Speed of Convergence in Heat Equation
Seminar Talk

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Preliminaries

The next differential equation we are going to analyse is the heat equation. Now, we will state the corresponding Cauchy problem and give an explicit formula for the solution in a form of a C_0-semigroup.

Definition

The system of equations defined the following way

$$\begin{cases} u'_t(t,x) = Lu(t,x) = a^2 u''_{xx}(t,x), \\ u(0,x) = u_0(x), \end{cases} \tag{1}$$

where $a > 0$, $x \in \mathbb{R}$, and $u_0 \in UC_b(\mathbb{R})$ for all $t \geq 0$, is called a Cauchy problem for the heat equation on the real line with constant coefficient of thermal conductivity.

Remark

The solution of the system (1), and, hence, the corresponding C_0-semigroup, is given by the Poisson integral

$$u(t,x) = (e^{tL}u_0)(x) = \int_{\mathbb{R}} \Phi(t, x-y)u_0(y)dy, \qquad (2)$$

where

$$\Phi(t,x) = \frac{1}{2a\sqrt{\pi t}} \exp\left(-\frac{x^2}{4a^2 t}\right). \qquad (3)$$

Chernoff functions. Part I

We will analyse the performance of approximations for the following three Chernoff functions:

$$(G(t)u_0)(x) = \frac{1}{4}u_0(x + 2a\sqrt{t}) + \frac{1}{4}u_0(x - 2a\sqrt{t}) + \frac{1}{2}u_0(x) \quad (4)$$

proposed by I. Remizov in [REM2018],

$$(G(t)u_0)(x) = \frac{1}{6}u_0(x + a\sqrt{6t}) + \frac{1}{6}u_0(x - a\sqrt{6t}) + \frac{2}{3}u_0(x) \quad (5)$$

proposed by A. Vedenin in [REM2020]

Chernoff functions. Part II

... and

$$(G(t)u_0)(x) = \frac{1}{30}u_0(x + a\sqrt{12t}) + \frac{1}{30}u_0(x - a\sqrt{12t}) \\ + \frac{3}{10}u_0(x + a\sqrt{2t}) + \frac{3}{10}u_0(x - a\sqrt{2t}) \\ + \frac{1}{3}u_0(x) \tag{6}$$

also proposed by A. Vedenin in oral communication to the author of the present paper in 2020. To simplify the complexity of computational algorithms one might need a general formula for the n-th composition degree for the above Chernoff functions.

Proposition

Let G be the Chernoff function given by (4) for all $f \in UC_b(\mathbb{R})$ and fixed $a > 0$. Then its n-th composition degree is given by the formula

$$\left(\left(G\left(\frac{t}{n}\right)\right)^n u_0\right)(x) = \frac{1}{4^n}\sum_{p=-n}^{n} \alpha_{p,n} u_0(x + 2ap\sqrt{t/n}), \qquad (7)$$

where for each $n \in \mathbb{N}$ and $p \in \{-n, \dots, n\}$ coefficients $\alpha_{p,n}$ are defined as

$$\alpha_{p,n} = \sum_{k=0}^{\left[\frac{n-|p|}{2}\right]} C_k^n C_{n-k}^{k+|p|} 2^{n-|p|-2k}.$$

Proposition

Let G be the Chernoff function given by (5) for all $f \in UC_b(\mathbb{R})$ and fixed $a > 0$. Then its n-th composition degree is given by the formula

$$\left(\left(G\left(\frac{t}{n}\right)\right)^n u_0\right)(x) = \frac{1}{6^n}\sum_{p=-n}^{n} \beta_{p,n} u_0(x + ap\sqrt{6t/n}), \qquad (8)$$

where for each $n \in \mathbb{N}$ and $p \in \{-n, \dots, n\}$ coefficients $\beta_{p,n}$ are defined as

$$\beta_{p,n} = \sum_{k=0}^{\left[\frac{n-|p|}{2}\right]} C_k^n C_{n-k}^{k+|p|} 4^{n-|p|-2k}.$$

Theorem

Suppose $u_0 = [x \mapsto \sin(x)]$. *Then the solution of the system* (1) *with initial condition* u_0 *is given by the formula* $u(t,x) = (e^{tL}u_0)(x) = e^{-a^2 t}\sin(x)$.

Prior to analysing the convergence rate for Chernoff function (4) and $u_0 = [x \to \sin(x)]$ let us compute its n-th composition degree.

Theorem

Suppose $u_0 = [x \mapsto e^{-|x|}]$. *Then the solution of the system* (1) *with initial condition* u_0 *and* $a = 1$ *is given by the formula*

$$u(t,x) = (e^{tL}u_0)(x) = e^{t-x}\left(1 - \frac{1}{2}\operatorname{erfc}\left(\frac{x}{2\sqrt{t}} - \sqrt{t}\right)\right) + e^{t+x}\frac{1}{2}\operatorname{erfc}\left(\frac{x}{2\sqrt{t}} + \sqrt{t}\right), \tag{9}$$

where

$$\operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}}\int_x^{+\infty} e^{-y^2}\,dy. \tag{10}$$

Proposition

Let G be the Chernoff function given by (5) (respectively (6)) for all $f \in UC_b(\mathbb{R})$ and fixed $a > 0$. Then its n-th composition degree for $u_0 = [x \mapsto \sin(x)]$ is given by

$$\left(\left(G\left(\frac{t}{n}\right)\right)^n u_0\right)(x) = \frac{1}{3^n}\left(2 + \cos\left(a\sqrt{\frac{6t}{n}}\right)\right)^n \sin(x) \qquad (11)$$

respectively

$$\left(\left(G\left(\frac{t}{n}\right)\right)^n u_0\right)(x) = \frac{1}{15^n}\left(5 + \cos\left(a\sqrt{\frac{12t}{n}}\right) + 9\cos\left(a\sqrt{\frac{2t}{n}}\right)\right)^n \cdot \sin(x). \qquad (12)$$

Proposition

Let G be the Chernoff function given by (4) for all $f \in UC_b(\mathbb{R})$ and fixed $a > 0$. Then its n-th composition degree for $u_0 = [x \mapsto \sin(x)]$ has the form

$$\left(\left(G\left(\frac{t}{n}\right)\right)^n u_0\right)(x) = \left(\cos\left(a\sqrt{\frac{t}{n}}\right)\right)^{2n} \sin(x). \qquad (13)$$

Theorem

Let $(e^{tL})_{t\geq 0}$ be the heat C_0-semigroup, and G be the Chernoff function given by (4) for all $f \in UC_b(\mathbb{R})$ and fixed $a > 0$. Then function $u_0 = [x \mapsto \sin(x)]$ belongs to the approximation subspace of order $\frac{1}{n}$.

Thank you for your attention!