Speed of convergence of Chernoff approximations
Seminar talk

Student: Pavel Prudnikov
Scientific supervisor: Ivan Remizov

Research group: Evolution equations and applications
National Research University Higher School of Economics

10.03.2021

Outline

Introduction. Part I

- What we will study: the Cauchy problem for the translation and heat equations with the corresponding C_0-semigroups, Chernoff functions based on translation operators;
- What we will do: approximate the speed of convergence of Chernoff approximations as well as the order of approximation subspaces for specific initial conditions.

Introduction. Part II

Goals:

1. analyse several initial conditions and find the explicit solutions of the Cauchy problem for the heat equation,
2. simplify the formulae that give Chernoff approximations,
3. find the order of approximation subspaces for specific initial conditions,
4. acquire similar results computationally to justify the proposed analysis.

Theorem (The Chernoff theorem)

Let $\mathcal{F}$ be a Banach space, and $\mathscr{L}(\mathcal{F})$ be a space of all linear bounded operators in $\mathcal{F}$. Let a mapping $G\colon [0,+\infty) \to \mathscr{L}(\mathcal{F})$, or the same as a family of linear bounded operators $(G(t))_{t\geq 0}$ in $\mathcal{F}$, also be given. Moreover, assume having a closed linear operator $L\colon (L) \subset \mathcal{F} \to \mathcal{F}$. Suppose that the following conditions are met:

- (E). *there exists a C_0-semigroup $(e^{tL})_{t\geq 0}$ with generator $(L, (L))$;*
- (CT). *function G is Chernoff tangent to operator L;*
- (N). *there exists number $\omega \in \mathbb{R}$ such that $\|G(t)\| \leq e^{\omega t}$ for all $t \geq 0$.*

Then for each $f \in \mathcal{F}$ we have $(G(t/n))^n f \xrightarrow{n\to\infty} e^{tL} f$ uniformly in $t \in [0, T]$ for any fixed $T \geq 0$, i.e., for each $f \in \mathcal{F}$ and each $T \geq 0$ we get

$$\lim_{n\to\infty} \sup_{t\in[0,T]} \left\| \left(G\left(\frac{t}{n}\right)\right)^n f - e^{tL} f \right\| = 0. \tag{1}$$

Analytic representation of convergence speed

Analyse the convergence rate for:

$$\begin{cases} u'_t(t,x) = u'_x(t,x), \\ u(0,x) = sin(x), \end{cases} \tag{2}$$

and

$$\begin{cases} u'_t(t,x) = u''_x(t,x), \\ u(0,x) = sin(x), \end{cases} \tag{3}$$

where $x \in \mathbb{R}$, $u_0 \in UC_b(\mathbb{R})$ for all $t \geq 0$.

Definition

Let $\tau \subset [0, +\infty) = \mathbb{R}^+$, and map $\psi \colon \mathbb{R}^+ \to \mathbb{R}^+$ be such that $\lim_{x \to +\infty} \psi(x) = 0$. Then the set

$$A^\tau_\psi = \left\{ f \in \mathcal{F} \mid \sup_{t \in \tau} \left\| \left(G\left(\frac{t}{n}\right)\right)^n f - e^{tL} f \right\| = O(\psi(n)) \text{ as } n \to \infty \right\}$$

is called an approximation subspace of order ψ.

Translation equation

For the linear one-dimensional translation equation the following theorem was proposed:

Theorem

Let $(e^{tL})_{t\geq 0}$ *be the translation semigroup on the real line*

$$(G(t)u_0)(x) = u_0(x+t), \tag{4}$$

for $u_0 \in UC_b(\mathbb{R})$ *and* $t \geq 0$, *and* G *be the Chernoff function*

$$(G(t)f)(x) = f(x + t + at^{k+1}) \tag{5}$$

for some fixed $a, k > 0$.
Then *function* $u_0 = [x \mapsto \sin(x)]$ *belongs to the approximation subspace of order* $\frac{1}{n^k}$, *and the error is given by the formula*

$$\left\| \left(G\left(\frac{t}{n}\right)\right)^n u_0 - e^{tL}u_0 \right\| = 2\left|\sin(at^{k+1}/2n^k)\right|. \tag{6}$$

Heat equation

For the heat equation the following theorem was stated:

Theorem

Let $(e^{tL})_{t\geq 0}$ *be the heat* C_0*-semigroup with the solution for* $u_0 = [x \mapsto \sin(x)]$ *given by*

$$u(t,x) = (e^{tL}u_0)(x) = e^{-a^2 t}\sin(x), \tag{20}$$

and G *be the Chernoff function given by*

$$\left(G(t)u_0\right)(x) = \frac{1}{4}u_0(x + 2a\sqrt{t}) + \frac{1}{4}u_0(x - 2a\sqrt{t}) + \frac{1}{2}u_0(x)$$

for all $f \in UC_b(\mathbb{R})$ *and fixed* $a > 0$.
Then *function* $u_0 = [x \mapsto \sin(x)]$ *belongs to the approximation subspace of order* $\frac{1}{n}$.

Initial condition and solution

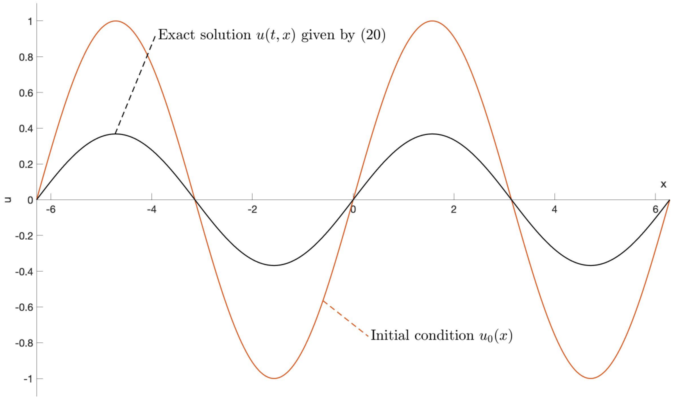

Figure 1: Graph of the initial condition $u_0 = [x \mapsto \sin(x)]$ (in orange) and the corresponding solution of the heat equation (in black)

Chernoff functions

We will analyse the convergence speed using the following Chernoff functions:

$$(G(t)u_0)(x) = \frac{1}{4}u_0(x + 2a\sqrt{t}) + \frac{1}{4}u_0(x - 2a\sqrt{t}) + \frac{1}{2}u_0(x) \quad (15)$$

$$(G(t)u_0)(x) = \frac{1}{6}u_0(x + a\sqrt{6t}) + \frac{1}{6}u_0(x - a\sqrt{6t}) + \frac{2}{3}u_0(x) \quad (16)$$

$$\begin{aligned}(G(t)u_0)(x) &= \frac{1}{30}u_0(x + a\sqrt{12t}) + \frac{1}{30}u_0(x - a\sqrt{12t}) \\ &+ \frac{3}{10}u_0(x + a\sqrt{2t}) + \frac{3}{10}u_0(x - a\sqrt{2t}) + \frac{1}{3}u_0(x)\end{aligned} \quad (17)$$

Simple form of Chernoff functions

Exploiting basic trigonometric identities, one can obtain the following closed forms of Chernoff functions:

$$\left(\left(G\left(\frac{t}{n}\right)\right)^n u_0\right)(x) = \left(\cos\left(a\sqrt{\frac{t}{n}}\right)\right)^{2n} \sin(x), \tag{18}$$

$$\left(\left(G\left(\frac{t}{n}\right)\right)^n u_0\right)(x) = \frac{1}{3^n}\left(2+\cos\left(a\sqrt{\frac{6t}{n}}\right)\right)^n \sin(x), \tag{19}$$

$$\left(\left(G\left(\frac{t}{n}\right)\right)^n u_0\right)(x) = \frac{1}{15^n}\left(5+\cos\left(a\sqrt{\frac{12t}{n}}\right)\right. \tag{20}$$
$$\left. + 9\cos\left(a\sqrt{\frac{2t}{n}}\right)\right)^n \sin(x).$$

Convergence visualization

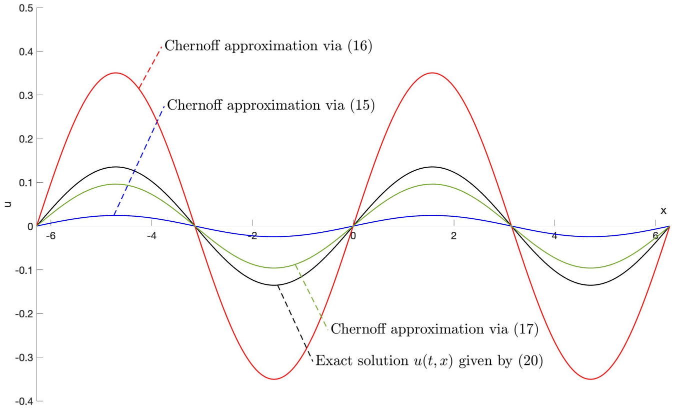

Figure 2: Graph of Chernoff approximations (in blue, red, and green) and the corresponding solution of the heat equation for $u_0 = [x \mapsto \sin(x)]$ (in black) when $n = 1$

Convergence speed approximation

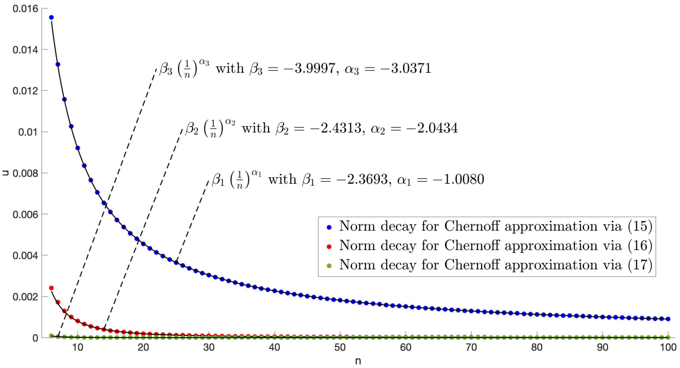

Figure 3: Graph of speed of convergence of Chernoff approximations (in blue, red, and green) to the solution of the heat equation for $u_0 = [x \mapsto \sin(x)]$, and the continuous approximations (in black) when $n = 5..100$

Convergence speed approximation

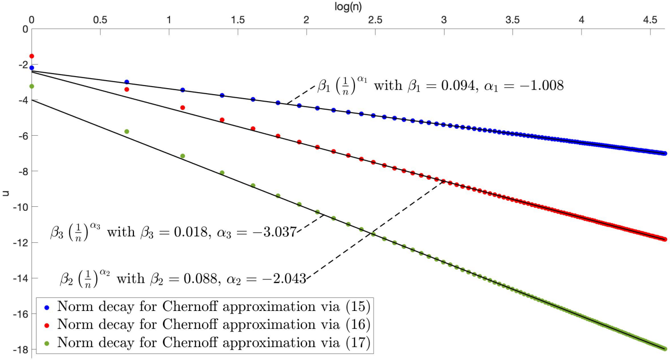

Figure 4: Graph of speed of convergence of Chernoff approximations (in blue, red, and green) to the solution of the heat equation for $u_0 = [x \mapsto \sin(x)]$, and the continuous approximations (in black) when $n = 5..100$ in log-log scale

Solution for the heat equation

Theorem

Suppose $u_0 = [x \mapsto e^{-|x|}]$. *Then the solution of the Cauchy problem for the heat equation with initial condition* u_0 *and* $a = 1$ *is given by the formula*

$$u(t,x) = (e^{tL}u_0)(x) = e^{t-x}\left(1 - \frac{1}{2}\operatorname{erfc}\left(\frac{x}{2\sqrt{t}} - \sqrt{t}\right)\right) + e^{t+x}\frac{1}{2}\operatorname{erfc}\left(\frac{x}{2\sqrt{t}} + \sqrt{t}\right), \tag{24}$$

where

$$\operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}}\int_x^{+\infty} e^{-y^2}\,dy.$$

Initial condition and solution

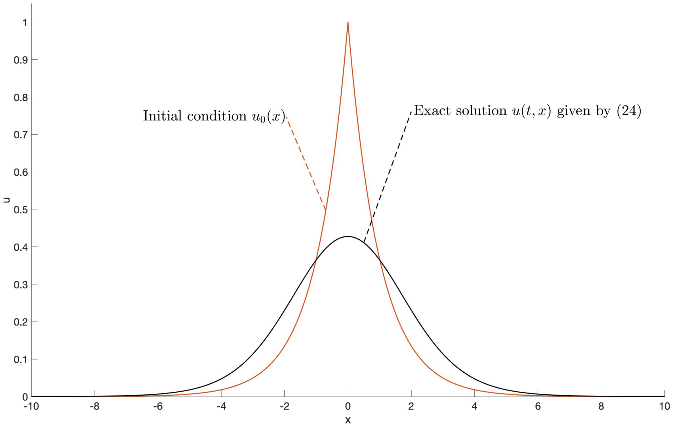

Figure 5: Graph of the initial condition $u_0 = [x \mapsto \exp(-|x|)]$ (in orange) and the corresponding solution of the heat equation (in black)

Simple form of Chernoff functions

Proposition

Let G be the Chernoff function given by (15) for all $f \in UC_b(\mathbb{R})$ and fixed $a > 0$. Then its n-th composition degree is given by the formula

$$\left(\left(G\left(\frac{t}{n}\right)\right)^n u_0\right)(x) = \frac{1}{4^n}\sum_{p=-n}^{n} \alpha_{p,n} u_0(x + 2ap\sqrt{t/n}), \qquad (7)$$

where for each $n \in \mathbb{N}$ and $p \in \{-n, \dots, n\}$ coefficients $\alpha_{p,n}$ are defined as

$$\alpha_{p,n} = \sum_{k=0}^{\left[\frac{n-|p|}{2}\right]} C_k^n C_{n-k}^{k+|p|} 2^{n-|p|-2k}.$$

Convergence visualization

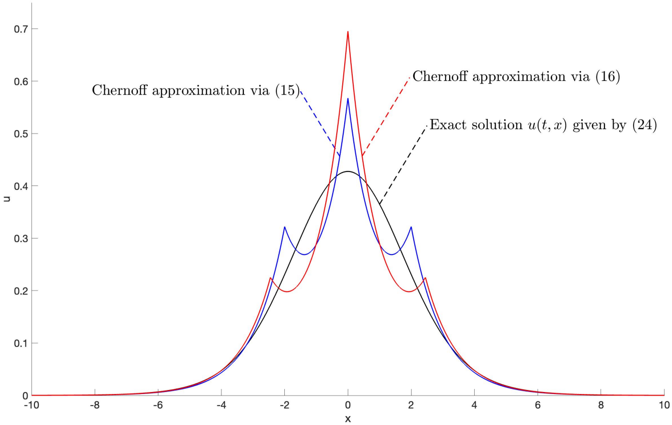

Figure 6: Graph of Chernoff approximations (in blue and red) and the corresponding solution of the heat equation for $u_0 = [x \mapsto \exp(-|x|)]$ (in black) when $n = 1$

Convergence speed approximation

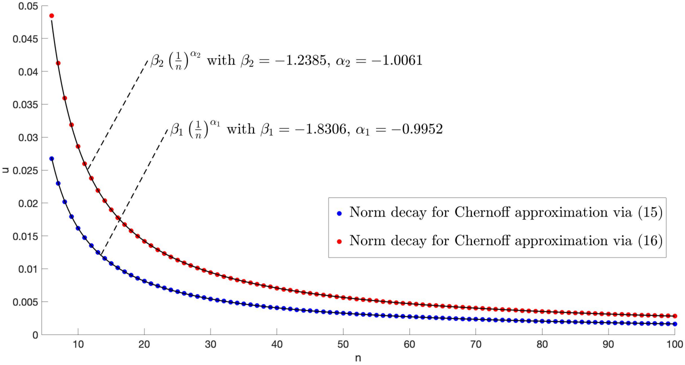

Figure 7: Graph of speed of convergence of Chernoff approximations (in blue and red) to the solution of the heat equation for $u_0 = [x \mapsto \exp(-|x|)]$, and the continuous approximations (in black) when $n = 5..100$

Convergence speed approximation

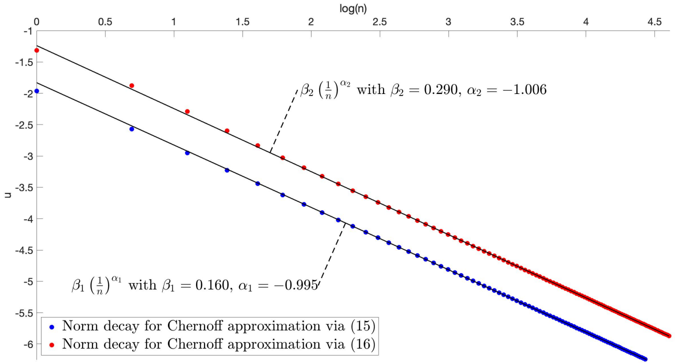

Figure 8: Graph of speed of convergence of Chernoff approximations (in blue and red) to the solution of the heat equation for $u_0 = [x \mapsto \exp(-|x|)]$, and the continuous approximations (in black) when $n = 5..100$ in log-log scale

Conclusion

So, the following goals were achieved:

1. Section 2: analytic approach of finding the approximation subspaces for $u_0 = [x \mapsto \sin(x)]$ within the translation semigroup and the heat equation;
2. Section 3: numerical experiments performed to analyse the order of approximation subspaces for $u_0 = [x \mapsto \sin(x)]$ and $u_0 = [x \mapsto \exp(-|x|)]$ for both translation and heat equations.

My publications on the topic

1. P. Prudnikov. Speed of convergence of Chernoff approximations for two model examples: heat equation and transport equation (arXiv:2012.09615; submitted to Applied Mathematics and Computation).

Thank you for your attention!