Change of variables in an integral over a manifold

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November 25, 2021

We observe a system

$$\frac{d\gamma_{x,A_j}(t)}{dt} = A_j(\gamma_{x,A_j}(t));$$

$$\gamma_{x,A_j}(0) = x.$$

Denote by $X_j(t)$ the vector field induced by the field $A_j(\gamma_{x,A_j}(t))$, for every $t \in [a,b]$ vector $X(t) \in T_{\gamma(t)}M$ exists and $X(t) = \dot{\gamma}(t)$ in local coordinates. Operator $\gamma = \gamma_{x,A_j}(t) : [a,b] \to M$ maps a segment onto a manifold M, where M is a smooth, connected manifold with bounded geometry of dimension n.

Let the set $\gamma * V$ be a set of all smooth vector fields among a curve γ and let the connectivity ∇ be defined on M, it means that $\nabla : V(M) \times V(M) \to V(M) : (X, Y) \to \nabla_X Y$. Then for every γ uniquely defined an operator $\frac{D}{dt} : \gamma * V \to \gamma * V$ such that:

1) $\frac{D(X+Y)}{dt} = \frac{DX}{dt} + \frac{DY}{dt}$;
2) if f(t) is a smooth function on $[a, b]$ then $\frac{D(fX)}{dt} = \frac{df}{dt}X + f\frac{DX}{dt}$;
3) if $X(t) = Y(\gamma(t))$ then $\frac{DX}{dt} = \nabla_{\dot{\gamma}} Y = \nabla_X Y$.

The operator $\frac{DX}{dt}$ is called absolute derivative of vector field X among γ. If $\frac{DX}{dt} = 0$ i.e. $\frac{dX^j}{dt} + \Gamma^j_{ik}\dot{\gamma}^i X^k = 0$. As is known, a solution to the Cauchy problem for such a system exists and is unique. That is why for every $t_0 \in [a, b]$ and for every $X_0 \in T_{\gamma(t_0)}M$ the vector field X(t) among the curve $\gamma : [a, b] \to M$ uniquely exists if $X(t_0) = X_0$ and called parallel field among a curve γ.

Let points p,q $\in M$ be connected by a curve $\gamma : [a, b] \to M$ such that $p = \gamma(a)$ and $q = \gamma(b)$. Let us define a shift operator $\tau_\gamma : T_pM \to T_qM$ satisfying the conditions $X(a) = X_0$ and $\tau_\gamma(X_0) = X(b)$.

Theorem

Shift operator τ_γ is a diffeomorphism of vector fields.

Proof.

Firstly, let us prove that τ_γ is a isomorphism.
Let $\tau_\gamma(X_0) = 0$ for $X_0 \in T_pM$ then $X(b) = 0$ consequently $X(t) = 0$ for all t due to uniqueness of solution to the Cauchy problem. So τ_γ is a linear injective mapping between vector spaces of the same dimension and that is why τ_γ is a isomorphism.
Since we are considering an isomorphism of smooth vector fields, then by definition, our mapping τ_γ is a diffeomorphism. □

Let M be an orientable smooth manifold of dimension n. Orientability of M means that the oriented atlas $A(M)$ is chosen from its smooth structure. Moreover for each $(U, \varphi), (V, \psi) \in A(M)$ such as $U \cap V \neq \emptyset$ transition function $y = \psi \circ \varphi^{-1}(x)$ everywhere on $\varphi(U \cap V)$ satisfies to inequality $det \left(\frac{\partial y}{\partial x} \right) > 0$.

Definition of a tensor

Let M be a smooth manifold, $a \in M$, $T_a M$ is the tangent space to M at point a, and $T_a^* M$ is the cotangent space. Consider numbers $p, q \in \mathbb{Z}, p, q \geq 0$. A mapping $L\colon (T_a^* M)^p \times (T_a M)^q \mathbb{R}$ linear in each argument is called a tensor of type (p, q) of the space $T_a M$.

Definition of a polylinear form

Mapping $\omega\colon (T_a(M))^q\mathbb{R}$ linear in each argument is called q-linear or polylinear form on T_aM. Polylinear form is a tensor of type $(0,q)$ i.e. $\omega \in T_q^0(T_aM)$.

Let us observe symmetric group S_q consisting of all bijections $\sigma\colon \{1,\ldots,q\} \to \{1,\ldots,q\}$. Composition of mappings is used as a group operation on S_q. For each element $\sigma \in S_q$, put $sgn(\sigma) = 1$ if the substitution is even and $sgn(\sigma) = -1$ otherwise.

Definition of an exterior form

Let ω be a q-linear form on T_aM and

$$\omega(X_{\sigma(1)}, \ldots, X_{\sigma(q)}) = sgn(\sigma)\omega(X_1, \ldots, X_q)$$

for all $X_1, \ldots, X_q \in T_aM$ and $\sigma \in S_q$. In this case ω is called an exterior form of degree q on T_aM. The set of all exterior q-forms on T_aM is denoted by $\Lambda^q(T_aM)$.

It is easy to notice that $\Lambda^q(T_aM)$ is a subspace of vector space $T_q^0(T_aM)$.

We denote by $\Lambda^q(M)$ a set of differential exterior forms of degree q on M. $\Lambda^q(M)$ is vector space over the field $\mathbb{R}$ and module over the ring of smooth functions $F(M)$.

Definition of a volume form

Let g be a Riemannian metric on M, then there is a positive definite and a symmetric tensor field of type (0,2). This metric g defines a volume form $\Omega \in \Lambda^n(M)$ that does not vanish anywhere and in each chart $(U, \varphi) \in A(M)$ has the form

$$\Omega = \sqrt{det(g_{ij})} d\varphi^1 \wedge \cdots \wedge d\varphi^n,$$

where (g_{ij}) is a matrix composed of the components of the tensor field g in the map (U, φ).

Definition of differential of mapping

Let $f\colon M \to \mathbb{R}$ be a smooth function, $a \in M$ and $b = f(a)$. Then $T_b\mathbb{R} = \mathbb{R}$. That is why a differential of mapping f at point a is called linear mapping $df_a\colon T_aM \to \mathbb{R}$. By df we denote exterior form on M.

Definition of codifferential of diffeomorphism

Let us observe smooth vector field A on M. If it is complete then it generates an one-parametric group of diffeomorphisms $\alpha_t \colon M \to M, t \in \mathbb{R}$. For all $a \in M$ we have

$$\frac{d\alpha_t(a)}{dt} = A(\alpha_t(a)).$$

Put $b = \alpha_t(a)$ and

$$(\alpha_t^* \Omega)_a = (\alpha_t^*)_b(\Omega_b),$$

where $(\alpha_t^*)_b \colon \Lambda^q(T_b M) \to \Lambda^q(T_a M)$ is a codifferential of diffeomorphism α_t at point b.

The codifferential is the adjoint of the exterior derivative.
By the way forms $(\alpha_t^*\Omega)_a \in \Lambda^n(T_aM)$ and $\alpha_t^*\Omega \in \Lambda^n(M)$ are defined. It is known that $dim\Lambda^n(T_aM) = 1$ and $\Omega \neq 0$ so there is a number $J_t(a) \in \mathbb{R}$ such that

$$(\alpha_t^*\Omega)_a = J_t(a)\Omega_a.$$

So we define a family of smooth functions $J_t \colon M \to \mathbb{R}$ smoothly depending on a parameter $t \in \mathbb{R}$. It is natural to call the function J_t the Jacobian of the diffeomorphism α_t with respect to the volume form Ω.

Lemma 1. We have an equation

$$J_t = 1 + t div A + o(t) \tag{1}$$

Proof. By construction of Jacobian we have

$$\alpha_t^* \Omega = J_t \Omega.$$

Differentiating the last equality with respect to t, we obtain

$$\frac{d}{dt}(\alpha_t^* \Omega) = \frac{dJ_t}{dt}\Omega. \tag{2}$$

By definition of Lie derivative of the tensor field Ω in the direction of the vector field A we have

$$\frac{d}{dt}(\alpha_t^* \Omega)|_{t=0} = L_A \Omega. \tag{3}$$

On the other hand, by definition of the divergence of vector field A

$$L_A \Omega = (div A)\Omega. \tag{4}$$

Consequently

$$(divA)\Omega = \left(\frac{dJ_t}{dt}\bigg|_{t=0} \right) \Omega,$$

it means that

$$divA = \frac{dJ_t}{dt}\bigg|_{t=0}. \tag{5}$$

Since $\alpha_0 = id_M$ and for all $a \in M$ we have $b = \alpha_0(a) = a$, then $\alpha_0^* = id_{\Lambda^n(M)}$ i.e. $\alpha_0^*\Omega = \Omega$. According to the definition of the Jacobian, this is means that $J_0 = 1$. Substituting the last equality and (5) into the Taylor formula for the function $J : t \to J_t$, we obtain (1).

Lemma 2. For all $a \in M$ and $t \in \mathbb{R}$ we have

$$J_t(a) = exp\left(\int_0^t divA(\alpha_s^*(a))ds\right). \tag{6}$$

Proof. By construction of Jacobian for all $t, s \in \mathbb{R}$ we have

$$\alpha_{t+s}^*\Omega = J_{t+s}\Omega \tag{7}$$

and

$$\left.\frac{dJ_{t+s}}{ds}\right|_{s=0} = \frac{dJ_t}{dt}. \tag{8}$$

On the other hand $\alpha_{t+s} = \alpha_t \circ \alpha_s$ so $\alpha^*_{t+s}\Omega = \alpha^*_s(\alpha^*_t\Omega)$. That is why

$$\frac{d}{ds}(\alpha^*_{t+s}\Omega)\bigg|_{s=0} = \frac{d}{ds}(\alpha^*_s(\alpha^*_t\Omega))\bigg|_{s=0} = L_A(\alpha^*_t\Omega),$$

where $L_A(\alpha^*_t\Omega)$ is the Lie derivative of form $\alpha^*_t\Omega$ in the direction of the vector field A. But $L_A(\alpha^*_t\Omega) = \alpha^*_t(L_A\Omega)$, from (4,7,8)

$$\frac{d}{ds}(\alpha^*_{t+s}\Omega)\bigg|_{s=0} = \alpha^*_t(L_A\Omega) = \alpha^*_t((divA)\Omega).$$

Since $\alpha_t^*((divA)\Omega) = (divA \circ \alpha_t)\alpha_t^*\Omega$ then

$$\frac{d}{ds}(\alpha_{t+s}^*\Omega)\bigg|_{s=0} = (divA \circ \alpha_t)\alpha_t^*\Omega = (divA \circ \alpha_t)J_t\Omega. \qquad (9)$$

Now we differentiate (7) with respect to s and substitute into the resulting equality $s = 0$. Then, by virtue of (8) and (9), for any point $a \in M$, we obtain

$$divA(\alpha_t(a))J_t\Omega_a = \frac{dJ_t(a)}{dt}\Omega_a.$$

Since $\Omega_a \neq 0$, this implies

$$\frac{dJ_t(a)}{dt} = divA(\alpha_t(a))J_t(a). \qquad (10)$$

Obviously, (6) is a solution to the differential equation (10) with the initial condition $J_0(a) = 1$.