CHERNOFF APPROXIMATIONS OF SOLUTIONS OF DIFFERENTIAL EQUATIONS ON MANIFOLDS

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Abstract

It is known that under some natural assumptions solution to linear evolution equation is given by the one-parameter semigroup of operators. Approximations to this semigroup on compact manifolds were found by Yana Butko in 2006-2008 using the Chernoff theorem. In resent paper by Mazzucchi, Moretti, Remizov and Smolyanov approximations were proposed for the case of parabolic equation on non-compact manifolds. The goal of the course work is to study the above-mentioned papers and to solve new problems in this field.

The main difficulty that we successfully overcome arises from the fact that we DO NOT assume that the manifold is compact.

Definition of a C_0-semigroup

Short definition. C_0-semigroup is a particular case of a semiflow in Banach space, i.e. function $(t,f) \longmapsto Q(t)f$ which is continuous in time $t \in [0,+\infty)$ for fixed vector f, satisfies the semigroup property and is continuous and linear in f for fixed t.

Long definition. Let $\mathcal{F}$ be a Banach space over the field $\mathbb{R}$. Let $\mathscr{L}(\mathcal{F})$ be the set of all bounded linear operators in $\mathcal{F}$. Suppose we have a mapping $Q : [0,+\infty) \to \mathscr{L}(\mathcal{F})$, i.e. $Q(t)$ is a bounded linear operator $Q(t)\colon \mathcal{F} \to \mathcal{F}$ for each $t \geq 0$. The mapping Q is called a C_0-semigroup, or a strongly continuous one-parameter semigroup of operators iff it satisfies the following conditions:

1. $Q(0)$ is the identity operator I, i.e. $\forall f \in \mathcal{F}\colon Q(0)f = f$;
2. Q maps the addition of numbers in $[0;+\infty)$ into a composition of operators in $\mathscr{L}(\mathcal{F})$, i.e. $\forall t \geq 0, \forall s \geq 0\colon Q(t+s) = Q(t) \circ Q(s)$;
3. Q is continuous at zero with respect to the strong operator topology in $\mathscr{L}(\mathcal{F})$, i.e. $\forall f \in \mathcal{F}$ function $t \longmapsto Q(t)f$ is continuous at 0 as a mapping $[0;+\infty) \to \mathcal{F}$, i.e. $\lim_{t\to 0} \| Q(t)f - f \| = 0$ for all $f \in \mathcal{F}$.

Definition of a generator of the C_0-semigroup

Short definition. Generator is a derivative of the semigroup at zero:

$$L = \frac{dQ(t)}{dt}\Big|_{t=0} = Q'(0).$$

Long definition. If $Q(t)_{t\geq 0}$ is a C_0-semigroup in Banach space $\mathcal{F}$, then define the set

$$\left\{\phi \in \mathcal{F} \colon \exists \lim_{t\to+0} \frac{Q(t)\phi - \phi}{t}\right\} = D(L).$$

The operator L defined on the domain $D(L)$ by the equality

$$L\phi = \lim_{t\to+0} \frac{Q(t)\phi - \phi}{t}$$

is called an infinitesimal generator (or generator) of the C_0-semigroup $(Q(t))_{t\geq 0}$.

Remark. If L is the generator of a C_0-semigroup Q then the notation $Q(t) = e^{tL}$ is used.

The Chernoff Theorem

Let $\mathcal{F}$ be a Banach space, and $\mathscr{L}(\mathcal{F})$ be the space of all linear bounded operators in $\mathcal{F}$ endowed with the operator norm. Let $L\colon D(L) \to \mathcal{F}$ be a linear operator defined on $D(L) \subset \mathcal{F}$, and Q be an $\mathscr{L}(\mathcal{F})$-valued function. Suppose that L and Q satisfy:

(E) There exists a C_0-semigroup $(e^{tL})_{t\geq 0}$ and its generator is $(L, D(L))$.

(CT1) The function Q is defined on $[0, +\infty)$, takes values in $\mathscr{L}(\mathcal{F})$, and the mapping $t \longmapsto Q(t)f$ is continuous for every vector $f \in \mathcal{F}$.

(CT2) $Q(0) = I$.

(CT3) There exists a dense subspace $\mathcal{D} \subset \mathcal{F}$ such that for every $f \in \mathcal{D}$ there exists a limit $Q'(0)f = \lim\limits_{t\to 0} \frac{(Q(t)f - f)}{t}$.

(CT4) The operator $(Q'(0), \mathcal{D})$ has a closure $(L, D(L))$.

(N) There exists $C \in \mathbb{R}$ such that $\|Q(t)\| \leq e^{Ct}$ for all $t \geq 0$.

Then for every $f \in \mathcal{F}$ we have $(Q\left(\frac{t}{n}\right))^n f \to e^{tL}f$ as $n \to \infty$, and the limit is uniform with respect to $t \in [0, t_0]$ for every fixed $t_0 > 0$.

If Q satisfies the Chernoff theorem then:
(a) Q is called a Chernoff function for operator L (or Chernoff-tangent to operator L) or Chernoff-equivalent to C_0-semigroup $(e^{tL})_{t\geq 0}$.
(b) The expression $Q(t/n)^n f$ is called a Chernoff approximation expression for $e^{tL}f$.
(c) The $\mathcal{F}$-valued function $u(t) := \lim_{n\to\infty} Q(t/n)^n u_0 = e^{tL}u_0$ is the classical solution of the Cauchy problem, so Chernoff approximation expressions become approximations to the solution with respect to norm in $\mathcal{F}$.

Remark. Every C_0-semigroup $Q(t) = e^{tL}$ is a Chernoff function for its generator L, actually it is the only one Chernoff function which has a semigroup composition property.

Main theorem

Let M be a Riemannian manifold of bounded geometry. Let function $c\colon M \to \mathbb{R}$ be measurable and bounded. Let A_j be the smooth and C^1 - bounded vector fields on M, for all j we have $divA_j(\alpha_s^*(x)) = 0$. Let us define for all $f \in L_p(M, \mathbb{C})$, $x \in M$ and $t \geq 0$:

$$(S(t)f)(x) = \frac{1}{4d} \sum_{j=1}^{d} \left(f\left(\gamma_{x,A_j}(\sqrt{2dt})\right) + f\left(\gamma_{x,-A_j}(\sqrt{2dt})\right)\right) +$$

$$+\frac{1}{2} f(\gamma_{x,A_0}(2t)) + tc(x)f(x),$$

where $\gamma_{x,A_j}\colon \mathbb{R}^+ \to M$ denotes the integral curve of the vector field A_j starting at time 0 at the point $x \in M$, namely the solution of the initial value problem

$$\begin{cases} \frac{d}{dt}\gamma_{x,A_j}(t) = A_j(\gamma_{x,A_j}(t)), \\ \gamma_{x,A_j}(0) = x. \end{cases}$$

We also assume that operator

$$(Lf)(x) = \frac{1}{2}\sum_{j=1}^{d}(A_jA_jf)(x) + A_0f(x) + c(x)f(x)$$

generates a C_0-semigroup $(e^{tL})_{t\geq 0}$. Then with respect to the norm $\|f\| = \left(\int_M |f(x)|^p \mu(dx)\right)^{1/p}$ in $L_p(M,\mathbb{R})$ the following is satisfied:

1. There exists such $C > 0$ that for all $t \geq 0$ we have $\|S(t)\| \leq e^{Ct}$.
2. $S(t)$ is Chernoff tangent to L.
3. The solution of Cauchy problem

$$\begin{cases} u_t'(t,x) = Lu(t,x), & x \in M, \quad t \in \mathbb{R}, \\ u(0,x) = u_0(x), \end{cases}$$

is given by

$$u(t,x) = (e^{tL}u_0)(x) = \left(\lim_{n\to+\infty} S\left(\frac{t}{n}\right)^n u_0\right)(x) \qquad (1)$$

for almost all $x \in M$.

Conclusion

Using the Chernoff theorem, tools of differential geometry and general theory of C_0-semigroups we found the solution to the Cauchy problem for second order parabolic equation on a manifold not assuming that manifold is compact, but assuming that it has a bounded geometry.

Thank you for your attention!

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