Integrable Hamiltonian systems. Geometry, topology, classification*, vols. 1, 2, «Udmurt University», Publ. House, Izhevsk 1999, 444 pp., 447 pp.; English transl., Chapman & Hall/CRC, Boca Raton, FL 2004, xvi+730 pp.

[2] Nguen T. Z.,*Symplectic topology of integrable Hamiltonian systems, I: Arnold - Liouville with singularities*, Compositio Mathematica. 1996 101 179 - 215.

[3] V. V. Fokicheva, *Description of singularities for billiard systems bounded by confocal ellipses or hyperbolas*, Moscow Univ. Math. Bull., 69:4 (2014), 148-158.

[1] The work was done at Moscow State University under the support of the Russian Science Foundation (project № 20-71-00155).

On the existence of an Ehresmann connection for foliations of codimension one on 3-manifolds

Bogatov A.A.

Department of Fundamental Mathematics
National Research University
Higher School of Economics

The concept of an Ehresmann connection was introduced by R. A. Blumenthal and J. J. Hebda [1]. It has a global differential-topological character. For a smooth foliation $(M, \mathcal{F})$ of codimention q on an n-dimentional manifold M, an Ehresmann connection is a q-dimentional smooth transverse distribution D admitting some homotopy property.

Ehresmann connections are used to describe the structure of foliations consistent with different geometries [4].

The aim of this work is to investigate the properties of codimension one foliations on three-dimensional manifolds admitting Ehresmann connection.

We give a new proof of the following known statement.Emphasize that compactness of foliated manifolds is not assumed.

Theorem 1. *Let $(M, \mathcal{F})$ be a smooth foliation of codimention 1 on three-dimensional manifold M admitting an Ehresmann connection. Then $(M, \mathcal{F})$ does not contain the Reeb component.*

S. P. Novikov proved that any C^r-smooth, $r \geq 2$, foliation of codimention 1 on a closed three-dimensional manifold M with a finite fundamental group contains the Reeb component [2] (see also [3, §25, Theorem 6.5]). Using the above result and Theorem 1, we obtain the following statement.

Corollary 1. *There are no two-dimentional foliations with an Ehresmann connection on a closed three-dimensional manifold M with a finite fundamental group.*

Corollary 1 implies that for every C^r-smooth, $r \geq 2$, foliation $(\mathbb{S}^3, \mathcal{F})$ of codimention 1 on the standard three-dimensional sphere $\mathbb{S}^3$, there is no an Ehresmann connection.

References

[1] Blumenthal R.A., Hebda J.J. Ehresmann connections for foliations // Indiana Univ. Math. J. 1984. V. 33, no 4. P. 597–611.

[2] Novikov S.P. Topology of foliations // Trans. Moscow Math. Soc. 1965. V. 14. P. 268–305.

[3] Tamura I. Topology of Foliations, Amer. Math. Soc., Providence, RI, 1992.

[4] Bazaikin Y.V., Gavaev A.S., Zhukova N.I. Chaos in Cartan foliations // Chaos. 2020. V. 30, no. 10.

Structure of parameter planes for Hodgkin-Huxley type of model, bifurcation mechanism of bistability emergency

Bagautdinova E. R.[1,2], Stankevich N. V.[1]

1. HSE University

2. Saratov State University

It is well known that the electrical activity of the beta cells of the pancreas and other biological cells depends on a number of different types of voltage and ligand driven ion channels that are permeable to inorganic ions such as sodium, potassium, chlorine, and calcium. The dynamics of the electric potential of the cell membrane is described by the Hodgkin-Huxley formalism and can be represented by a system of ordinary differential equations with nonlinearity, which are characterized by a number of nonlinear effects. One of these effects is multistability, which is the coexistence of different modes of functioning of a cell model. Of all the types, of particular interest is the multistability between the resting state, and the burst state, which, with certain parameters, can be realized in the model proposed in [1].

The model is a system of three ordinary differential equations:

$$\tau\dot{V} = \leqslant -I_{Ca}(V) \leqslant -I_K(V,n) \leqslant -I_{K2}(V) \leqslant -I_S(V,S);$$
$$\tau\dot{n} = \sigma(n_\infty(V) - n);$$
$$\tau_S\dot{S} = S_\infty(V) - S.$$

where V represents membrane potential, n is potassium concentration, and S is calcium concentration. The functions $\leqslant I_{Ca}(V) = \leqslant g_{Ca} \leqslant m_\infty(V)(V - V_{Ca})$, $\leqslant I_K(V,n) = \leqslant g_K \leqslant n(V - V_K)$, $\leqslant I_S(V,n) = \leqslant g_S \leqslant S(V - V_K)$, define three current passing through the cell membrane, fast current created by calcium and potassium channels and slow from potassium ions. The equation for the current passing through the additional potassium channel with a non-monotonic characteristic is written as follows: $\leqslant I_{K2}(V) = \leqslant g_{K2} \leqslant p_\infty(V)(V - V_K)$ The function describing the opening of ion channels in the classical form has the following form: $\omega_\infty(V) = [1 + exp\frac{V_\omega - V}{\theta_\omega}]^{-1}$, $\omega = \leqslant m, \leqslant n, \leqslant S$. For an additional ion channel, this characteristic is non-monotonic: $\leqslant p_\infty(V) = [exp\frac{V - V_p}{\theta_p} + exp\frac{V_p - V}{\theta_p]}]^{-1}$. [2]

A feature of this modification is the appearance of a communication defect in the cell, which is achieved by taking into account an additional potassium ion channel, the probabilistic characteristic of the opening of which is non-monotonic.

In this work, the planes of the parameters of the original model (at gK2 = 0) are studied, and the mechanisms of the emergence of the burst attractor are studied. Maps of dynamic regimes on various planes of parameters for the system with modification, including on the plane of parameters responsible for the additional ion channel, where the bistability regions are localized, have been constructed and analyzed. A numerical bifurcation analysis has been carried out, as a result of which a bifurcation scenario for the emergence of bistability has been determined.

This work was supported by the Russian Science Foundation grant 20-71-10048.

References

[1] Sherman A., Rinzel J., Keizer J. 54(3) (1988) 411-425.

[2] Stankevich N.V., Mosekilde E. Chaos. 27(12) (2017) 123101

On Functional Equations for the Elliptic Dilogarithm

Bolbachan V.

Faculty of Mathematics

National research university higher school of economics

Let $E = \mathbb{C}/\langle 1, \tau\rangle$ be an elliptic curve over $\mathbb{C}$. The elliptic dilogarithm was defined by Spencer Bloch([1, Lemma 8.1.1]). The equivalent representation is given by the following formula:

$$D_\tau(\xi) = \sum_{n=-\infty}^{\infty} D(e^{2\pi i\xi + 2\pi i\tau n}).$$

This formula defines a real analytic function on E satisfying the following antisymmety relation:

$$D_\tau(\xi) + D_\tau(-\xi) = 0. \tag{1}$$

Now we want to formulate the so-called elliptic Bloch relations([1, Theorem 9.2.1]). Let f be a rational function on E of degree n such that

$$(f) = \sum_{i=1}^{n}([\alpha_i] - [\gamma_i]), (1-f) = \sum_{i=1}^{n}([\beta_i] - [\gamma_i]).$$

Then the following relation for D_τ holds:

$$\sum_{i,j=1}^{n}\left(D_\tau\left(\alpha_i - \beta_j\right) + D_\tau\left(\beta_i - \gamma_j\right) + D_\tau\left(\gamma_i - \alpha_j\right)\right) = 0. \tag{2}$$

Consider the following geometric example(see [2], Lemma 3.29). Let us realize the elliptic curve E as a cubic plane curve in $\mathbb{P}^2$ and consider three different lines $l, m, n \subset \mathbb{P}^2(\mathbb{C})$ intersecting at a point in the complement of E. Let h_l, h_m, h_n be homogeneous equations of these lines such that $h_m = h_n + h_l$. Denote by $\alpha_i, \beta_i, \gamma_i, i = 1, 2, 3$ the intersection points of the lines l, n, m with E. It is easy to see that the relation (2) for the function h_l/h_m takes the following form:

$$\sum_{i,j=1}^{3}\left(D_\tau\left(\alpha_i - \beta_j\right) + D_\tau\left(\beta_i - \gamma_j\right) + D_\tau\left(\gamma_i - \alpha_j\right)\right). \tag{3}$$

Our main result is the following statement: For any rational function f on E the relation (2) can be represented as a linear combination with integer coefficient of relations of the form (1) and (3).

This statement is a positive solution of Conjecture 3.30 from [2]. Denote by K the field of rational functions on E. The definition of the pre-Bloch group can be found in [3]. We recall that *degree* of a non-zero rational function $f \in K$ is the number of zeros of f counting with multiplicities. (In particular, degree of any non-zero constant is equal to 0.) The previous theorem is an easy consequence of the following statement:

Let E be an elliptic curve over algebraically closed field of characteristic 0. The group $B_2(K)$ is generated by elements of the form $[f]$ with $\deg f \leq 3$.

References

[1] Bloch, S.: Higher Regulators, Algebraic K-theory, and Zeta Functions of Elliptic Curves, *CRM monograph series*, vol. 11. American Mathematical Society, Providence (2000)

[2] Goncharov, A.B., Levin, A.M.: Zagier's conjecture on $L(E, 2)$. Inventiones mathematicae **132**(2), 393–432 (1998)

[3] Suslin, A.A.: K_3 of a field and the Bloch group. Proceedings of the Steklov Institute of Mathematics **183**, 217–239 (1991)

Bifurcation analysis of the 1D Lorenz map for various separatrix values.

Voevodkin V. S.[1]

1. Department of Fundamental Mathematics 1
National Research University Higher School of Economics 1

In this work we present some results of two-parameter bifurcations analysis for the well-known 1D Lorenz map:

$$X_{n+1} = (-\mu + A|X_n|^{\nu})\text{sign}(X_n). \tag{1}$$

Here μ is a separatrix splitting parameter, ν is a saddle index of the equilibrium ($\nu = \lambda/\gamma$, where γ and $-\lambda$ are the nearest to the origin unstable and stable eigenvalues of the saddle), and A is the separatrix value.

Recall that $\mu = 0$ corresponds to the homoclinic butterfly bifurcation (when a pair of homoclinic loops of the saddle equilibrium exists). Also note that we call the saddle equilibrium is neutral when $\nu = 1$.

In accordance with the Shilnikov criterion [1] when $\mu = 0, \nu = 1$, and $0 < |A| < 2$ there exists an open region in the parameter space with a Lorenz attractor of the Afraimovich-Bykov-Shilnikov model [2, 3] and the line $\mu = 0, \nu = 1$ belongs to its boundary.

In this work, with help on analytical and numerical bifurcation analysys, we show that, depending on the separatrix value A, there exist three different cases of the birth of region with chaotic dynamics in the (μ, ν)- parameter plane. Namely:

1. If $0 < |A| < 1$ then only the region with Lorenz attractor originates from the point $(\mu, \nu) = (0, 1)$;
2. If $1 < |A| < 2$ then both regions with the Lorenz and Rovella (contracting Lorenz) attractors originates from the point $(\mu, \nu) = (0, 1)$;
3. When $|A| > 2$ only the region with the Rovella attractor originates from the point $(\mu, \nu) = (0, 1)$.

Acknowledgment The author thanks Alexey Kazakov for setting the problem and fruitful discussion.

References

[1] Shilnikov, L. P. "The bifurcation theory and quasi-hyperbolic attractors." Uspehi Mat. Nauk 36 (1981): 240-241.

[2] Afraimovich, V. S., Bykov, V. V. and Shilnikov, L. P. "On the origin and structure of the Lorenz attractor." Akademiia Nauk SSSR Doklady. Vol. 234. 1977.

[3] Afraimovich, V. S., Bykov, V. V., Shilnikov, L. P. (1982). Attractive nonrough limit sets of Lorenz-attractor type. Trudy Moskovskoe Matematicheskoe Obshchestvo, 44, 150-212.

Estimation of the average time profit for a probabilistic model of population dynamics

Egorova A. V.

Department of Functional Analysis and Its Applications
Vladimir State University named after Alexander and Nikolai Stoletovs

Currently, the problems of optimal collection of renewable resources in probabilistic models are of great interest. For example, the paper [1], models of collecting a renewable resource described by differential equations with impulse effects are considered; the method of extracting such a resource in the long term, in which a part of the population is preserved, is determined, and the function of the average time profit is estimated.

In this abstract, we investigate a model of population dynamics, which is determined by difference equations that depend on random parameters, and also solve the problem of choosing a control.

Suppose that the development of a population in the absence of exploitation is described by the equation

$$x(k+1) = f\big(x(k)\big), \quad k = 0, 1, 2, \dots, \tag{1}$$

where $x(k)$ — population size or resource quantity at a given time k, $f(x)$ — a real non-negative function given for all $x \in R_+ \doteq [0, +\infty)$, that satisfies the condition $f(0) = 0$ and $f \in C^2(R_+)$. We also consider the functions $f \in C^2(I)$, where $I = [0, a]$, such that $f(0) = 0$ and $f(I) \subseteq I$.

Let us assume that at the moment k some random share of the resource $\omega(k) \in \Omega \subseteq [0, 1]$ is extracted from the population. We assume that it is possible to influence the harvesting process in such a way as to stop harvesting in the event that its share turns out to be large enough and to preserve the largest possible remainder of the population. Then, the share of the extracted resource will be equal to $\ell(k) = \ell(\omega(k), u(k)) = \min\big\{\omega(k), u(k)\big\}$. In this case, the model of the exploited homogeneous population has the form $x(k+1) = f\big((1 - \ell(\omega(k), u(k)))x(k)\big)$, where $\big(x(k), \omega(k), u(k)\big) \in R_+ \times \Omega \times [0, 1]$.

If the equation (1) has a solution of the form $x(k) \equiv const = x^*$, then this solution is called *the equilibrium position* of the given equation. The set of initial points $x(0)$, for which $\lim\limits_{k\to\infty} \big\|x(k) - x^*\big\| = 0$, is called *the set of attraction of the point* x^*, we will denote it $S(x^*)$.

Consider the probability model $(\Sigma, \mathfrak{A}, \mu)$, described in [1]. Here $\Sigma \doteq \{\sigma : \sigma = (\omega(1), \dots, \omega(k), \dots)\}$. Assume that a probability space is given $(\Omega, \widetilde{\mathfrak{A}}, \widetilde{\mu})$, where $\widetilde{\mathfrak{A}}$ — the sigma algebra of subsets $\Omega \subseteq [0, 1]$ on which the probability measure $\widetilde{\mu}$ is defined.

Let us denote $U \doteq \{\overline{u} : \overline{u} = (u(1), \dots, u(k), \dots)\}$, $L(\sigma, \overline{u}) \doteq \big(\ell(\omega(1), u(1)), \dots, \ell(\omega(k), u(k)), \dots\big)$, $X(k) = X\big(\ell(1), \dots, \ell(k-1), x(0)\big)$ — the amount of the resource before harvesting the time k, depending on the share of the resource $\ell(j)$, $j = 1, \dots, k-1$, the population size harvested at previous points in time and the initial population size value $x(0)$. For all $x(0) \geqslant 0$ consider the function

$$H_*\big(L(\sigma, \overline{u}), x(0)\big) \doteq \varliminf_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} X\big(\ell(1), \dots, \ell(k-1), x(0)\big)\ell(k),$$

that we call *the average time profit* from resource extraction.

For any $x \in R_+$, we introduce a random variable $\ell(\omega, u(x)) = \min\big\{\omega, u(x)\big\}$ and denote by $M\ell(\omega, u(x))$ its mathematical expectation.

Let be (b_1, b_2) — maximum interval containing a point x^*, for all points of which the inequality is satisfied $0 < f'(x) < 1$; (c_1, c_2) — maximum interval containing a point x^*, for all points of which $-1 < f'(x^*) < 0$.

Theorem 1. *Let the equation (1) has an equilibrium position $x^* > 0$, $0 < f'(x^*) < 1$. Then for any $x(0) \in S(x^*)$, $\widetilde{x} \in [b_1, x^*]$ exist a control $\overline{u}(\widetilde{x}) \in U$ such that for almost everyone $\sigma \in \Sigma$ the inequalities are satisfied*

$$f(\widetilde{x})M\ell(\omega, u(\widetilde{x})) \leq H_*(L, x(0)) \leq x^* M\ell(\omega, u(\widetilde{x}));$$

where $u(\widetilde{x}) = 1 - \frac{\widetilde{x}}{f(\widetilde{x})}$.

Theorem 2. *Let the equation (1) has an equilibrium position $x^* > 0$ and $-1 < f'(x^*) < 0$. Then for any $x(0) \in S(x^*)$, $\widetilde{x} \in [c_1, x^*]$ exist a control $\overline{u}(\widetilde{x}) \in U$ such that for almost everyone $\sigma \in \Sigma$ the inequalities are satisfied*

$$f(f(\widetilde{x}))M\ell(\omega, u_1(\widetilde{x})) \leq H_*(L, x(0)) \leq f(\widetilde{x})M\ell(\omega, u_2(\widetilde{x}));$$

where $u_1(\widetilde{x}) = 1 - \frac{\widetilde{x}}{f(f(\widetilde{x}))}$, $u_2(\widetilde{x}) = 1 - \frac{\widetilde{x}}{f(\widetilde{x})}$.

References

[1] L. I. Rodina. Optimization of average time profit for a probability model of the population subject to a craft, Vestn. Udmurtsk. Univ. Mat. Mekh. Komp. Nauki, 2018, V. 28 (1), PP. 48-58 (in Russian).

Necessary and sufficient conditions of topological conjugacy rough circle's transformations product

Golikova I. V., Pochinka O. V.

Department of Mathematics, Informatics and Computer Sciences
Chair of Fundamental Mathematics 1
National Research University Higher School of Economics 1

The results were obtained in collaboration with O.V. Pochinka and are devoted to the topological classification of rough circles transformations product.

As A. G. Mayer showed in [1], the topological conjugacy class of the orientation-preserving rough circle transformation is uniquely determined by the parameters n, k, l, where k is the period of periodic points, $2n$ is the number of periodic orbits and $\frac{l}{k}$ is the rotation number of the transformation. Thus, any such diffeomorphism is topologically conjugate to some model transformation $\phi_{n,k,l} : \mathbb{S}^1 \to \mathbb{S}^1$. Accordingly, rough circles transformations product is a gradient-like diffeomorphism on a two-dimensional torus, which are topologically conjugated to a model transformation $\phi_{n_1,k_1,l_1} \times \phi_{n_2,k_2,l_2} : \mathbb{T}^2 \to \mathbb{T}^2$. Each model diffeomorphism has $4n_1n_2k_1k_2$ periodic points, and their period is equal to $q = HOK(k_1, k_2)$.

The main result of the work is the proof of the following theorem.

Theorem. *Diffeomorphisms* $\phi_{n_1,k_1,l_1} \times \phi_{n_2,k_2,l_2}$, $\phi_{n'_1,k'_1,l'_1} \times \phi_{n'_2,k'_2,l'_2} : \mathbb{T}^2 \to \mathbb{T}^2$ *are topologically conjugate if and only if* $n_1k_1 = n'_1k'_1$, $n_2k_2 = n'_2k'_2$ *and* $q = q'$.

Thus, the rotation numbers of the Cartesian products of circles rough transformations are not topological invariants. The proof of the classification theorem relies heavily on the results of work [2] in which it is established that the complete topological invariant of the gradient-like diffeomorphism of a surface is a tri-colored graph. When numerical equalities are performed in the theorem above, the isomorphism of graphs is constructed by methods of work [3], where the classification of rough circles transformations product is obtained to the precision of a conjugacy by linear transformation.

Acknowledgements. The author is partially supported by Laboratory of Dynamical Systems and Applications NRU HSE, of the Ministry of science and higher education of the RF grant ag. № 075-15-2019-1931.

References

[1] Mayer, A. G. (1939). Rough transformation of a circle into a circle. *Scientific notes GSU*, 12, 215–229 (In Russ.).

[2] Grines, V. Z., Kapkaeva S. K., Pochinka O. V. (2014). A three-colored graph as a complete topological invariant for gradient-like diffeomorphisms of surfaces. *Sbornik: Mathematics*, 10, p. 20-46 (In Russ.).

[3] Kurenkov, E. D., Ryazanova K. A. (2017). On periodic translations on n-torus. *Dinamicheskie Sistemy*, 7(35), p. 113-118.

Investigation of the stability of the Hill equation in critical cases

Isaeva K.S.

Department of Mathematics and Information Technology

Bashkir State University

We consider the Hill equation of the form, depending on the parameters α and ε

$$x'' + (\alpha + \varepsilon\varphi(t))x = 0, \tag{1}$$

wherein $\alpha \geqslant 0$, but ε – small parameter. Function $\varphi(t)$ continuously depends on t and is T -periodic, that is $\varphi(t+T) \equiv \varphi(t)$.

it is known that (see for example [1]) in the problem of studying the stability of the equation (1) critical are the parameter values α:

$$\alpha_k = \left(\frac{\pi k}{T}\right)^2, \qquad (k = 0, 1, 2, 3, \ldots).$$

In this paper, we propose new formulas for the first approximation (in powers of the small parameter ε) for the multipliers of the equation (1), performance-based [2]. These formulas, in particular, allow one to analyze stability of the equation (1).

Using standard variable substitution$x_1 = x$, $x_2 = x'$ we reduce the equation to the system (1) to the system

$$\left.\begin{array}{rcl} x_1' &=& x_2\,, \\ x_2' &=& -(\alpha + \varepsilon\varphi(t))x_1 \end{array}\right\},$$

i.e to a system of the form

$$\frac{dx}{dt} = J[A_0(\alpha) + S(t,\varepsilon)]x\,, \quad x \in R^2\,, \tag{2}$$

wherein

$$J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \qquad A_0(\alpha) = \begin{bmatrix} \alpha & 0 \\ 0 & 1 \end{bmatrix}, \qquad S(t,\varepsilon) = \varepsilon\varphi(t)\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

We will confine ourselves to considering the case $\alpha_0 = 0$. In this case, we will consider a more particular problem, namely, we will study the question of the stability of the equation (1) for values (α, ε), ocated on a straight line $\varepsilon = m\alpha$, where m – auxiliary parameter.

Substituting equality $\varepsilon = m\alpha$ to (2), we get the equation:

$$\frac{dx}{dt} = J[A_0(\alpha) + S(t, m\alpha)]x\,, \quad x \in R^2\,, \tag{3}$$

wherein m - fixed, and α plays the role of a small parameter.

Equation(3) represent in the form:

$$\frac{dx}{dt} = J[A_0 + \alpha\tilde{S}(t,m)]x\,, \quad x \in R^2\,, \tag{4}$$

where

$$A_0 = A_0(0) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad \tilde{S}(t,m) = (1 + m\varphi(t))D_0\,, \quad D_0 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

Matrix $JA_0 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ has a nonsemisimple zero eigenvalue $\lambda_0 = 0$ multiplicity 2. Monodromy matrix $V_0 = e^{TJA_0}$ "unperturbed" (i.e for $\alpha = 0$) system (4) has nonsemisimple eigenvalue $\mu_0 = 1$ multiplicity 2. Then for small $|\alpha|$ monodromy matrix $V(\alpha)$ "disturbed" system (4) has a pair of eigenvalues $\mu_1(\varepsilon)$ и $\mu_2(\varepsilon)$ such that functions $\mu_1(\varepsilon)$ и $\mu_2(\varepsilon)$ are continuous, moreover $\mu_1(0) = \mu_2(0) = \mu_0$. furthermore, they can be represented as a decomposition Puisier:

$$\mu_1(\varepsilon) = 1 + \mu_1^{(1)}\varepsilon^{1/2} + O(\varepsilon)\,, \quad \mu_2(\varepsilon) = 1 + \mu_2^{(1)}\varepsilon^{1/2} + O(\varepsilon)\,. \tag{5}$$

Assuming that

$$\varphi_0 = \int_0^T \varphi(t)dt.$$

Theorem 2. *Coefficient* $\mu_1^{(1)}$ *и* $\mu_2^{(1)}$ *in decomposition* (5) *– is numbers:*

$$\mu_1^{(1)}(m) = \sqrt{-T(1 + m\varphi_0)}\,, \quad \mu_2^{(1)}(m) = -\mu_1^{(1)}(m)\,. \tag{6}$$

Theorem 3. *Let* $\varepsilon(1 + m\varphi_0) > 0$ *(*$\varepsilon(1 + m\varphi_0) < 0$*). Then for a given perturbation* $\varphi(t)$ *with corresponding small* $|\varepsilon|$ *the equation* (1) *is stable (unstable).*

References

[1] Yakubovich V. A., Starzhinsky V. M. *Linear differential equations with periodic coefficients and their applications.* (The science, M., 1972).

[2] Yumagulov M. G., Ibragimova L. S., Belova A. S. *Methods for studying the stability of linear periodic systems, depending on a small parameter.* // Results of Science and Techn. Ser. Modern. math. and its app. Thematic review. T. **163**. 2019. S. 113–126.

Realization of Fomenko-Zieschang invariants by integrable billiard books

Kharcheva I. S.[1]

1. Department of differential geometry and applications
Lomonosov Moscow State University

Let us consider a free motion of a particle in some fixed domain Ω in a plane with elastic reflection at the boundary $P = \partial\Omega$. We obtain a Hamiltonian dynamical system with a Hamiltonian that equals to the scalar square of the velocity vector. Such dynamical systems and their generalizations usually are called billiards

If the domain's boundary P is a piecewise curve and consist of several arcs of confocal ellipses and hyperbolas then the billiard has a following special property: the straight lines containing the segments of the billiard trajectory are tangents to a certain quadric (ellipse or hyperbola). The parameter of this quadric is the value of the additional integral Λ (see [1]). Thus this billiard is integrable and called an elementary billiard.

A billiard book is a generalization of elementary billiards obtained by glueing them along the boundaries. A billiard book is still an integrable Hamiltonian system on a piecewise smooth phase space (see [2]).

Integrable Hamiltonian systems can be classified by topological invariants: 3-atoms, f-graphs, coarse and marked molecules (see [3]). Such invariants allow us to speak about Liouville equivalence of different dynamical systems.

Researching billiard books we try to realize well-known classical dynamical systems in terms of topological invariants (see, for example, [4]). So V. V. Vedyushkina and I. S. Kharcheva found an algorithm that constructs a billiard book which realizes 3-atoms and f-graphs. This algorithm can be extended to another one that realizes any coarse molecule. Details of this result will be presented.

The work was done at Moscow State University under the support of the Russian Science Foundation (project no. 17-11-01303).

References

[1] *V. V. Kozlov, D. V. Treshchev*, Billiards. A genetic introduction to the dynamics of systems with impacts. // Transl. Math. Monogr., 89, Amer. Math. Soc., Providence, RI, 1991, viii+171 pp.

[2] *I. S. Kharcheva*, Isoenergetic manifolds of integrable billiard books, Moscow University Mathematics Bulletin, 2020, Vol. 75, No. 4, pp. 149?160.

[3] *A. V. Bolsinov, A. T. Fomenko*, Integrable Hamiltonian systems. Geometry, topology, classification. // Chapman & Hall/CRC, Boca Raton (2004).

[4] *V. V. Vedyushkina* and *I. S. Kharcheva*, Billiard books model all three-dimensional bifurcations of integrable Hamiltonian systems. // Sbornik: Mathematics, 209(12), 1690-1727.

Kovalevskaya system in pseudo-Euclidean space: compactness and non-compactness of common level 2-surfaces

Kibkalo V. A.[1,2]

*1. Faculty of Mechanics and Mathematics
Lomonosov Moscow State University*

2. Moscow Center for Fundamental and Applied Mathematics

1. A.V. Borisov and I.S. Mamaev considered [1] class of analogs of well-known integrable Hamiltonian systems of rigid body dynamics for the case of motion in a pseudo-Euclidean space.

Recall that a Kovalevskaya integrable system can be defined on Lie algebra $e(3)$, i.e. a linear space $\mathbf{R}^6(x_1, x_2, x_3, J_1, J_2, J_3)$ with Lie–Poisson bracket $\{J_i, J_j\} = c_{ijk}J_k, \{J_i, x_j\} = c_{ijk}x_k, \{x_i, x_j\} = 0$ for $c_{ijk} = \operatorname{sgn}(123) \to (ijk)$.

Complex mapping $\hat{J}_j = i \cdot J_j \,/\, k, \hat{x}_j = i \cdot x_j \,/\, k,\ j = 1, 2, 3$ transform this real-analytic system to a system on Lie algebra $e(2,1)$. Separation of variables for new system was constructed by S.V. Sokolov [2].

Two Casimir functions (geometric and area integrals f_1 and f_2 respectively), the Hamiltonian H and additional first integral F are the following

$$f_1 = x_1^2 + x_2^2 - x_3^2 = a,$$

$$f_2 = x_1J_1 + x_2J_2 - x_3J_3 = b,$$

$$H = \frac{1}{2}\left(J_1^2 + J_2^2 - 2J_3^2\right) - b_1x_1 = h,$$

$$F = \frac{1}{4}\left(J_1^2 - J_2^2 + 2b_1x_1\right)^2 + \frac{1}{4}\left(2J_1J_2\right)^2 = f.$$

Integral F is the same as in Euclidean case and f_1, f_2, H have another sign of the terms x_3^2, x_3J_3 and $2J_3^2$ respectively.

2. According to A.T. Fomenko school approach [3] we describe phase topology of this system and will look for and calculate new topological invariants of it. First problem is to describe common level surface $T_{a,b,h,f}$ of four first integrals f_1, f_2, H, F.

$$T_{a,b,h,f} = \{x \in R^6 | f_1 = a, f_2 = b, H = h, F = f\}.$$

In the talk **criterion of the compactness** of a common level surface $T_{a,b,h,f}$ is formulated. Problem is not trivial because quadratic forms in f_1 and H are not positive and thus $|\vec{x}|^2 + |\vec{J}|^2$ are not bounded by simple formula on a, h (in the Euclidean case it guaranteed compactness of such level).

Theorem (V. Kibkalo). *Let $f_2 = b \neq 0$. Then for $f \geq h^2$ the common level surface $T_{a,b,h,f}$ has at least one unbounded connected component. For $0 \leq f < h^2$ and every b the surface $T_{a,b,h,f}$ is bounded and thus compact.*

Image of critical points set under the momentum map (H, F) of Kovalevskaya systems (both non-Euclidean and Euclidean) are mapped on the same bifurcation curves (may be on different arcs of them) in Ohk plane as functions on a, b, h, f. For $b \neq 0$ these curves do not coincide with the parabola $f = h^2$ discussed above.

Remark. Thus the bifurcation from a compact level surfaces (Liouville torus or tori) to a non-compact one does not "usually" contain critical points of the momentum map of this system. This 3-singularity (i.e. foliated preimage of a small vertical interval intersecting parabola $F = H^2$ in a point $f > 0$) belongs to the class of "noncompact" singularities discussed in [5]. They are much more complicated for detection because does not relate to the *fall of the rank of the momentum map* $(\mathrm{d}\,H, \mathrm{d}\,F)$. We show that in Kovalevskaya system this effect is related to *fall of the degree of some polynomial.*

References

[1] A.V. Borisov, I.S. Mamaev, “Rigid body dynamics in non-Euclidean spaces” // Rus. J. of Math. Phys., **23**:4 (2016), 431–454.

[2] S.V. Sokolov, “The integrable case of Kovalevskaya in a non-Euclidean space: separation of variables” // Tr. MAI., **100** (2018), 1–13.

[3] A.T. Fomenko, H. Zieschang, “A topological invariant and a criterion for the equivalence of integrable Hamiltonian systems with two degrees of freedom” // Math. USSR Izv., **36** (1991), 567–596.

[4] V.A. Kibkalo “Noncompactness property of fibers and singularities of non-Euclidean Kovalevskaya system on pencil of Lie algebras” // Moscow Univ. Math. Bull., **75**:6 (2020), 263–267.

[5] D.A. Fedoseev, A.T. Fomenko, “Noncompact bifurcations of integrable dynamic systems” // Fundam. Prikl. Mat., **21**:6 (2016), 217–243.

Astrocyte-induced intermittent synchronization in the ring of inhibitors neurons

Makovkin S.Yu.[1]**, Gordleeva S. Yu**[1,2]**, Ivanchenko M.V.**[1]

1. Department of Applied Mathematics, Institute of Information Technology, Mathematics and Mechanics
Lobachevsky University, Nizhny Novgorod, Russia

2. Neuroscience and Cognitive Technology Laboratory, Center for Technologies in Robotics and Mechatronics Components
Innopolis University, Innopolis, Russia

We study a model of a neuron-astrocyte ensemble that consists of two hundred pairs of reciprocal neurons and astrocytes. Dynamics of intracellular Ca^{2+} concentration in astrocyte is described by the Ullah-Jung model [1] and the membrane potential of a neuron evolves according to the Hodgkin-Huxley model [2] with Mainen modification for mammals brain [3], [4].

The dynamics of membrane potential is described by the HH model [2]. Nonlinear functions for gating variables α_x and β_x, are taken as the follows [3]. The concentration of neurotransmitter that was released from the synapse of the (i)-neuron, can be expressed as follows [5]. The state variables of the i-th astrocyte include Ca^{2+} intracellular concentration, Ca_i, the fraction of activated $IP3$ receptors on the endoplasmic reticulum, z_i, and intracellular concentration of inositol 1,4,5-trisphosphate $IP3$, $IP3_i$. They evolve according to the following equations [1]. Biophysical interpretation of all nonlinear functions and parameters and their values that are determined experimentally can be found in [1]. Calcium elevations in astrocytes trigger the release of gliotransmitters, such as glutamate, GABA, ATP and D-serine. Gliotransmitter can modulate the synaptic strength by binding with pre- or postsynaptic terminals. Among the variety of experimental manifestations of different gliotransmitters [6], we focus on the effect of the astrocyte-induced enhancment of synaptic transmission. It can be described with the simple model proposed in [7]. Detailed biophysical description of the neuron-astrocyte interaction can be found in [8], [9].

We find that astrocytic modulation of neuronal synaptic connectivity can lead to the emergence of a special regime of intermittent synchronization between spiking neurons. In a generic system of coupled oscillators, intermittent synchronization is observed as an intermediate regime between the phase locking mode and asynchronous mode [10]. Typically, this regime is restricted to a narrow parameter interval, where the frequency of phase slips changes gradualy. Strikingly, we show that astrocytes can induce the intermittent synchronization of neurons in a wide range of system parameters, and its characteristics remain robust under variation of parameters.

The mechanism behind astrocyte-mediated intermittent synchronization is as follows. First, spiking neurons induce calcium oscillations in astrocytes, that originally resided in the steady state. Thus, a new slow time scale appears in the system, shaped by the astrocytic calcium dynamics. When intracellular Ca^{2+} concentration comes over a certain threshold, the synaptic strength between the neurons gets enhanced. Eventually, they become synchronized in anti-phase on the time interval equal to duration of calcium pulses in astrocyte. In absence of the astrocytic influence, the system stays in the beating regime, as prescribed by the frequency detuning between coupled oscillators.

In summary, we investigated the influence of the astrocytic modulation on the synchronization of a pair synaptically coupled spiking HH neuronal oscillators for excitatory and inhibitory synaptic connectivities. We showed that astrocytes do not noticeably modify the region of complete synchronization between spiking neurons, but induce an extended region of intermittent synchronization. The characteristics of intermittent synchronization, such as frequency and duration of synchronization intervals, are determined essentially by the slow calcium dynamics in astrocytes and do not dependent on neuronal frequency detuning or coupling [11].

References

[1] Ullah G., Jung P., Cornell-Bell A. H. (2006). Anti-phase calcium oscillations in astrocytes via inositol (1,4,5)-trisphosphate regeneration. Cell Calcium, 39(197–208).

[2] Hodgkin, A. L., Huxley, A. F. (1952). A quantitative description of membrane current and its application to conduction and excitation in nerve. The Journal of physiology, 117(4) 500-544.

[3] Mainen, Z. F., Joerges, J., Huguenard, J. R., Sejnowski, T. J. (1995). A model of spike initiation in neocortical pyramidal neurons. Neuron, 156(6), 1427-1439.

[4] Mainen, Z. F., Sejnowski, T. J. (1996). Influence of dendritic structure on firing pattern in model neocortical neurons. Nature, 382(6589), 363.

[5] Gordleeva, S. Y., Ermolaeva, A. V., Kastalskiy, I. A., Kazantsev, V. B. (2019). Astrocyte as spatiotemporal integrating detector of neuronal activity. Frontiers in physiology, 10, 294.

[6] Araque, A., Carmignoto, G., Haydon, P. G., Oliet, S. H., Robitaille, R., and Volterra, A. (2014). Gliotransmitters travel in time and space. Neuron, 81, 728-739.

[7] Kanakov, O., Gordleeva, S., Ermolaeva, A., Jalan, S., Zaikin, A. (2019). Astrocyte-induced positive integrated information in neuron-astrocyte ensembles. Physical Review E, 99(1), 012418.

[8] Gordleeva, S. Y., Lebedev, S. A., Rumyantseva, M. A., Kazantsev, V. B. (2018). Astrocyte as a detector of synchronous events of a neural network. JETP Letters, 107(7), 440-445.

[9] Pankratova, E. V., Kalyakulina, A. I., Stasenko, S. V., Gordleeva, S. Y., Lazarevich, I. A., Kazantsev, V. B. (2019). Neuronal synchronization enhanced by neuron–astrocyte interaction. Nonlinear Dynamics, 1-16.

[10] Pikovsky, A., Kurths, J., Rosenblum, M., Kurths, J. (2003). Synchronization: a universal concept in nonlinear sciences (Vol. 12). Cambridge university press.

[11] Makovkin, S. Y., Shkerin, I. V., Gordleeva, S. Y., Ivanchenko, M. V. (2020). Astrocyte-induced intermittent synchronization of neurons in a minimal network. Chaos, Solitons & Fractals, 138, 109951.

Inhibitory oscillators in the multiplex models of neuron-glial systems

Makovkin S.Yu.[1], **Lapteva T.**[2], **Ivanchenko M.V.**[1]

1. *Department of Applied Mathematics, Institute of Information Technology, Mathematics and Mechanics*
Lobachevsky University, Nizhny Novgorod, Russia

2. *Departament of Control Theory and Systems Dynamics, Institute of Information Technology, Mathematics and Mechanics*
Lobachevsky University, Nizhny Novgorod, Russia

We study a two-layer multiplex network of Kuramoto oscillators. One layer mimics an ensemble of “glial” (astrocyte) cells, which are low-frequency, locally coupled oscillators, the other layer mimics “neuronal” cells, which are fast-frequency oscillators with long-range random interactions. Both of these networks are of the same size $N \times N$ and are located one “above” the other. This is a qualitative, but realistic enough model of the cerebral cortex, reflecting the multi-timescale nature and multiplex topology of neuronal and glial connections.

The “glial" layer consists of ordered elements combined into a lattice, where each element has a connection to its eight neighbors (bottom, right, top, left, bottom-right, bottom-left, top-right, top-left). In addition, each of the elements of the “glial” layer together with each of its eight neighbors are coupled to the corresponding “mirror” element of the “neuronal” layer.

The dynamical evolution of the nodes in this multiplex network is given by bidirectionally coupled phase oscillators. The degree of synchronization in the network is characterized in two ways: by Kuramoto order parameter and standard deviation of time-averaged oscillator frequencies.

Since astrocyte-astrocyte and astrocyte-neural interactions are based on calcium signaling, the respective coupling strengths in normal conditions are not independent. For simplicity we assume that the coupling strengths in the “glial” layer and between the different layers is identical.

When neural and glial layers are coupled via biological behaviour: $\sigma_G = \sigma_{GN}$. Synchronization and desynchronization effects were shown in paper [1].

Here we concluded several points about synchronization case:

- Mean field in glial layer is born with the interaction of neural layer.
- There is particle desynchronization in glial and neuron layers.
- There is abrupt transition to synchronization.

Another important observation is that when there are no random connections in the “neural” subnetwork (when the “neural” network has regular and periodic topology like the “glial” subnetwork), multistable solutions of the ρ_N parameter are occur. If we increase the level of random connections in the neural subnetwork by 10% (when the $p_{rewiring}$ parameter increases from 0.0 to 0.1), instantly extract multistable solutions and there is only a stationary solution for the ρ_N parameter.

The main features of synchronization in the multiplex network with completely random “neural” layer are as follows:

1. The “glial” lattice can develop mean-field synchonization due to coupling to the “neural” layer, where such regime is possible on its own;
2. The road to synchronization in the “neural” layer is non-monotonous, and includes partial desynchronization due to interaction with the “glial” layer and a sharp transition to synchronization afterwards.

Exploring the transition by rewiring the "neural" layer from the regular lattice to completely random connectivity, we find the following:

3. When the "neural" layer is taken a periodic lattice, $p = 0$, a bi-stability of synchronization is observed: while the average frequencies are synchronized, different spatial patterns of phases may develop, and the order parameter can vary from $\rho \sim 1$ to $\rho \sim 0.3 \div 0.5$;

4. The rate of random rewiring characteristic of the small-world transition in the "neural" layer, $p \sim 0.1$ is sufficient to eliminate this bistability and stabilize $\rho \sim 1$ in the synchronization regime, like in the case of completely random "neural" layer.

Adding the inhibitory elements with $A_n = -1$ in neuronal subnetwork, mimicking inhibitory neurons get another Kuramoto mean field distribution and frequency dispersion. p_{rew} parameter is fixed here at point 0.3. Increasing inhibitory level decrease the level of common synchronization in ρ_N data, but increases the frequency spread suddenly. Increasing frequency dispersion can be seen after fixing system parameters as $p_{rew} = 0.3$, $p_{inh} = 0.2$.

Further, we assign the reverse signature to outcoming links from some nodes in the "neural" layer, $A_n = -1$, mimicking inhibitory neurons. The ratio of such nodes is a control parameter.

The effects of "inhibitory" oscillators is as follows:

5. Increasing the number of inhibitory elements in the "neuronal" layer leads to a substantial drop in the level of both global synchronization in the system and in the "neuronal" layer itself.

6. Increasing the inhibition level indirectly affects synchronization in the "glial" layer, shifting the synchronization boundary.

.

References

[1] Makovkin, S., Kumar, A., Zaikin, A., Jalan, S., Ivanchenko, M. (2017). Multiplexing topologies and time scales: The gains and losses of synchrony. *Physical Review E*, **96**(5), 052214.

On perfect basic sets of surface A-diffeomorphisms

Mints D. I.[1]

1. International Laboratory of Dynamical Systems and Applications
HSE University, Russian Federation

Let M^2 be a closed surface.

A nontrivial basic set Ω of an A-diffeomorphism $f : M^2 \to M^2$ is said to be perfect if its complement $M^2 \backslash \Omega$ consists of a finite number of domains, each of which is homeomorphic to a disk. According to the results of [1], [2], perfect basic set of a surface A-diffeomorphism is one-dimensional connected set which is either attractor or repeller and has local structure of the direct product of interval and Cantor set. A-diffeomorphisms such that their non-wandering set contains perfect basic set naturally arise as a result of generalized S. Smale's surgery on generalized pseudo-Anosov diffeomorphisms of an arbitrary surface (this surgery was announced in [3]). Specifically it leads to the appearance of a structurally stable diffeomorphism of the same surface with non-wandering set consisting of one perfect basic set (attractor) and a finite number of source periodic points. It is worth noting that recently, in [1], it was obtained a topological classification of A-diffeomorphisms of orientable surfaces of genus greater than one, non-wandering set of which consists of a one-dimensional widely situated (this definition is given below) attractor and zero-dimensional sources.

The following theorem is the main result of the given report.

Theorem 4. *Let $f : M^2 \to M^2$ be a structurally stable diffeomorphism all trivial basic sets of which are source periodic points $\alpha_1, ..., \alpha_k$, where $k \geq 1$. Then the non-wandering set $NW(f)$ of the diffeomorphism f consists of points $\alpha_1, ..., \alpha_k$ and exactly one perfect one-dimensional attractor Λ.*

Let us note that DA-diffeomorphisms of a two-dimensional torus, as well as diffeomorphisms of a two-dimensional sphere whose non-wandering set contains Plykin's attractor, are examples of diffeomorphisms satisfying the conditions of theorem 4.

A nontrivial basic set Ω of an A-diffeomorphism $f : M^2 \to M^2$ is said to be widely situated if there is no loop homotopic to zero formed by a segment of the stable manifold and a segment of the unstable manifold of any point $x \in \Omega$.

Let g be genus of surface M^2.

Corollary 2. *Let the conditions of theorem 4 be satisfied and, in addition, the attractor Λ has no bunches of degree one and $g \geq 1$ if surface M^2 is orientable, $g \geq 3$ if surface M^2 is nonorientable. Then the attractor Λ is widely situated on surface M^2.*

Corollary 3. *Let the conditions of theorem 4 be satisfied and, in addition, $k = 1$ (that is the point α_1 is fixed). Then surface M^2 has genus $g \geq 1$ if it is orientable, $g \geq 3$ if it is nonorientable, the attractor Λ is widely situated on surface M^2 and has no bunches of degree one.*

The results of this report were obtained in collaboration with V. Z. Grines.

This report was carried out with the financial support of the Russian Science Foundation (project 17-11-01041).

References

[1] V. Z. Grines, E. D. Kurenkov, "Diffeomorphisms of 2-manifolds with one-dimensional spaciously situated basic sets", Izv. RAN. Ser. Mat., 84:5 (2020), 40-97; Izv. Math., 84:5 (2020), 862-909.

[2] R. V. Plykin, ”The topology of basis sets for Smale diffeomorphisms”, Mat. Sb. (N.S.), 84(126):2 (1971), 301-312; Math. USSR-Sb., 13:2 (1971), 297-307.

[3] A. Yu. Zhirov, R. V. Plykin, ”On the relationship between one-dimensional hyperbolic attractors of surface diffeomorphisms and generalized pseudo-Anosov diffeomorphisms”, Mat. Zametki, 58:1 (1995), 149-152; Math. Notes, 58:1 (1995), 779-781.

Investigation of the synchronization of Hodgkin-Huxley neurons through the influence of synaptic communication

Pushkarev D. S., Makovkin S.Yu.[1], Gordleeva S. Yu[1,2], Ivanchenko M.V.[1]

1. Department of Applied Mathematics, Institute of Information Technology, Mathematics and Mechanics
Lobachevsky University, Nizhny Novgorod, Russia

2. Neuroscience and Cognitive Technology Laboratory, Center for Technologies in Robotics and Mechatronics Components
Innopolis University, Innopolis, Russia

The objective of this work is to study the response of a Hodgkin-Huxley neuron [1] to the effect of an electric current on it at different values.

Let us investigate the system for how it will behave depending on the change in the I_{app} parameter.

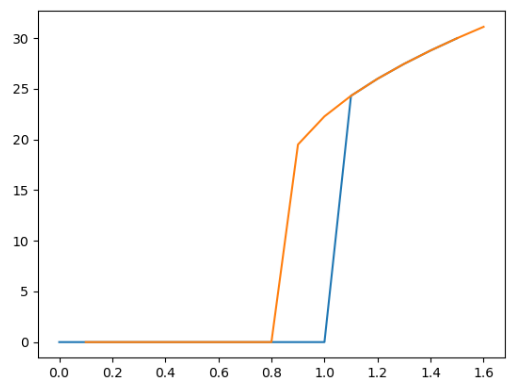

Figure 1. dependence of the average frequency w at I_{app} from 0 to 1.6.

Let's add a second neuron and connect it to the first.
Let's build a synchronization zone and see how the system will behave, depending on the different values of I_{app1}, I_{app2}, g_{syn}.

The fixed parameter is $I_{app1} = 1.2$, and I_{app2} will vary from 1.05 to 1.5 in 0.01 increments. For each such step, we will consider synchronization by changing the values of the synaptic binding strength g_{syn} from 0 to 0.2 with a step of 0.002. Synchronization is indicated by one on the graph (Figure 2), if there is no synchronization, then 0. [2]

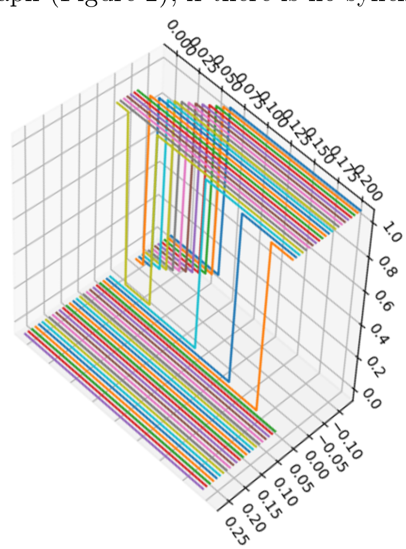

Figure 2. $E_{syn} = 0$ (excitatory synaptic communication)

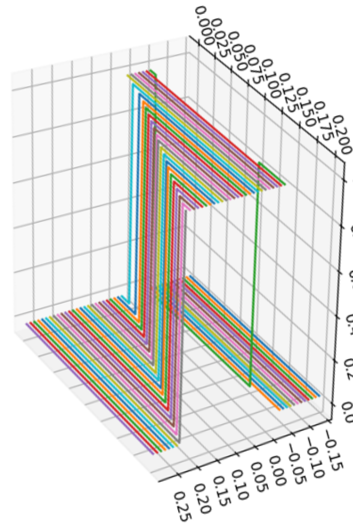

Figure 3. $E_{syn} = -90$ (fading synaptic connection)

References

[1] Hodgkin A. L., Huxley A. F. A quantitative description of membrane current and its application to conduction and excitation in nerve //The Journal of physiology. – 1952. – T. 117. – №. 4. – C. 500-544.

[2] Makovkin S. Y. et al. Astrocyte-induced intermittent synchronization of neurons in a minimal network //Chaos, Solitons & Fractals. – 2020. – T. 138. – C. 109951.

Topology of the Liouville foliation of an integrable magnetic billiard.

Pustovoitov S.E.

Department of differential geometry and applications
Moscow state university

Mathematical billiard is a dynamical system that describes the motion of a material point inside a closed bounded domain (billiard table). The material point moves along a smooth trajectory until it hits a boundary of the table and then reflects from it according to the usual reflection law. We know that the classical planar mathematical billiard, where a velocity vector of the point is constant, is a Hamiltonian system. The integrability of this billiard depends on shape of the table. For example, the billiard inside an ellipse or some domain bounded by confocal ellipses and hyperbolas is integrable, i.e. there exist a first integral that is functionally independent with the Hamiltonian. Such billiards were considered by V. V. Vedyushkina in [2].

Consider a billiard obtained from the classical planar billiard by adding a magnetic field forcing on a material point. Suppose the magnetic field is uniform in signature b and orthogonal to the billiard table. It terns out that if the table is bounded by an ellipse then there doesn't exist two functionally independent first integrals, i.e. system is not integrable anymore [3]. Also the general problem of the integrability of the magnetic billiard was investigated by M. Bialy and A. E. Mironov [4]. The following theorem guarantees the integrability of a circular billiard.

Theorem 1. *Consider a planar billiard inside a circle with orthogonal uniform magnetic field. A trajectory of the material point is piecewise smooth curve which consists of circular arcs (they are called the Larmor circles) of a common radius A and centers of these circles are equidistant from the center of the table at a distance R. Therefore there are two first integrals R and A, which are functionally independent.*

Theorem 2. *Pre-image of every regular value of the integrals R and A is homeomorphic to a torus in a phase space.*

That means one can treat the system using the Fomenko-Zieschang theory of marked molecules [1]. In particular, by fixing some values of the integral R one can obtain a marked molecule of a kind A–A with $r = \infty$ and $\epsilon = -1$, which was not observed for billiards before.

References

[1] A.V.Bolsinov, A.T.Fomenko. Integrable Hamiltonian systems. Geometry, topology, classification. Chapman & Hall/CRC, Boca Raton (2004).

[2] V. V. Fokicheva, "A topological classification of billiards in locally planar domains bounded by arcs of confocal quadrics", Sb. Math., 206:10 (2015), 1463–1507

[3] M. Robnik, M. V. Berry, "Classical billiards in magnetic fields", J. Phys. A, 18:9 (1985), 1361–1378.

[4] Misha Bialy, Andrey E Mironov, "Algebraic non-integrability of magnetic billiards", J. Phys. A: Math. Theor. 49 455101

Waves of COVID-19

Rassadin A.E.

National Research University Higher School of Economics, Nizhnii Novgorod
brat_ ras@list.ru

The sudden appearance of novel coronavirus COVID-19 gives rise to a series of investigations devoted to theory of epidemics (see [1] and references therein). The majority of these researchs deals with temporal dynamics only. But it is obvious that knowledge of spatio-temporal dynamics of disease gives one more satisfactory description of the pandemic spreading.

Let us consider the following Cauchy problem on the circle $\mathbf{S}^1$:

$$\frac{\partial u}{\partial t} + \omega\,\frac{\partial u}{\partial \varphi} = -\,u\,\ln u\,, \qquad u(\varphi,0) = u_0(\varphi)\,, \qquad u_0(\varphi + 2\,\pi) = u_0(\varphi)\,, \tag{1}$$

where $u(\varphi, t)$ is a density of infected on the circle $\mathbf{S}^1$, ω is angular velocity of disease rotation on configuration space, and right side of the quasilinear transport equation (1) corresponds to the Gompertz law [2].

It is easy to check that exact solution of problem (1) is equal to:

$$u(\varphi,t) = \exp[\,\exp(-\,t)\,\ln u_0(\varphi - \omega\,t)\,]. \tag{2}$$

If one choose initial condition for problem(1) as $u_0(\varphi) = \exp(-a + b\,\cos m\,\varphi)$ with $m \in \mathbf{N}$ and $0 < b < a$ then it is not difficult to find explicitly from formula (2) the next Fourier series for the density of infected:

$$u(\varphi,t) = U_0(t) + 2\,\sum_{n=1}^{\infty} U_n(t)\,\cos n\,m\,(\varphi - \omega\,t), \tag{3}$$

where amplitudes $U_n(t)$ of the Fourier spectrum for expansion (3) are expressed via the modified Bessel functions of the first kind as follows:

$$U_n(t) = \exp(-a\,\exp(-t))\,I_n(b\,\exp(-t))\,, \qquad n \in \mathbf{N}_0\,. \tag{4}$$

Graphs of functions (3) and (4) under $a = 1.5$, $b = 1.0$, $m = 3$ and $\omega = 7.0$ are shown on Figures 1 and 2 respectively.

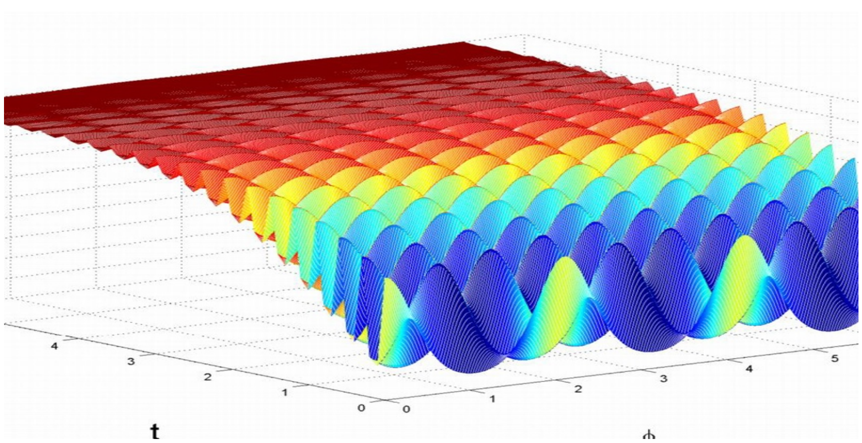

Figure 1: Spatio-temporal evolution of the density of infected

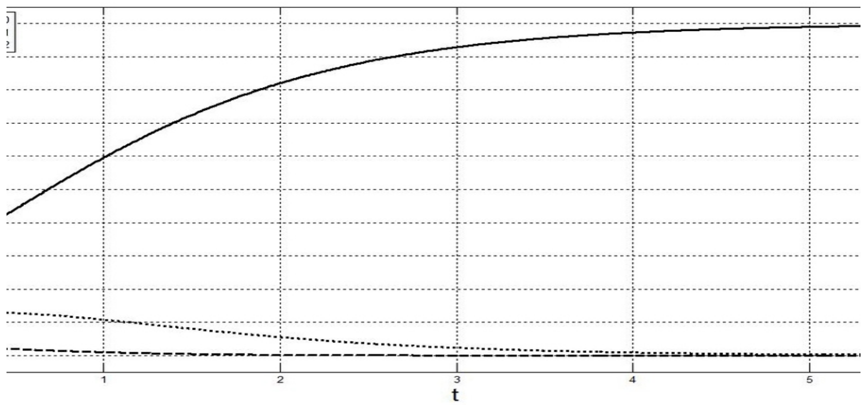

Figure 2: Temporal evolution of the first Fourier harmonics

References

[1] Pelinovsky E., Kurkin A., Kurkina O., Kokoulina M., Epifanova A. Logistic Equation and COVID-19 // Chaos, Solitons and Fractals: Nonlinear Science, and Nonequilibrium and Complex Phenomena. 2020. Vol. 140. P. 110241-1-13.

[2] Gompertz B. On the Nature of the Function Expressive of the Law of Human Mortality, and on a New Mode of Determining the Value of Life Contingensies // Philosophical Transactions of the Royal Society of London. 1825. Vol. 115. P. 513-585.

Stability of a linear nonautonomous periodic Hamiltonian system depending on a small parameter and construction of multipliers

Shafigullina D. A.

Department of Mathematic and Information Tehnology
Bashkir State University

Рассматривается линейная периодическая гамильтонова система, зависящая от малого параметра ε вида: We consider a linear periodic Hamiltonian system that depends on a small parameter *varepsilon* of the form:

$$x' = J(A_0 + \varepsilon S(t))x, x \in R^4, \tag{1}$$

где A_0 - постоянная симметрическая матрица, а $S(t)$ - симметрическая матрица, элементы которой являются непрерывными T-периодическими функциями (т.е. $S(t+T) \equiv S(t)$). Матрица J определена равенством: where A_0 is a constant symmetric matrix, and $S(t)$ is a symmetric matrix whose elements are continuous T -periodic functions (i.e., $S(t+T) \equiv S(t)$). The matrix J is defined by the equality:

$$J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix};$$

here 0 and I are, respectively, the zero and unit matrices of order 2.

It is assumed that the matrix JA_0 has a pair of multiples (multiplicities of 2) of purely imaginary eigenvalues *pmi*w_0, where $w_0 > 0$. The problem of the stability conditions of the system is studied (1) for small values of the parameter ε, as well as for the approximate construction of the multipliers of this system.

There are 2 possible cases: either the eigenvalues of *pmi*w_0 are semisimple, or they are non-semisimple.

Let us first consider case 1. Let the matrix JA_0 have the form:

$$JA_0 = \begin{pmatrix} 0 & w_0 & 0 & 0 \\ -w_0 & 0 & 0 & 0 \\ 0 & 0 & 0 & w_0 \\ 0 & 0 & -w_0 & 0 \end{pmatrix}$$

As eigenvectors corresponding to a multiple of the semisimple eigenvalue $\lambda = iw_0$ of the matrix JA_0, select vectors:

$$e = \frac{1}{2}\begin{pmatrix} -i \\ 1 \\ 1 \\ i \end{pmatrix}, \qquad g = \frac{1}{2}\begin{pmatrix} -i \\ 1 \\ -1 \\ -i \end{pmatrix}$$

Тогда $(iJe, e) = -1$ и $(iJg, g) = -1$. Отсюда и из [1] получим, что соответствующий мультипликатор $\mu_0 = e^{iw_0T}$ (кратности 2) системы (1) является индефинитным. Then $(iJe, e) = -1$ and $(iJg, g) = -1$. From this and from [1] we get that the corresponding multiplier *mu*$_0 = e^{iw_0T}$ (multiplicity 2) of the system (1) is indefinite.

According to [2], the multipliers of system (2) are represented as

$\mu_1(\varepsilon) = \mu_0 + \mu_1^{(1)}(\varepsilon) + O(\varepsilon^{3/2}), \mu_2(\varepsilon) = \mu_0 + \mu_1^{(2)}(\varepsilon) + O(\varepsilon^{3/2})$,

where $\mu_1^{(1)} = -i\mu_0\lambda_1, \mu_1^{(2)} = -i\mu_0\lambda_2$;

Here λ_1, λ_2 are the roots of the quadratic equation

$$\lambda^2 + (a-b)\lambda - ab + c\bar{c} = 0, \tag{2}$$

The coefficients in (2) are calculated by the formulas

$$a = (S_0 e, e), b = (S_0 g, g), c = (S_0 g, e);$$

Here $S_0 = \int_0^T S(t)\, dt$.

Denote by Δ the discriminate of equation (2).

Theorem 1: Let $Delta > 0 (< 0)$, then system (1) for small ε will be stable (unstable).

References

[1] Yakubovich V. A., Starzhinsky V. M. *Linear differential equations with periodic coefficients and their applications.* (Nauka, M., 1972).

[2] Юмагулов М. Г., Ибрагимова Л. С., Белова А. С. *Методы исследования устойчивости линейных периодических систем, зависящих от малого параметра.* // Итоги науки и техн. Сер. Соврем. мат. и ее прил. Темат. обз. Т. **163**. 2019. С. 113–126. Yumagulov M. G., Ibragimova L. S., Belova A. S. *Methods for studying the stability of linear periodic systems that depend on a small parameter.* // Results of science and technology. Ser. Sovrem. mat. and its adj. Subject. obz. t. **163**. 2019. p. 113-126.

Umbilical singularity of the solution of polytropic gas dynamics equations

Shavlukov A. M.

1. Faculty of Mathematics and Information Technologies
Bashkir State University

2. Department of Differential Equations
Institute of Mathematics with Computing Centre - Subdivision of the Ufa Federal Research Centre of Russian Academy of Science

In the following paper we have discussed typical umbilical singularity of the formal asymptotic solution of the system of one-dimensional polytropic gas flow

$$u_T + uu_X + \alpha(\rho)\rho_X = 0, \rho_X + (\rho u)_X = 0, \tag{1}$$

for the gas state $p = a^2\rho^\gamma$, $a = const$, ρ is gas density, γ is constant polytropic index. We have studied cases $\gamma = 3$, $\gamma = \frac{5}{3}$ (monatomic gas flow).

The system above was rewritten in terms of Riemann invariants. We have coupled Hopf equations in the case $\gamma = 3$.

The formal asymptotic solution is described by canon equation of a truncated hyperbolic umbilic in the neighborhood of the point where the solution is not smooth. The perturbation of the germ is different from the one explored in [5] in the case of $\gamma = \frac{5}{3}$. We have put forward a hypothesis that known classification of Riemann invariants isn't fully correct.

The result was obtained in work with B.I. Suleimanov.

References

[1] Rakhimov, A.K. Singularities of Riemannian invariants. Funct Anal Its Appl 27, 39–50 (1993)

On the arrangement of a pair of nonsingular curves of degree 2 and nonsingular curve of degree 3 in the real projective plane

Sokolova I.M.

Department of Mathematics, Informatics and Computer Sciences
National Research University Higher School of Economics

At the II International Congress of Mathematicians in Paris in 1900, David Hilbert formulated 23 mathematical problems that should be solved in the XX century. The sixteenth problem of D. Hilbert is called "The problem of topology of algebraic curves and surfaces". It is divided into two parts, the first of which is the study of the topology of nonsingular algebraic curves on the projective plane $\mathbb{R}P^2$ and nonsingular algebraic surfaces in the projective space $\mathbb{R}P^3$. The topic of Hilbert's sixteenth problem is related to the problem of studying the topology of real algebraic varieties of dimension d in the real projective space $\mathbb{R}P^q$, where $q \geqslant 3, 1 \leqslant d \leqslant (q-1)$, as well as the study of real algebraic varieties with singularities. Thus, the problem of the topology of decomposable plane algebraic curves, a fragment of which is considered in this work, is also included in the range of questions related to Hilbert's sixteenth problem.

In this work, we will consider the problem of topological classification of arrangements in the real projective plane of the union of a pair of nonsingular curves of degree 2 and nonsingular curve of degree 3, assuming that certain conditions of maximality and general position are met (we will take into account the known information about the mutual arrangements of a pair of nonsingular curves of degree 2 and nonsingular curve of degree 3 given in the articles [2] and [3]). We will find a list of topological models of such curves that do not contradict to the topological consequences of the Bezu theorem and the prohibitions proved by the theory of braids and links (article [4]). It is proved that the classification contains no more than two specific types of arrangements in the real projective plane of the studied type, the question of the realizability of which is still open.

References

[1] Topology of real algebraic varieties. Part 1 – 6 / Methodological development, comp. Gudkov D.A., Polotovsky G.M., Shustin E.I. – Gorky: GSU, 1986-1987.

[2] Polotovsky G. M. Topological classification of decomposable curves of degree 6 // Dissertation for the degree of candidate of physical and mathematical sciences, Gorky, 1979. 261 p.

[3] Kuzmenko T., Polotovsky G. Classification of curves of degree 6 decomposing into a product of M-curves in general position // AMS Translations, Ser. 2, Vol. 173 "Topology of real algebraic Vsrieties and Related Topics". 1996.P.165-177

[4] Orevkov S.Yu., Link theory and oval arrangments of real algebraic curve, Topology, 38 (1999), 779-810

Routes to hyperchaos in a three-dimensional Mirá map.

Shykhmamedov A.I.

International Laboratory of Dynamical Systems and Applications
National Research University Higher School of Economics

In this work, we study scenarios of the appearance of chaotic and hyperchaotic attractors in three-dimensional Mirá map of the form

$$\begin{cases} \bar{x} = y \\ \bar{y} = z \\ \bar{z} = Bx + Cy + Az - y^2. \end{cases} \tag{1}$$

where x, y, z are map variables and A, B, C are parameters. Note, that the determinant of the Jacoby matrix of this map is equal to B. From the papers [1] and [2], it is known that this map demonstrates discrete Shilnikov attractors. These attractors appear in accordance with the scenario presented in [3, 1]. The main stage of this scenario is the absorption of the saddle-focus fixed point with two-dimensional unstable manifold which appears after supercritical Neimark-Sacker bifurcation. In [4] this scenario was extended to the systems demonstrating secondary Neimark-Sacker bifurcation with stable periodic orbits emerging inside Arnold's tongues. In accordance with this scenario, a chaotic attractor absorbs the periodic saddle-focus orbit which appears via secondary Neimark-Sacker bifurcations and the discrete Shilnikov-like attractor containing this periodic orbit appears. Also, it was shown in paper [4] that this scenario can lead to the birth of hyperchaotic attractors.

In this work, we show that the above scenario leads to the emergence of spiral chaos and hyperchaos in the three-dimensional Mirá map under consideration. Various types of discrete Shilnikov-like attractors containing different period orbits are found. We also discuss that depending on the measure of saddle-focus periodic orbits belonging to the attractor comparing with the saddles with one-dimensional unstable manifold the resulting Shilnikov-like attractors may be chaotic or hyperchaotic.

In the second part of this work, we show that for some values of parameters, e.g. for small Jacobian B, the map under consideration demonstrates hyperchaotic attractors without saddle-focus periodic orbit. We propose for this case a new scenario. The key part of this scenario is the cascades of period-doubling bifurcations with periodic saddle orbits belonging to the Hénon-like chaotic attractor which, in its turn, appears from the stable fixed point due to the Feigenbaum's cascade followed by the cascade of heteroclinic bifurcations of band merging. Note, that after the period-doubling cascades with the saddle orbits almost all periodic orbits belonging to the attractor are saddles with two-dimensional unstable manifold and, thus, the created attractor becomes hyperchaotic one.

This work is supported by the RSF grant 19-71-10048.

References

[1] Gonchenko, A., Gonchenko, S., Kazakov, A., & Turaev, D. (2014). Simple scenarios of onset of chaos in three-dimensional maps. International Journal of Bifurcation and Chaos, 24(08), 1440005.

[2] Gonchenko, A. S., & Gonchenko, S. V. (2016). Variety of strange pseudohyperbolic attractors in three-dimensional generalized Hénon maps. Physica D: Nonlinear Phenomena, 337, 43-57.

[3] Gonchenko, A. S., Gonchenko, S. V., & Shilnikov, L. P. (2012). Towards Scenarios of Chaos Appearance in Three-Dimensional Maps, Nelin.

[4] Garashchuk, I. R., Sinelshchikov, D. I., Kazakov, A. O., & Kudryashov, N. A. (2019). Hyperchaos and multistability in the model of two interacting microbubble contrast agents. Chaos: An Interdisciplinary Journal of Nonlinear Science, 29(6), 063131.